HIGHER SPIN DIRAC OPERATORS

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Abstract. This manuscript is a résumé of the article [HT2], which is based on the talk in the workshop "Geometric Structures and Differential Equations –Symmetry, Singularity, and Quantization–". In [HT2], we give the Weitzenböck-type formulas among the geometric first order differential operators on the spinor fields with spin $j+1/2$ over Riemannian spin manifolds of constant curvature. Then we find an explicit factorization formula of the Laplace operator raised to the power $j+1$ and understand how the spinor fields with spin $j+1/2$ are related to the spinors with lower spin. As an application, we calculate the spectra of the operators on the standard sphere and clarify the relation among the spinors from the viewpoint of representation theory. Next we discuss the spinor fields coupled with differential forms and give a kind of Hodge-de Rham decomposition on spaces of constant curvature. Lastly we study the case of trace-free symmetric tensor fields with an application to Killing tensor fields.

1. What is higher spin Dirac operators

Higher spin Dirac operators are a generalization of the Dirac operator and the Rarita-Schwinger operator. These operators appear in the following context:

- Dirac operator: to describe spin 1/2 fermion (electron)
  keywords: index theorem, spin geometry
- Rarita-Schwinger operator: to describe spin 3/2 fermion (ex. gravitino)
  keywords: deformation theory, PSU(3)-structure
- higher spin Dirac operator: to describe spin $j+1/2$ fermion
  keywords: Clifford analysis (on Euclidean space)

2. Notation of Spin(n)-representation

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space and $\langle , \rangle$ be a standard inner product on $\mathbb{R}^n$, $\{e_i\}$ be the standard basis of $\mathbb{R}^n$.
Then

$$\Lambda^2(\mathbb{R}^n) \cong \mathfrak{so}(n); (u \wedge v)(w) = \langle u, w\rangle v - \langle v, w\rangle u,$$

where $\mathfrak{so}(n)$ is the Lie algebra of the special orthogonal group SO(n) or the spin group Spin(n).
Let $\{e_{ij} = e_i \wedge e_j \mid 1 \leq i < j \leq n\}$ be a basis of $\Lambda^2(\mathbb{R}^n) \cong \mathfrak{so}(n)$.
We choose a Cartan subalgebra

$$\mathfrak{t} = \mathrm{span}_{\mathbb{R}}\{e_{2k-1,2k} \mid 1 \leq k \leq m = \lfloor n/2 \rfloor\}$$

of $\mathfrak{so}(n)$ and a basis of the dual of the Cartan subalgebra $\mathfrak{t}^*$ as $\{\widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_m\}$ with

$$\widehat{e}_l(e_{2k-1,2k}) = \delta_{kl}.$$

We regard $i\mathfrak{t}^*$ as the m-dimensional Euclidean space and the basis $\{\mathbf{e}_k = i\widehat{e}_k\}$ as

$$\mathbf{e}_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{m-k}).$$

There is one-to-one correspondence between finite dimensional irreducible unitary representation (π, V) of Spin(n) and $\lambda = \sum_{i=1}^m \lambda_i \mathbf{e}_i = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ in $\mathbb{Z}^m$ or $(\mathbb{Z}+1/2)^m$ with *the* *dominant condition*,

$$\begin{aligned} &\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m| && \text{for } n = 2m, \\ &\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-1} \geq \lambda_m \geq 0 && \text{for } n = 2m+1. \end{aligned}$$

We call λ *the highest weight* of V and write the irreducible unitary representation with highest weight λ as (π_λ, V_λ).

For example, the highest weight of $\Lambda^p(\mathbb{R}^n) \otimes \mathbb{C}$ is

$$(\underbrace{1, \ldots, 1}_{p}, 0, \ldots 0).$$

and the spinor representation is the irreducible representation with highest weight

$$(1/2, 1/2, \ldots, 1/2) \text{ for } n = 2m+1,$$

or the direct sum of the irreducible representations with highest weights

$$(1/2, 1/2, \ldots, 1/2, 1/2) \text{ and } (1/2, 1/2, \ldots, 1/2, -1/2) \text{ for } n = 2m.$$

3. The Dirac operator

Let (M, g) be an n-dimimensional spin manifold (oriented Riemannian manifold with $w_2(M) = 0$).

A spin structure on M is a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(M)$ which satisfies the following diagram.

$$\begin{array}{ccc} \mathrm{Spin}(M) \times \mathrm{Spin}(n) & \xrightarrow{\Phi \times \mathrm{Ad}} & \mathrm{SO}(M) \times \mathrm{SO}(n) \\ \downarrow & & \downarrow \\ \mathrm{Spin}(M) & \xrightarrow{\Phi} & \mathrm{SO}(M) \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M \end{array}$$

The spinor space is

$$W_0 = \begin{cases} V_{(1/2,1/2,\ldots,1/2)} & \text{for } n\colon \text{odd}, \\ V_{(1/2,1/2,\ldots,1/2,1/2)} \oplus V_{(1/2,1/2,\ldots,1/2,-1/2)} & \text{for } n\colon \text{even}. \end{cases}$$

The spinor bundle is

$$S_0 := \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} W_0.$$

The Dirac operator is defined by

$$D := \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} : \Gamma(S_0) \to \Gamma(S_0)$$

where $\{e_i\}$ is a local o.n.f. and $e_i \cdot$ is the Clifford multiplication.

Remark 3.1. The Dirac operator is a formally self-adjoint elliptic first order differential operator.

The Dirac operator satisfies the Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{\mathrm{Scal}}{4} = \Delta_{S_0} + \frac{\mathrm{Scal}}{8},$$

where Δ is the standard (Lichnerowicz) Laplacian. This leads a vanishing theorem:

$$\text{compact and } \mathrm{Scal} > 0 \Rightarrow \mathrm{Ker}\, D = \{0\}.$$

4. The standard Laplacian

The standard Laplacian is a generalization of the Laplacian on the associated vector bundle with $\mathrm{SO}(M)$ or $\mathrm{Spin}(M)$:

$$\Delta = \nabla^*\nabla + q(R),$$

where $q(R) = \frac{1}{2}\sum_{i,j}(e_i \wedge e_j)_*(R(e_i, e_j))_*$. For example,

$$\Delta = d\delta + \delta d$$

on $\Lambda^*(M)$,

$$\Delta = \nabla^*\nabla + \frac{\mathrm{Scal}}{8}$$

on S_0,

$$\Delta = (\text{Casimir operator})$$

on a homogeneous vector bundle over a Riemannian symmetric space.

5. The Rarita-Schwinger operator

Let TM^c be the complexified tangent bundle. Then, by the action of $\mathrm{Spin}(n)$,

$$S_0 \otimes TM^c \cong S_0 \oplus S_1,$$

where $S_1 = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} W_1$ is the associated vector bundle of $\mathrm{Spin}(M)$ by the representation with highest weight $(3/2, 1/2, \ldots, 1/2)$.

This decomposition realized as follows;

$$i\colon S_0 \to S_0 \otimes TM^c;\ \phi \mapsto -\frac{1}{n}\sum_i e_i \cdot \phi \otimes e_i,$$

$$\mu\colon S_0 \otimes TM^c \to S_0;\ \phi \otimes X \mapsto X \cdot \phi,$$

and $S_1 := \mathrm{Ker}\,\mu$ lead

$$S_0 \otimes TM^c = i(S_0) \oplus S_1.$$

$D_{TM} = \sum_i (e_i \cdot \otimes \mathrm{id}) \circ \nabla_{e_i} \colon \Gamma(S_0 \otimes TM^c) \to \Gamma(S_0 \otimes TM^c)$: the twisted Dirac operator on $S_0 \otimes TM^c$.

Along the decomposition $S_0 \otimes TM^c \cong S_0 \oplus S_1$, the twisted Dirac operator is decomposed as the 2×2-matrix

$$D_{TM} = \begin{pmatrix} \frac{2-n}{n} D & 2P^* \\ \frac{2}{n} P & Q \end{pmatrix},$$

where $P\colon \Gamma(S_0) \to \Gamma(S_1)$ is the twistor (Penrose) operator.

Definition 1. $Q : \Gamma(S_1) \to \Gamma(S_1)$ is called the Rarita-Schwinger (RS) operator.

Remark 5.1. The Rarita-Schwinger operator is a formally self-adjoint elliptic first order differential operator.

6. Results of Homma-Semmelmann [HS]

Let (M, g) be a compact Einstein spin manifolds. The (twisted) Lichnerowicz formula

$$D_{TM}^2 = \Delta_{S_0 \otimes TM^c} + \frac{\mathrm{Scal}}{8} - \mathrm{Id}_{S_0} \otimes \mathrm{Ric}.$$

gives that

$$\begin{pmatrix} \left(\frac{2-n}{n}\right)^2 D^2 + \frac{4}{n}P^*P & \frac{2(2-n)}{n}DP^* + 2P^*Q \\ \frac{2(2-n)}{n^2}PD + \frac{2}{n}QP & Q^2 + \frac{4}{n}PP^* \end{pmatrix}$$
$$= \begin{pmatrix} \Delta_{S_0} + \frac{\mathrm{Scal}}{8} - \frac{\mathrm{Scal}}{n} & 0 \\ 0 & \Delta_{S_1} + \frac{\mathrm{Scal}}{8} - \frac{\mathrm{Scal}}{n} \end{pmatrix}.$$

Since the twistor operator $P\colon \Gamma(S_0) \to \Gamma(S_1)$ is an overdetermined elliptic operator, the following decomposition holds;

$$\Gamma(S_1) = \operatorname{Ker} P^* \oplus \operatorname{Im} P,$$

and Q and Δ preserves the above decomposition.

Proposition 2 (Homma-Semmelmann [HS])**.** *Let (M, g) be an Einstein spin manifolds, then*

$$Q \circ P = \frac{n-2}{n} P \circ D, \quad P^* \circ Q = \frac{n-2}{n} D \circ P^*,$$
$$\Delta_{S_0} \circ D = D \circ \Delta_{S_0}, \quad \Delta_{S_0} \circ P^* = P^* \circ \Delta_{S_0},$$
$$\Delta_{S_1} \circ P = P \circ \Delta_{S_0}, \quad \Delta_{S_1} \circ Q = Q \circ \Delta_{S_1}.$$

Proposition 3 (Homma-Semmelmann [HS])**.** *Let (M, g) be an compact Einstein spin manifolds, then*

- $Q^2 = \Delta_{S_1} + \frac{n-8}{8n}\mathrm{Scal}$ *on* $\operatorname{Ker} P^*$,
- $Q^2 = \left(\frac{n-2}{n}\right)^2 (\Delta_{S_1} + \frac{\mathrm{Scal}}{8})$ *on* $\operatorname{Im} P$.

7. The higher spin Dirac operator

Let (M, g) be an n-dimimensional spin manifold, S_0 be the spinor bundle, and Sym^j_0 be the primitive irreducible component of the j-th symmetric tensor product of TM^c with highest weight $(j, 0, \ldots, 0)$.

Then, by the action of $\mathrm{Spin}(n)$,

$$S_0 \otimes \mathrm{Sym}^j_0 \cong S_{j-1} \oplus S_j,$$

where the spin $j + 1/2$ bundle $S_j = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} W_j$ is the associated vector bundle of $\mathrm{Spin}(n)$ by the representation with highest weight $(j + 1/2, 1/2, \ldots, 1/2)$
or the direct sum of the representations with highest weights $(j + 1/2, 1/2, \ldots, 1/2, 1/2)$ and $(j + 1/2, 1/2, \ldots, 1/2, -1/2)$.

Let $D(j)\colon \Gamma(S_0 \otimes \mathrm{Sym}^j_0) \to \Gamma(S_0 \otimes \mathrm{Sym}^j_0)$ be the twisted Dirac operator. Along the decomposition $S_0 \otimes \mathrm{Sym}^j_0 \cong S_{j-1} \oplus S_j$, the twisted Dirac operator $D(j)$ is considered as 2×2-matrix

$$D(j) = \begin{pmatrix} D'_{j-1} & T^+_{j-1} \\ T^-_j & D_j \end{pmatrix}.$$

$D_j\colon \Gamma(S_j) \to \Gamma(S_j)$ is called the higher spin Dirac (HSD) operator.

Remark 7.1. D_j is a formally self-adjoint elliptic first order differential operator.

Remark 7.2. $T^+_{j-1} : \Gamma(S_{j-1}) \to \Gamma(S_j)$ satisfies $(T^+_{j-1})^* = T^-_j$.

Remark 7.3. When $j = 1$, D'_0 is the constant multiple of the Dirac operator, D_1 is the Rarita-Schwinger operator, T^+_0 is the constant multiple of the twistor operator P.

In Bureš-Sommen-Souček-Lancker [BSSL1],[BSSL2], they considered to generalize spherical harmonic analysis on Euclidean space to spinor fields with higher spin;

- Polynomial solutions (called monogenic function),
- Clifford Cauchy kernel,

- Factorization formula
  that is to constitute the fundamental solution of the HSD operator. On $\mathbb{R}^n$, D_j is the factor of Δ_j^{j+1}, i.e. by using a (2j+1)-st order differential operator A_{2j+1},
$$\Delta_j^{j+1} = D_j \circ A_{2j+1}.$$

By the (twisted) Lichnerowicz formula
$$D(j)^2 = \Delta_{S_0 \otimes \mathrm{Sym}_0^j} + \frac{\mathrm{Scal}}{8} - \frac{1}{2}\mathrm{Id}_{S_0} \otimes R_{\mathrm{Sym}_0^j},$$
we obtain various formulas on spaces of constant curvature.

Let (M, g) be Riemannian manifold of constant sectional curvature $K = c$ with a spin structure, Δ_j be the standard Laplacian on $\Gamma(S_j)$.

Proposition 4.
$$D_j^2 + T_{j-1}^+(T_{j-1}^+)^* = \Delta_j - (j(n+j-2) - \frac{n(n-1)}{8})c,$$
$$(T_{j-1}^+)^* T_{j-1}^+ + (D'_{j-1})^2 = \Delta_{j-1} - (j(n+j-2) - \frac{n(n-1)}{8})c,$$
$$(T_{j-1}^+)^* D_j + D'_{j-1}(T_{j-1}^+)^* = 0, \quad D_j T_{j-1}^+ + T_{j-1}^+ D'_{j-1} = 0.$$

Proposition 5.
$$\Delta_j D_j = D_j \Delta_j, \quad \Delta_j T_{j-1}^+ = T_{j-1}^+ \Delta_{j-1}, \quad \Delta_{j-1} T_j^- = T_j^- \Delta_j.$$

8. Generalized gradients

Composing ∇ and the projection
$$\Pi_j : S_j \otimes TM^c \to S_j,$$
we have so-called *the higher spin Dirac operator*,
$$\widetilde{D}_j := \Pi_j \circ \nabla, \quad \Gamma(S_j) \xrightarrow{\nabla} \Gamma(S_j \otimes TM^c) \xrightarrow{\Pi_j} \Gamma(S_j).$$
In this manner we construct four generalized gradients on $\Gamma(S_j)$ and name them as follows;

$\widetilde{T}_j^+ : \Gamma(S_j) \to \Gamma(S_{j+1})$	*the (first) twistor operator,*
$U_j : \Gamma(S_j) \to \Gamma(S_{j,1})$	*the (second) twistor operator,*
$\widetilde{D}_j : \Gamma(S_j) \to \Gamma(S_j)$	*the higher spin Dirac opeator,*
$\widetilde{T}_j^- : \Gamma(S_j) \to \Gamma(S_{j-1})$	*the co-twistor opeator.*

By using Weitzenböck foumula for the generalized gradients $\left\{\widetilde{D}_j, \widetilde{T}_j^-, \widetilde{T}_{j-1}^+, \widetilde{D}_{j-1}\right\}$,
$$\frac{(n+2j)(n-2)}{n+2j-2}\widetilde{D}_j^2 + \frac{4(n+j-2)}{n+2j-2}(\widetilde{T}_j^-)^*\widetilde{T}_j^- = \Delta_j + \mathrm{curv},$$
$$\frac{4j}{n+2j-2}(\widetilde{T}_{j-1}^+)^*\widetilde{T}_{j-1}^+ + \frac{(n+2j-4)(n-2)}{n+2j-2}(\widetilde{D}_{j-1})^2 = \Delta_{j-1} + \mathrm{curv}.$$

Proposition 6.
$$D_j = \sqrt{\frac{(n+2j)(n-2)}{n+2j-2}}\widetilde{D}_j, \ D'_{j-1} = -\sqrt{\frac{(n+2(j-1)-2)(n-2)}{n+2(j-1)}}\widetilde{D}_{j-1}.$$

Proposition 7.
$$D'_j = -\frac{n+2j-2}{n+2j}D_j.$$

9. Factorization formula

Theorem 8 (Factorization formula)**.** *On a spin manifold (M, g) of constant sectional curvature $K = c$,*

$$\prod_{s=0}^{j}\left(D_j^2 - \frac{(n+2s-2)^2}{(n+2j-2)^2}\left(\Delta_j - \left(s(n+s-2) - \tfrac{n(n-1)}{8}\right)c\right)\right) = 0.$$

This formula is a generalization of the factorization formula on $\mathbb{R}^n$.

Remark 9.1. When $j = 0$, Theorem 8 is the Lichnerowicz formula

$$D^2 = \Delta + \frac{\mathrm{Scal}}{8}.$$

Sketch proof. Let

$$B(s;j) := D_j^2 - \frac{(n+2s-2)^2}{(n+2j-2)^2}\left(\Delta_j - \left(s(n+s-2) - \frac{n(n-1)}{8}\right)c\right).$$

We prove the theorem by induction for j. We start from the equation for $j = 0$,

$$\prod_{s=0}^{0} B(s;0) = B(0;0) = D_0^2 - (\Delta_0 + \frac{n(n-1)}{8}c) = 0.$$

Sandwich by T_j^+ and $(T_j^+)^*$, and we obtain

$$\begin{aligned} 0 =& T_j^+\left(\prod_{s=0}^{j} B(s;j)\right)(T_j^+)^* = \left(\prod_{s=0}^{j}\frac{(n+2j)^2}{(n+2j-2)^2}B(s;j+1)\right)T_j^+(T_j^+)^* \\ =& -\frac{(n+2j)^{2(j+1)}}{(n+2j-2)^{2(j+1)}}\left(\prod_{s=0}^{j} B(s;j+1)\right)B(j+1;j+1). \end{aligned}$$

Thus we have proved theorem holds for $j + 1$. □

10. The meaning of the factorization formula

We assume that the manifold is compact. We put

$$B(s,j) = D_j^2 - \frac{(n+2s-2)^2}{(n+2j-2)^2}\left(\Delta_j - \left(s(n+s-2) - \tfrac{n(n-1)}{8}\right)c\right),$$

then we obtain the filtration

$$F_j = \operatorname{Ker} B(j;j) \subset \cdots \subset F_k = \operatorname{Ker}\prod_{s=k}^{j} B(s;j) \subset \cdots \subset F_0 = \Gamma(S_j).$$

Furthermore, we put

$$W_s = \begin{cases} T_{j-1}^+ \cdots T_0^+(\Gamma(S_0)), & s = 0, \\ T_{j-1}^+ \cdots T_s^+(\operatorname{Ker} T_s^-), & 1 \le s \le j-1, \\ \operatorname{Ker} T_j^-, & s = j, \end{cases}$$

then we get the direct sum decomposition from the filtration $\{F_j\}$,

$$\Gamma(S_j) = \bigoplus_{0 \le s \le j} W_s, \quad F_s = W_s \oplus F_{s+1} \quad (0 \le s \le j).$$

In the case of the sphere,

$$\mathbf{V}_j(s)' := \bigoplus_{k \ge 0} V_j(k,s)' = \bigoplus_{k \ge 0} V_{(k+j+1/2, s+1/2, 1/2, \ldots, 1/2)}.$$

$$
\begin{array}{ccccccccc}
L^2(S_0) & = & \mathbf{V}_0(0)' & & & & & & \\
 & & \Big\downarrow T_0^+ & & & & & & \\
 & & \vdots & \ddots & & & & & \\
L^2(S_{j-1}) & = & \mathbf{V}_{j-1}(0)' & \oplus\cdots\oplus & \mathbf{V}_{j-1}(j-1)' & \oplus & \{0\} & & \\
 & & T_j^- \Big\uparrow\Big\downarrow T_{j-1}^+ & & T_j^- \Big\uparrow\Big\downarrow T_{j-1}^+ & & T_j^- \Big\uparrow & & \\
L^2(S_j) & = & \mathbf{V}_j(0)' & \oplus\cdots\oplus & \mathbf{V}_j(j-1)' & \oplus & \mathbf{V}_j(j)' & \oplus & \{0\} \\
 & & T_{j+1}^- \Big\uparrow\Big\downarrow T_j^+ & & T_{j+1}^- \Big\uparrow\Big\downarrow T_j^+ & & T_{j+1}^- \Big\uparrow\Big\downarrow T_j^+ & & T_{j+1}^- \Big\uparrow \\
L^2(S_{j+1}) & = & \mathbf{V}_{j+1}(0)' & \oplus\cdots\oplus & \mathbf{V}_{j+1}(j-1)' & \oplus & \mathbf{V}_{j+1}(j)' & \oplus & \mathbf{V}_{j+1}(j+1)' \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
.
$$

11. Spectra of the HSD operator on the sphere

By the factorization formula, we get spectra of the HSD operator on the sphere

Theorem 9 (Branson [Bra])**.** *The eigenvalues of the square of the higher spin Dirac operator on the sphere* $S^n = \mathrm{Spin}(n+1)/\mathrm{Spin}(n)$ *are*

$$\frac{(n+2s-2)^2}{(n+2j-2)^2}\left(k+j+\frac{n}{2}\right)^2 \quad (s=0,\dots,j, \quad k=0,1,\dots).$$

Our method would be easier to understand than in [Bra].

12. On spinor fields with differential forms

Let $\Lambda^j = \Lambda^j(T^*M^c)$ be the bundle of (complexified) differential forms with highest weights $(\underbrace{1,\dots,1}_{j},0,\dots,0)$.

Then, by the action of $\mathrm{Spin}(n)$,

$$S_0 \otimes \Lambda^j \cong E_j \oplus E_{j-1} \oplus \cdots \oplus E_0$$

where $E_j = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} W_j$ is the associated vector bundle of $\mathrm{Spin}(n)$ by the representation with highest weight $(\underbrace{3/2,\dots,3/2}_{j},1/2,\dots,1/2)$.

Let $D(j)\colon \Gamma(S_0\otimes\Lambda^j) \to \Gamma(S_0\otimes\Lambda^j)$ be the twisted Dirac operator. Along the decomposition, the twisted Dirac operator $D(j)$ is considered as $(j+1)\times(j+1)$-matrix

$$D(j) = \begin{pmatrix}
D(j)_j & T(j)_{j-1}^+ & 0 & 0 & \dots & 0 \\
T(j)_j^- & D(j)_{j-1} & T(j)_{j-2}^+ & 0 & \dots & 0 \\
0 & T(j)_{j-1}^- & D(j)_{j-2} & T(j)_{j-3}^+ & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \dots & D(j)_1 & T(j)_0^+ \\
0 & 0 & 0 & \dots & T(j)_1^- & D(j)_0
\end{pmatrix}.$$

We write

$$D_j := D(j)_j, \quad T_j^- := T(j)_j^-, \quad T_{j-1}^+ = T(j)_{j-1}^+.$$

Let (M, g) be Riemannian manifold of constant sectional curvature $K = c$ with a spin structure, Δ_j be the standard Laplacian on $\Gamma(E_j)$.

Theorem 10 (Factorization formula)**.** *On a spin manifold (M, g) of constant sectional curvature $K = c$,*

$$\left(D_j^2 - \frac{(n-2j)^2}{(n-2j+2)^2}\left(\Delta_j - \left((j-1)(n-j+1) - \tfrac{n(n-1)}{8}\right)c\right)\right)$$
$$\times\left(D_j^2 - \left(\Delta_j - \left(j(n-j) - \tfrac{n(n-1)}{8}\right)c\right)\right) = 0.$$

Proposition 11.

$$D_j T_{j-1}^+ = \frac{n-2j}{n-2j+2} T_{j-1}^+ D_{j-1},$$
$$T_{j+1}^+ T_j^+ = 0, \quad T_{j-1}^- T_j^- = 0, \quad T_{j-1}^+ = (T_j^-)^*,$$
$$\Delta_j D_j = D_j \Delta_j, \quad \Delta_j T_{j-1}^+ = T_{j-1}^+ \Delta_{j-1}, \quad \Delta_{j-1} T_j^- = T_j^- \Delta_j.$$

$$\cdots \underset{T_{j-1}^-}{\overset{T_{j-2}^+}{\rightleftarrows}} \underset{\circlearrowleft D_{j-1}}{\Gamma(E_{j-1})} \underset{T_j^-}{\overset{T_{j-1}^+}{\rightleftarrows}} \underset{\circlearrowleft D_j}{\Gamma(E_j)} \underset{T_{j+1}^-}{\overset{T_j^+}{\rightleftarrows}} \underset{\circlearrowleft D_{j+1}}{\Gamma(E_{j+1})} \underset{T_{j+2}^-}{\overset{T_{j+1}^+}{\rightleftarrows}} \cdots$$

Theorem 12. *On a spin manifold (M, g) of constant sectional curvature $K = c$, we have the Hodge-de Rham decomposition for spinor fields coupled with differential forms,*

$$\Gamma(E_j) = T_{j-1}^+(\Gamma(E_{j-1})) \oplus T_{j+1}^-(\Gamma(E_{j+1})) \oplus \operatorname{Ker}(\Delta_j - \frac{n(n+1)}{8}c),$$
$$\operatorname{Ker} T_j^+ = T_{j-1}^+(\Gamma(E_{j-1})) \oplus \operatorname{Ker}(\Delta_j - \frac{n(n+1)}{8}c),$$
$$\operatorname{Ker} T_j^- = T_{j+1}^-(\Gamma(E_{j+1})) \oplus \operatorname{Ker}(\Delta_j - \frac{n(n+1)}{8}c).$$

13. Spectra on the sphere

Theorem 13 (Branson [Bra])**.** *The eigenvalues of D_j^2 on the sphere $S^n = \mathrm{Spin}(n+1)/\mathrm{Spin}(n)$ are*

(1) *on* $\operatorname{Im} T_{j-1}^+ \cong \operatorname{Ker} T_j^+$,

$$\left(\frac{n-2j}{n-2j+2}\right)^2 \left(k + \frac{n}{2}\right)^2 \quad (k = 1, 2, \dots).$$

(2) *on* $\operatorname{Ker} T_j^-$,

$$\left(l + 1 + \frac{n}{2}\right)^2 \quad (l = 0, 1, 2, \dots).$$

14. Generalized gradients on symmetric tensor fields

Let Sym_0^j be the primitive irreducible component of the j-th symmetric tensor product of TM.

By the action of $\mathrm{Spin}(n)$,

$$\mathrm{Sym}_0^j \otimes TM^c \cong \mathrm{Sym}_0^{j+1} \oplus \mathrm{Sym}_0^{j-1} \oplus \mathrm{Sym}_0^{j,1}$$

where $\mathrm{Sym}_0^{j,1}$ is an irreducible vector bundle with the highest weight $(j, 1, 0, \dots, 0)$.

We consider generalized gradients

$$T_j^+ : \Gamma(\mathrm{Sym}_0^j) \xrightarrow{\nabla} \Gamma(\mathrm{Sym}_0^j \otimes T^*M^c) \cong \Gamma(\mathrm{Sym}_0^j \otimes TM^c) \xrightarrow{\text{proj.}} \Gamma(\mathrm{Sym}_0^{j+1}),$$

$$T_j^- : \Gamma(\mathrm{Sym}_0^j) \overset{\nabla}{\to} \Gamma(\mathrm{Sym}_0^j \otimes T^*M^c) \cong \Gamma(\mathrm{Sym}_0^j \otimes TM^c) \overset{\text{proj.}}{\to} \Gamma(\mathrm{Sym}_0^{j-1}),$$
$$U_j : \Gamma(\mathrm{Sym}_0^j) \overset{\nabla}{\to} \Gamma(\mathrm{Sym}_0^j \otimes T^*M^c) \cong \Gamma(\mathrm{Sym}_0^j \otimes TM^c) \overset{\text{proj.}}{\to} \Gamma(\mathrm{Sym}_0^{j,1}).$$

Let (M, g) be Riemannian manifold of constant sectional curvature $K = c$ with a spin structure and Δ_j be the standard Laplacian on $\Gamma(\mathrm{Sym}_0^j)$.

Proposition 14.

$$\Delta_j = (j+1)(T_j^+)^* T_j^+ - (n+j-3)(T_j^-)^* T_j^- + 2j(n+j-2),$$
$$(T_{j+1}^-)^* T_{j+1}^- = \frac{(j+1)(n+2j-2)}{(n+j-2)(n+2j)} T_j^+ (T_j^+)^*.$$

Theorem 15 (Factorization formula)**.** *On a spin manifold (M, g) of constant sectional curvature $K = c$,*

$$\prod_{s=0}^{j} \Big((T_j^+)^* T_j^+ - a(s;j)(\Delta_j - b(s;j)c) \Big) = 0,$$

where $a(s;j) = \frac{(j-s+1)(n+j+s-2)}{(j+1)(n+2j-2)}$, $b(s;j) = j(n+j-1) + s(n+s-3)$.

Let $K^j(S^n)$ be the space of the Killing tensor fields with degree j on the sphere and $P^{j-2i}(S^n)$ be the space of the primitive Killing tensor fields with degree j on the sphere. By using the factorization formula, we obtain the following result in a different way from original [Tak].

Theorem 16 (Takeuchi [Tak])**.**

$$K^j(S^n) = \bigoplus_{0 \le j \le \lfloor j/2 \rfloor} g^i \cdot P^{j-2i}(S^n).$$

15. future work

As a future work, we want to extend the factorization formula to Riemannian symmetric spaces. If we obtain the factorization formula on Riemannian symmetric spaces, we can calculate the eigenvalues of the higher spin Dirac operator not only the sphere but also many Riemannian symmetric spaces by using the method in [HT1].

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