Obstructions to integrability of nearly integrable dynamical systems near regular level sets

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Abstract. We consider analytical, nearly integrable systems which may be non-Hamiltonian, and discuss their nonintegrability in the non-Hamiltonian sense. We give a sufficient condition for them to be analytically nonintegrable such that the commutative vector fields and first integrals depend on the perturbation parameter analytically. We compare our results with classical results of Poincaré and Kozlov for systems written in action and angle coordinates and discuss their relationships with the subharmonic and homoclinic Melnikov methods for periodic perturbations of single-degree-of-freeedom Hamiltonian systems. This is joint work with Kazuyuki Yagasaki (Kyoto University).

1 Introduction

Consider a dynamical systems of the form

$$\dot{x} = X_\varepsilon(x), \quad x \in \mathscr{M}, \tag{1.1}$$

where ε is a small parameter such that $|\varepsilon| \ll 1$, $\mathscr{M}$ is an n-dimensional analytic manifold for $n \geq 2$ and the vector field X_ε is analytic in x and ε. Let $X_\varepsilon = X^0 + \varepsilon X^1 + O(\varepsilon^2)$ for $|\varepsilon| > 0$ sufficiently small. When $\varepsilon = 0$, the system (1.1) becomes

$$\dot{x} = X^0(x), \tag{1.2}$$

which is assumed to be *analytically* $(q, n-q)$*-integrable* in the following sense of Bogoyavlenskij [2] for some positive integer $q \leq n$.

Definition 1.1 (Bogoyavlenskij)**.** *The system* (1.2) *is called* $(q, n-q)$-integrable *or simply* integrable *if there exist q vector fields $Y_1(x)(:= X^0(x)), Y_2(x), \ldots, Y_q(x)$ and $n-q$ scalar-valued functions $F_1(x), \ldots, F_{n-q}(x)$ such that the following two conditions hold:*

(i) *$Y_1(x), \ldots, Y_q(x)$ are linearly independent almost everywhere and commute with each other, i.e., $[Y_j, Y_k](x) \equiv 0$ for $j, k = 1, \ldots, q$, where $[\cdot, \cdot]$ denotes the Lie bracket;*

(ii) *$F_1(x), \ldots, F_{n-q}(x)$ are functionally independent, i.e., $dF_1(x), \ldots, dF_{n-q}(x)$ are linearly independent almost everywhere, and $F_1(x), \ldots, F_{n-q}(x)$ are first integrals of $Y_1, \ldots, Y_q$, i.e., $dF_k(Y_j) = 0$ for $j = 1, \ldots, q$ and $k = 1, \ldots, n-q$.*

If $Y_1, Y_2, \ldots, Y_q$ and $F_1, \ldots, F_{n-q}$ are analytic, then Eq. (1.2) *is said to be* analytically integrable.

Note that Definition 1.1 is considered as a generalization of Liouville-integrability for Hamiltonian systems [1] since an m-degree-of-freedom Liouville-integrable Hamiltonian system with $m \geq 1$ has not only m functionally independent first integrals but also m linearly independent commutative (Hamiltonian) vector fields generated by the first integrals.

The system (1.1) is regarded as a perturbation of the analytically $(q, n-q)$-integrable system (1.2). Under some assumptions (see Section 2), we give sufficient conditions for nonexistence of $n-q$ real-analytic first integrals and for real-analytic nonintegrability of the perturbed system (1.1) near a regular level set such that the first integrals and commutative vector fields depend analytically on ε near $\varepsilon = 0$. The persistence of such first integrals and commutative vector fields in the perturbed system (1.1) along with that of periodic and homoclinic orbits was previously discussed in [10]. Our approach is based on the technique of [10] and different from those of Poincaré [11, 12] and Kozlov [5, 6]. We also describe a consequence of our results to nearly integrable systems in the action-angle coordinates and show how it improves the results of Poincaré [12] and Kozlov [5, 6]. Moreover, motivated by Ziglin [15] and Morales-Ruiz [8], we discuss relationships between our results and the subharmonic and homoclinic Melnikov methods [3, 7, 13] for time-periodic perturbations of single-degree-of-freedom analytic Hamiltonian systems. Here we prove that if the homoclinic Melnikov function is not identically constant, then the perturbed systems are not Bogoyavlenskij-integarble such that the commutative vector fields and first integrals depend analytically on ε near $\varepsilon = 0$.

As an application, we apply our theory to the periodically forced Duffing oscillator and show nonexistence of real-analytic first integrals and real-analytic nonintegrability of the system even if it does not have transverse homoclinic orbits to a periodic orbit.

2 Main Results

In this section, we state our main results for (1.1). We first make the following assumptions on the unperturbed system (1.2):

(A1) For some positive integer $q < n$, the system (1.2) is analytically $(q, n-q)$ integrable, i.e., there exist q analytic vector fields $Y_1(x)(:= X^0(x)), \dots, Y_q(x)$ and $n-q$ analytic scalar-valued functions $F_1(x), \dots, F_{n-q}(x)$ such that conditions (i) and (ii) of Definition 1.1 hold.

(A2) Let $F(x) = (F_1(x), \dots, F_{n-q}(x))$. There exists a regular value $c \in \mathbb{R}^{n-q}$ of F, i.e., rank $dF(x) = n-q$ when $F(x) = c$, such that $F^{-1}(c)$ has a connected and compact component and $Y_1(x), \dots, Y_q(x)$ are linearly independent on $F^{-1}(c)$.

We say that the level set $F^{-1}(c)$ is *regular* if $c \in \mathbb{R}^{n-q}$ is a regular value of F.

Under (A1) and (A2), by the Liouville-Miuner-Arnold-Jost theorem (see [2, 16, 17] for the details), we can transform (1.2) into the following system:

$$\dot{I} = 0, \quad \dot{\theta} = \omega(I), \quad (I, \theta) \in U \times \mathbb{T}^q, \tag{2.1}$$

where $\mathbb{T}^q = \prod_{j=1}^q \mathbb{S}^1$ with $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, U is an open subset in $\mathbb{R}^{n-q}$ and $\omega(I)$ is analytic in I. The variables I and θ are called the *action* and *angle variables* as in Hamiltonian systems, and $\omega(I)$ is referred to as the *angular frequency vector*. Let $\omega_j(I)$ be the jth component of $\omega(I)$ for $j = 1, \dots, q$. Moreover, we assume the following:

(A3) If $r \in \mathbb{Z}^m$ and $r \cdot \omega(I) = 0$ for any $I \in \mathbb{R}^\ell$, then $r = 0$.

(A4) There exists a key set D_{R} for $C^\omega(U)$ such that for $I \in D_{\mathrm{R}}$ a resonance of multiplicity $q-1$,

$$\dim_{\mathbb{Q}} \langle \omega_1(I), \dots, \omega_q(I) \rangle = 1,$$

occurs with $\omega(I) \neq 0$, i.e., there exists a positive constant $\omega_0(I)$ depending on I such that

$$\frac{\omega(I)}{\omega_0(I)} \in \mathbb{Z}^q \setminus \{0\}.$$

Moreover, there exists a q-parameter (but essentially $(q-1)$-parameter) family of periodic orbits $\gamma_\tau^I(t)$ for $(I, \tau) \in D_{\mathrm{R}} \times \mathbb{T}^q$ with the period $T^I = 2\pi/\omega_0(I)$ in the unperturbed system (1.2).

Note that a set $\Delta \subset U$ is called a *key set* (or *uniqueness set*) for $C^\omega(U)$ if any analytic function vanishing on Δ vanishes on U. For example, any dense set in U is a key set for $C^\omega(U)$. Moreover, we easily see that if $\operatorname{rank} \mathrm{D}\omega(I^*) = q$ for some $I^* \in U$, then both (A3) and (A4) hold in a neighborhood U of I^*.

Define the integrals

$$\mathscr{I}_{F_k}^I(\tau) := \int_0^{T^I} dF_j(X^1)_{\gamma_\tau^I(t)} dt, \quad k = 1, \dots, n-q, \tag{2.2}$$

for $I \in D_{\mathrm{R}}$ and set $\mathscr{I}_F^I(\tau) := (\mathscr{I}_{F_1}^I(\tau), \dots, \mathscr{I}_{F_{n-q}}^I(\tau))$. Note that

$$\mathscr{I}_{F_k}^I(\tau + \omega(I)t) = \mathscr{I}_{F_k}^I(\tau)$$

for $\tau \in \mathbb{T}^q$ and $t \in \mathbb{R}$. We now state the first of our main results as follow.

Theorem 2.1. *Suppose that assumptions (A1)-(A4) hold. If there exists a key set $D \subset D_{\mathrm{R}}$ for $C^\omega(U)$ such that $\mathscr{I}_F^I(\tau)$ is not identically zero for any $I \in D$, then the perturbed system* (1.1) *does not have $n-q$ real-analytic first integrals in a neighborhood of $F^{-1}(c)$ near $\varepsilon = 0$ such that they are functionally independent in x for $|\varepsilon| \neq 0$ and depend analytically on ε.*

For integrability of non-Hamiltonian systems (1.1), we have to consider not only first integrals but also commutative vector fields. So we need the following assumption additionally:

(A5) For some $I^* \in U$, the differential $d\omega(I^*)$ is injective, i.e.,

$$\operatorname{rank} d\omega(I^*) = n - q.$$

Note that (A4) holds only when $n - q \leq q$. Finally, we state the second main result as follows.

Theorem 2.2. *Suppose that assumptions (A1)-(A5) hold. If there exists a key set* $D \subset D_{\mathrm{R}}$ *for* $C^{\omega}(U)$ *such that* $\mathscr{I}_F^I(\tau)$ *is not constant for any* $I \in D$*, then for* $|\varepsilon| \neq 0$ *sufficiently small the perturbed system* (1.1) *is not real-analytically integrable in the Bogoyavlenskij sense near* $F^{-1}(c)$ *such that the first integrals and commutative vector fields depend analytically on* ε *near* $\varepsilon = 0$.

Remark 2.1. Theorems 2.1 and 2.2 say nothing about the nonexistence of real-analytic first integral and real-analytic nonintegrability of (1.1) for $\varepsilon > 0$ fixed.

3 Nearly integrable systems written in action and angle coordinates

In this section, we consider nearly integrable systems written in the action-angle coordinates

$$\dot{I} = \varepsilon h(I, \theta; \varepsilon), \quad \dot{\theta} = \omega(I) + \varepsilon g(I, \theta; \varepsilon), \quad (I, \theta) \in U \times \mathbb{T}^q, \tag{3.1}$$

where ε is a small parameter such that $|\varepsilon| \ll 1$ and $h(I, \theta; \varepsilon), g(I, \theta; \varepsilon)$ are analytic in (I, θ, ε), and describe consequences of Theorems 2.1 and 2.2 to it. The unperturbed system (2.1) is $(q, n-q)$-integrable in the Bogoyavlenskij sense and has $n-q$ first integrals $I_1, \dots, I_{n-q}$ and q commutative vector fields $\partial/\partial\theta_1, \dots, \partial/\partial\theta_q$. Thus, conditions (A1) and (A2) already hold.

We first discuss consequences of Theorem 2.1 to (3.1) and assume that conditions (A3) and (A4) hold. For $I \in D_{\mathrm{R}}$ the unperturbed system (2.1) has an q-parameter family of periodic orbits given by

$$(I, \theta) = (I, \omega(I)t + \tau), \quad \tau \in \mathbb{T}^q. \tag{3.2}$$

The integrals given by (2.2) for the $n-q$ first integrals $I = (I_1, \dots, I_{n-q})$ become

$$\mathscr{I}_I^I(\tau) = \int_0^{T^I} h(I, \omega(I)t + \tau; 0) dt,$$

where $\tau \in \mathbb{T}^q$.

Using the Fourier expansion of $h(I, \theta; 0)$ given by

$$h(I, \theta; 0) = \sum_{r \in \mathbb{Z}^m} \hat{h}_r(I) \exp(ir \cdot \theta),$$

where $\hat{h}_r(I)$, $r \in \mathbb{Z}^m$, are the Fourier coefficients and "$\cdot$" represents the inner product, we rewrite the above integral as

$$\mathscr{I}_I^I(\tau) = \int_0^{T^I} \sum_{r \in \mathbb{Z}^m} \hat{h}_r(I) \exp(ir \cdot (\omega(I)t + \tau)) dt = T^I \sum_{r \in \Lambda_I} \hat{h}_r(I) e^{ir \cdot \tau}, \tag{3.3}$$

where $\Lambda_I = \{r \in \mathbb{Z}^m \mid r \cdot \omega(I) = 0\}$. Applying Theorem 2.1, we obtain the following result for (3.1).

Theorem 3.1. *Suppose that assumptions (A3) and (A4) hold. If there exists a key set* $D \subset D_{\mathrm{R}}$ *for* $C^{\omega}(U)$ *such that* $\hat{h}_r(I) \neq 0$ *for some* $r \in \Lambda_I$ *with* $I \in D$*, then the perturbed system* (3.1) *does not have* $n-q$ *real-analytic first integrals near* $\varepsilon = 0$ *such that they are functionally independent in* x *for* $|\varepsilon| \neq 0$ *and depend analytically on* ε.

Remark 3.1.

(i) We easily see that there exists a key set $D \subset D_{\mathrm{R}}$ for $C^\omega(U)$ such that $\hat{h}_r(I) \neq 0$ with some $r \in \Lambda_I$ for $I \in D$ if and only if what is called *Poincaré set* $\mathscr{P}_{n-q-1} \subset U$ is a key set for $C^\omega(U)$. Note that the latter is the assumption of Kozlov's nonintegrablity theorem [6], which is an improvement of Poincaré's one [12].

(ii) From Theorem 2.2 of [10] and Eq. (3.3) we see that the first integrals $I_1, \ldots, I_{n-q}$ do not persist in (3.1) near the resonant torus $\{I\} \times \mathbb{T}^q$ if $\hat{h}_r(I) \neq 0$ for some $r \in \Lambda_I$.

We next apply Theorem 2.2 to (3.1).

Theorem 3.2. *Suppose that assumptions (A3), (A4) and (A5) hold. If there exists a key set $D \subset D_{\mathrm{R}}$ for $C^\omega(U)$ such that $\hat{h}_r(I) \neq 0$ for some $r \in \Lambda_I \setminus \{0\}$ with $I \in D$, then for $|\varepsilon| \neq 0$ sufficiently small the perturbed system* (3.1) *is not real-analytically integrable in the sense of Theorem 2.2.*

Remark 3.2. We emphasize that Poincaré's theorem [12] and Kozlov's one [6] do not mention about Bogoyavlenskij integrability.

4 Relationships with the Melnikov Methods

In this section, we discuss relationships of our main results in Section 2 with the subharmonic and homoclinic Melnikov methods for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems. See [3, 7, 13, 14] for the details of the Melnikov methods.

Consider a systems of the form

$$\dot{x} = J\mathrm{D}H(x) + \varepsilon u(x, \nu t), \quad x \in \mathbb{R}^2, \tag{4.1}$$

where ε is a small parameter as in the preceding sections, $\nu > 0$ is a constant, $H : \mathbb{R}^2 \to \mathbb{R}$ and $u : \mathbb{R}^2 \times \mathbb{S} \to \mathbb{R}^2$ are analytic, and J is the 2×2 symplectic matrix,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Equation (4.1) represents a time-periodic perturbation of the single-degree-of-freedom Hamiltonian system

$$\dot{x} = J\mathrm{D}H(x), \tag{4.2}$$

with the Hamiltonian $H(x)$. Letting $\phi = \nu t \mod 2\pi$ such that $\phi \in \mathbb{S}^1$, we rewrite (4.1) as an autonomous system,

$$\dot{x} = J\mathrm{D}H(x) + \varepsilon u(x, \phi), \quad \dot{\phi} = \nu. \tag{4.3}$$

We easily see that assumptions (A1) and (A2) hold in (4.3) with $\varepsilon = 0$: $H(x)$ is a first integral and $(0, 1) \in \mathbb{R}^2 \times \mathbb{R}$ is a commutative vector field. We make the following assumptions on the unperturbed system (4.2):

(M1) There exists a one-parameter family of periodic orbits $x^\alpha(t)$ with period $\hat{T}^\alpha > 0$, $\alpha \in (\alpha_1, \alpha_2)$, for some $\alpha_1 < \alpha_2$. Moreover, $\hat{T}^\alpha$ is not constant as a function of α.

(M2) $x^\alpha(t)$ is analytic with respect to $\alpha \in (\alpha_1, \alpha_2)$.

Note that in (M1) $x^\alpha(t)$ is automatically analytic with respect to t since the vector field of (4.2) is analytic.

We assume that at $\alpha = \alpha^{l/n}$

$$\frac{2\pi}{\hat{T}^\alpha} = \frac{n}{l}\nu, \tag{4.4}$$

where l and n are relatively prime integers. We define the *subharmonic Melnikov function* as

$$M^{l/n}(\phi) = \int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t)) \cdot u(x^\alpha(t), \nu t + \phi)\mathrm{d}t, \tag{4.5}$$

where $\alpha = \alpha^{l/n}$. Let $T^\alpha = n\hat{T}^\alpha = 2\pi l/\nu$ for $\alpha = \alpha^{l/n}$. If $M^{l/n}(\phi)$ has a simple zero at $\phi = \phi_0$ and $d\hat{T}^\alpha/d\alpha \neq 0$ at $\alpha = \alpha^{l/n}$, then for $|\varepsilon| > 0$ sufficiently small there exists a T^α-periodic orbit near $(x, \phi) = (x^\alpha(t), \nu t + \phi_0)$ in (4.3). See Theorem 3.1 of [14]. A similar result is also found in [3, 13].

On the other hand, since it is a single-degree-of-freedom Hamiltonian system, the unperturbed system (4.2) is integrable, so that it can be transformed into the form (2.1). So the perturbed system (4.3) is transformed into the form (3.1). Here we take $I = \alpha$ and have $\omega(I) = (\Omega(I), \nu)$, where

$$\Omega(\alpha) = \frac{2\pi}{\hat{T}^\alpha}.$$

We remark that the transformed system is not Hamiltonian even when $\varepsilon = 0$. Choose a point $\alpha = \alpha_0 \in (\alpha_1, \alpha_2)$ such that $d\hat{T}^\alpha/d\alpha \neq 0$, and let U be a neighborhood of α_0. We see that assumptions (A3) and (A4) hold for

$$D_{\mathrm{R}} = \{\alpha^{l/n} \mid l, n \in \mathbb{N}\} \cap U.$$

Let $\alpha = \alpha^{l/n}$ and let $\gamma_\tau^\alpha(t) = (x^\alpha(t + \tau_1), \nu(t + \tau_1) + \tau_2)$. We see that $\gamma_\tau^\alpha(t)$ is a T^α-periodic orbit in (4.3) with $\varepsilon = 0$. Note that $\gamma_\tau^\alpha(t)$ is essentially parameterized by a single parameter, say $\phi := \nu\tau_1 + \tau_2$. So we write $\gamma_\phi^\alpha(t) = (x^\alpha(t), \nu t + \phi)$. The integral (2.2) for $H(x)$ along $\gamma_\phi^\alpha(t)$ becomes

$$\mathscr{I}_H^\alpha(\phi) = \int_0^{2\pi l/\nu} \mathrm{D}H(x^\alpha(t)) \cdot u(x^\alpha(t), \nu t + \phi)dt = M^{l/n}(\phi) \tag{4.6}$$

by (4.5). As stated above, if $M^{l/n}(\phi)$ has a simple zero at $\phi = \phi_0$, then there exists a T^α-periodic orbit near $\gamma_{\phi_0}^\alpha(t)$.

We additionally assume the following on the unperturbed system (4.2):

(M3) There exists a hyperbolic saddle x_0 with a homoclinic orbit $x^{\mathrm{h}}(t)$ such that

$$\lim_{\alpha \to \alpha_2} \sup_{t \in \mathbb{R}} d(x^\alpha(t), \Gamma) = 0,$$

where $\Gamma = \{x^{\mathrm{h}}(t) \mid t \in \mathbb{R}\} \cup \{x_0\}$ and $d(x, \Gamma) = \inf_{y \in \Gamma} |x - y|$.

We define the *homoclinic Melnikov function* as

$$M(\phi) = \int_{-\infty}^{\infty} \mathrm{D}H(x^{\mathrm{h}}(t)) \cdot u(x^{\mathrm{h}}(t), t+\phi)dt. \tag{4.7}$$

If $M(\phi)$ has a simple zero, then for $|\varepsilon| > 0$ sufficiently small there exist transverse homoclinic orbits to a periodic orbit near $\{x_0\} \times \mathbb{S}^1$ in (4.3) [3, 7, 13]. The existence of such transverse homoclinic orbits implies that the system (4.3) exhibits chaotic motions by the Smale-Birkhoff theorem [3,13] and has no real-analytic (additional) first integral (see, e.g., Chapter III of [9]). However, we cannot exclude the possibility of analytical integrability of (4.3) since it may have two additional linearly independent commutative vector fields. We easily show that

$$\lim_{l\to\infty} M^{l/1}(\phi) = M(\phi) \tag{4.8}$$

for each $\phi \in \mathbb{S}^1$ (see Theorem 4.6.4 of [3]). Thus it follows from (4.8) that if $M(\phi)$ is not identically zero or constant, then for $l > 0$ sufficiently large neither is $M^{l/1}(\phi)$. Applying Theorems 2.1 and 2.2 with (4.8), we have the following two results.

Theorem 4.1. *Suppose that $M(\phi)$ is not identically zero. Then for $|\varepsilon| \neq 0$ sufficiently small the system* (4.3) *has no real-analytic first integral such that it depends analytically on ε near $\varepsilon = 0$.*

Theorem 4.2. *Suppose that $M(\phi)$ is not constant. Then for $|\varepsilon| \neq 0$ sufficiently small the system* (4.3) *is not real-analytically integrable in the sense of Theorem 2.2.*

5 Example: Periodically forced Duffing oscillator

We now apply Theorems 4.1 and 4.2 to the periodically forced Duffing oscillator [3, 4, 13].

Consider the periodically forced Duffing oscillator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - x_1^3 + \varepsilon(\beta\cos\nu t - \delta x_2), \tag{5.1}$$

where $\nu > 0$ and $\beta, \delta \geq 0$ are constants. The system (5.1) has the form (4.1) with

$$H = -\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

and the autonomous system (4.3) becomes

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - x_1^3 + \varepsilon(\beta\cos\theta - \delta x_2), \quad \dot{\phi} = \nu, \tag{5.2}$$

where $(x, \phi) \in \mathbb{R}^2 \times \mathbb{S}$.

When $\varepsilon = 0$, in the phase plane there exist a pair of homoclinic orbits

$$x_{\pm}^{\mathrm{h}}(t) = (\pm\sqrt{2}\,\mathrm{sech}\, t, \mp\sqrt{2}\,\mathrm{sech}\, t\, \tanh t).$$

Moreover, the homoclinic Melnikov function (4.7) for $x_{\pm}^{\mathrm{h}}(t)$ becomes

$$M_{\pm}(\tau) = -\frac{4}{3}\delta \pm \sqrt{2}\pi\nu\beta\,\mathrm{csch}\left(\frac{\pi\nu}{2}\right)\sin\tau,$$

(See [3, 13] for the computations of the Melnikov functions). Therefore, applying Theorems 4.1 and 4.2, we have the following.

Proposition 5.1. *The periodically forced Duffing oscillator* (5.2) *has no real-analytic first integral depending analytically on* ε *if* $\delta \neq 0$.

Proposition 5.2. *For* $|\varepsilon| \neq 0$ *sufficiently small the periodically forced Duffing oscillator* (5.2) *is not real-analytically integrable in the sense of Theorem 2.2 if* $\beta \neq 0$.

If $\beta \neq 0$ and

$$\frac{\delta}{\beta} < \frac{3}{4}\sqrt{2}\pi\nu \operatorname{csch}\left(\frac{\pi\nu}{2}\right), \tag{5.3}$$

then $M_{\pm}(\tau)$ has a simple zero, so that for $|\varepsilon| > 0$ sufficiently small there exist transverse homoclinic orbits to a periodic orbit near the origin and chaotic dynamics may occur in (5.2). From Proposition 5.2 we see that the system (5.2) is nonintegrable even if condition (5.3) does not hold, i.e., there may exist no transverse homoclinic orbit to the periodic orbit.

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