Some Recent Results on Nonintegrability of Dynamical Systems

Kazuyuki Yagasaki
Graduate School of Informatics, Kyoto University

1 Introduction

In this article, we briefly review some recent results on nonintegrability of dynamical systems [19, 26–30]. See these references for the details of the results.

Consider systems of the general form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f(x)$ is meromorphic or analytic. Here we adopt the following concept of integrability.

Definition 1.1 (Bogoyavlenskij [4]). *For any integer $n \geq 1$, the n-dimensional system* (1.1) *is called* $(m, n-m)$-integrable *or simply* integrable *for some integer $m \in [1, n]$ if there exist m vector fields $f_1(x)(:= f(x)), f_2(x), \dots, f_m(x)$ and $n-m$ scalar-valued functions $F_1(x), \dots, F_{n-m}(x)$ such that the following two conditions hold*:

(i) *$f_1(x), \dots, f_m(x)$ are linearly independent almost everywhere and commute with each other, i.e., $[f_j, f_k](x) := \mathrm{D}f_k(x)f_j(x) - \mathrm{D}f_j(x)f_k(x) \equiv 0$ for $j, k = 1, \dots, m$, where $[\cdot, \cdot]$ denotes the Lie bracket*;

(ii) *The derivatives $\mathrm{D}F_1(x), \dots, \mathrm{D}F_{n-m}(x)$ are linearly independent almost everywhere and $F_1(x), \dots, F_{n-m}(x)$ are first integrals of $f_1, \dots, f_m$, i.e., $\mathrm{D}F_k(x)^{\mathrm{T}} f_j(x) \equiv 0$ for $j = 1, \dots, m$ and $k = 1, \dots, n-m$, where the superscript 'T' represents the transpose operator.*

We say that the system (1.1) *is* meromorphically (*resp.* analytically) integrable *if the first integrals and commutative vector fields are meromorphic* (*resp. analytic*).

Definition 1.1 is considered as a generalization of Liouville-integrability for Hamiltonian systems [13] since an n-degree-of-freedom Liouville-integrable Hamiltonian system with $n \geq 1$ but also n linearly independent commutative (Hamiltonian) vector fields generated by the first integrals.

The outline of this article is as follows. In Section 2, we briefly describe a generalized version due to Ayoul and Zung [3] of the Morales-Ramis theory [13, 14], which is a powerful tool to prove the nonintegrability of dynamical systems in the Bogoyavlenskij sense. We also give the result of [29] obtained by the theory for the SEIR epidemic model (see, e.g., [5]), which has recently attracted much attention from the viewpoint of combating COVID-19. In Section 3, we present a technique to prove the meromorphic nonintegrability of nearly integrable systems [26]. The technique is based on the Morales-Ramis theory and its extension, the Morales-Ramis-Simó theory [16], and treats the integrability of these systems such that the related first integrals and commutative vector fields also depend on the small parameter analytically or meromorphically. We apply the technique to the restricted three-body problem and succeed in proving its nonintegrability for any mass ratio of the two primaries [26]. This result is very outstanding compared with the famous one of Poincaré [20]. Moreover, we give the result of [27] obtained by the technique for periodic perturbations of single-degree-of-freedom Hamiltonian systems and see that it is closely related to the subharmonic Melnikov method [10,23]. In Section 4, we outline recent results on nonintegrability of three- or four-dimensional systems near degenerate equilibria. The results do not rely on the Morales-Ramis theory, and two novel results play key roles in their proofs.

After the RIMS symposium "Geometric Structures and Differential Equations – Symmetry, Singularity, and Quantization", further results on nonintegrability of dynamical systems were reported in [31,32].

2 Generalized Morales-Ramis Theory

The generalized version due to Ayoul and Zung [3] of the Morales-Ramis and Morales-Ramis-Simó theories [13,14,16] were used to obtain the results reviewed here. We briefly describe the generalized Morales-Ramis theory.

Consider the general system (1.1). Let $x = \phi(t)$ be its nonstationary particular solution. The variational equation (VE) of (1.1) along $x = \phi(t)$ is given by

$$\dot{\xi} = \mathrm{D}f(\phi(t))\xi, \quad \xi \in \mathbb{C}^n. \tag{2.1}$$

Let $\mathscr{C}$ be a curve given by $x = \phi(t)$ and let $\overline{\mathscr{C}}$ be its closure containing the infinity in the phase space, i.e., points at which a component of $x(t)$ becomes infinite. Assume that the vector field $f(x)$ can be meromorphically extended to a region containing $\overline{\mathscr{C}}$. We take the meromorphic function field on $\overline{\mathscr{C}}$ as the coefficient field of (2.1). Using arguments given by Morales-Ruiz and Ramis [13,14] and Ayoul and Zung [3], we have the following result.

Theorem 2.1. *Suppose that the VE* (2.1) *has no irregular singularity at infinity in the phase space and let $\mathscr{G}$ denote the differential Galois group of* (2.1)*. If Eq.* (1.1) *is meromorphically integrable near $\mathscr{C}$, then the identity component $\mathscr{G}^0$ of $\mathscr{G}$ is commutative.*

Remark 2.2. (i) *The differential Galois group contains the monodromy group. If the VE* (2.1) *is Fuchsian, then the Zariski closure of the latter is equivalent to the former.*

(ii) *If the system* (1.1) *is Hamiltonian, then Bogoyavlenskij-integrability is replaced by Liouville-integrability.*

(iii) *If the VE* (2.1) *has an irregular singularity at infinity in the phase space, then we must replace the word "meromorphically" with "rationally" in the conclusion of Theorem* 2.1*. See Section* 4.2 *of* [13] *or Section* 5.2 *of* [14] *for the details.*

(iv) *A higher-order theory beyond* (2.1) *was also developed by Morales, Ramis and Simó, and is called the* Morales-Ramis-Simó theory*.*

(v) *The Morales -Ramis and Morales-Ramis-Simó theories have been applied successfully to many systems: Henon-Heiles system, general N-body problems ($N \geq 3$), heavy top, homogeneous potentials, Lorentz equation and so on. See* [6,8,9,12–16] *and references therein for the details.*

We now consider the SEIR epidemic model (see, e.g., [5])

$$\dot{S} = -rSI, \quad \dot{E} = rSI - bE, \quad \dot{I} = bE - aI, \quad \dot{R} = aI, \tag{2.2}$$

where the state variables S, E, I and R represent the numbers of susceptible, exposed, infected and removed individuals, respectively, while r, b and a represent the infection, latent and removal rates, respectively. The variable R is not essential in (2.2) since it does not appear in the first three equations. We see that

$$F(S,E,I) = S \exp\left(-\frac{r}{a}(S+E+I)\right)$$

is a first integral for the (S,E,I)-components of (2.2). We have a particular solution $(S,E,I) = (0,0,C_1 e^{-at})$ with $C_1 \in \mathbb{C}$ is a constant. Introducing the new variables $X = e^{r(S+E+I)/a}$, $Y = e^{r(S+E)/a}$ and $Z = e^{rS/a}$ and removing the inessential R-component, we rewrite (2.2) as

$$\begin{aligned}
&\dot{S} = -rSI, \quad \dot{E} = rSI - bE, \quad \dot{I} = bE - aI,\\
&\dot{X} = -rIX, \quad \dot{Y} = -\frac{rb}{a}EY, \quad \dot{Z} = -\frac{r^2}{a}SIZ,
\end{aligned}$$

for which the VE (2.1) along the particular solution becomes

$$\begin{aligned}
&\delta\dot{S} = -rC_1 e^{-at}\delta S, \quad \delta\dot{E} = rC_1 e^{-at}\delta S - b\delta E,\\
&\delta\dot{I} = b\delta E - a\delta I, \quad \delta\dot{X} = -r\bar{X}(t)\delta I - rC_1 e^{-at}\delta X,\\
&\delta\dot{Y} = -\tfrac{rb}{a}\delta E, \quad \delta\dot{Z} = -\tfrac{r^2}{a}C_1 e^{-at}\delta S,
\end{aligned}$$

and has irregular singularity at ∞, where $\bar{X}(t) = \exp\left(\frac{r}{a}C_1 e^{-at}\right)$. Letting $\mathbb{K} = \mathbb{C}(e^{-at}, \bar{X}(t))$ and $\Gamma = \mathbb{P}^1$, we apply Theorem 2.1 to prove the following [29].

Theorem 2.3. *If $a \neq 0$ and $b/a \notin \mathbb{Q} \setminus \{1\}$, then the (S,E,I)-components of (2.2) is not Bogoyavlenskij-integrable near $(S,E) = (0,0)$ such that the first integrals and commutative vector fields are rational functions of S, E, I, e^S, e^E and e^I. Here we may take $a,b,r \in \mathbb{C}$.*

Remark 2.4. *When $b/a = -1$, the (S,E,I)-components of (2.2) is integrable since*

$$\hat{F}(S,E,I) = -\frac{a}{r}S + \tfrac{1}{2}aI^2 - (a+1)(S+E+I) + \tfrac{1}{2}(S+E+I)^2$$

is another first integral.

3 Nearly Integrable Systems

In this section we review the recent results on nonintegrability of nearly integrable systems [26–28]. We begin with the general theory [26].

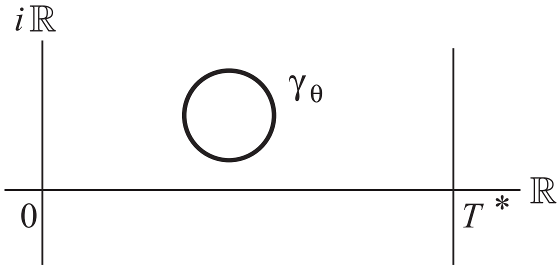

Figure 1: Assumption (A2).

3.1 General theory

Let $\ell, m \in \mathbb{N}$ and consider

$$\dot{I} = \varepsilon h(I, \theta; \varepsilon), \quad \dot{\theta} = \omega(I) + \varepsilon g(I, \theta; \varepsilon), \tag{3.1}$$

in action-angle coordinates $(I, \theta) \in \mathbb{R}^\ell \times \mathbb{T}^m$, where $|\varepsilon| \ll 1$ and $h : \mathbb{R}^\ell \times \mathbb{T}^m \times \mathbb{R} \to \mathbb{R}^\ell$, $\omega : \mathbb{R}^\ell \to \mathbb{R}^m$ and $g : \mathbb{R}^\ell \times \mathbb{T}^m \times \mathbb{R} \to \mathbb{R}^m$ are analytic. We make the following assumptions:

(A1) For some $I^* \in \mathbb{R}^\ell$ there exists a constant $\omega^* > 0$ such that $\omega(I^*)/\omega^* \in \mathbb{Z}^m \setminus \{0\}$.

(A2) Let $T^* = 2\pi/\omega^*$. For some $k \geq 0$ and $\theta \in \mathbb{T}^m$, there exists a closed loop γ_θ such that $\gamma_\theta \cap (i\mathbb{R} \cup (T^* + i\mathbb{R})) = \emptyset$ (see Fig. 1) and

$$\mathscr{I}^k(\theta) := \mathrm{D}\omega(I^*) \int_{\gamma_\theta} \mathrm{D}_\varepsilon^k h(I^*, \omega(I^*)\tau + \theta; 0)\mathrm{d}\tau \neq 0.$$

Under these assumptions we have the following.

Theorem 3.1. *Suppose that* (A1) *and* (A2) *hold. Then the system* (3.1) *is not meromorphically integrable near the resonant periodic orbit* $(I^*, \omega(I^*)t + \theta)$ *in the Bogoyavlenskij sense such that the first integrals and commutative vector fields also depend meromorphically on* ε *near* $\varepsilon = 0$.

See [26] for a proof of Theorem 3.1. In the proof, a perturbation approach was used along with the theory of Section 2.

Remark 3.2. *The real-analytical integrability of* (3.1) *such that the first integrals and commutative vector fields also depend real-analytically on* ε *near* $\varepsilon = 0$ *was also recently discussed in* [18] *under more restricted conditions in some meaning.*

3.2 Restricted three-body problem

We now consider the restricted three-body problem. Here we only treat the planar case but can extend the result to the spacial case without much difficulty. The equation of motion in the rotational frame (see Fig. 2) is given by

$$\dot{x} = p_x + y, \quad \dot{y} = p_y - x, \quad \dot{p}_x = p_y + \frac{\partial U_2}{\partial x}(x, y), \quad \dot{p}_y = -p_x + \frac{\partial U_2}{\partial y}(x, y), \tag{3.2}$$

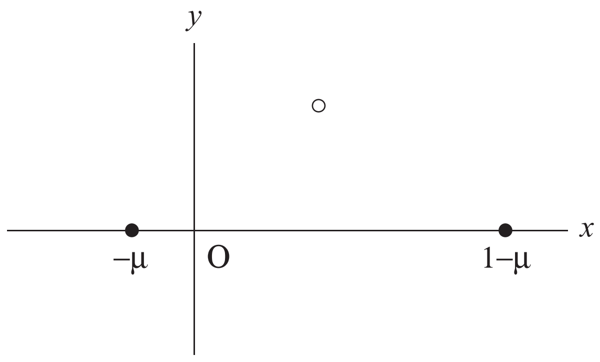

Figure 2: Restricted three-body problem.

where

$$U_2(x,y) = \frac{\mu}{\sqrt{(x-1+\mu)^2+y^2}} + \frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}}.$$

Equation (3.2) is a two-degree-of-freedom Hamiltonian system with the Hamiltonian

$$H = \tfrac{1}{2}(p_x^2+p_y^2) + (p_x y - p_y x) - U_2(x,y),$$

so that it is Liouville-integrable if it has an additional first integral. Poincaré [20] proved the following.

Theorem 3.3 (Poincaré)**.** *The Hamiltonian system* (3.2) *is not analytically integrable such that the additional first integral also depends analytically on* μ *near* $\mu = 0$*.*

Regarding (3.2) as a Hamiltonian system on the four-dimensional complex manifold

$$\begin{aligned}\mathscr{S}_2 =&\{(x,y,p_x,p_y,u_1,u_2)\in\mathbb{C}^6\\ &\mid u_1^2-(x-1+\mu)^2-y^2 = u_2-(x+\mu)^2-y^2=0\},\end{aligned}$$

we rewrite it as a meromorphic (rational) system:

$$\begin{aligned}&\dot x = p_x + y, \quad \dot y = p_y - x,\\ &\dot p_x = p_y - \mu(x-1+\mu)/u_1^3 - (1-\mu)(x+\mu)/u_2^2,\\ &\dot p_y = -p_x - \mu y/u_1^3 - (1-\mu)y/u_2^2,\\ &\dot u_1 = ((x-1+\mu)(p_x+y)+y(p_y-x))/u_1,\\ &\dot u_2 = ((x+\mu)(p_x+y)+y(p_y-x))/u_2.\end{aligned}$$

Consider a neighborhood of $(x,y)=(-\mu,0)$. Let

$$\varepsilon^2\xi = x+\mu, \quad \varepsilon^2\eta = y, \quad \varepsilon^{-1}p_\xi = p_x, \quad \varepsilon^{-1}p_\eta = p_y+\mu.$$

After scaling $t\to t/\varepsilon^3$, up to the order of ε^6, we obtain

$$\begin{aligned}&\dot\xi = p_\xi + \varepsilon^3\eta, \quad \dot p_\xi = -\frac{(1-\mu)\xi}{(\xi^2+\eta^2)^{3/2}} + \varepsilon^3 p_\eta + 2\varepsilon^6\mu\xi,\\ &\dot\eta = p_\eta - \varepsilon^3\xi, \quad \dot p_\eta = -\frac{(1-\mu)\eta}{(\xi^2+\eta^2)^{3/2}} - \varepsilon^3 p_\xi - \varepsilon^6\mu\eta,\end{aligned} \tag{3.3}$$

which is a Hamiltonian system with the Hamiltonian

$$H = \tfrac{1}{2}(p_\xi^2 + p_\eta^2) - \frac{1-\mu}{\sqrt{\xi^2+\eta^2}} + \varepsilon^3(\eta p_\xi - \xi p_\eta) - \tfrac{1}{2}\varepsilon^6\mu(2\xi^2 - \eta^2).$$

Using Delaunay elements, we rewrite the above system in action-angle coordinates. Let $\pi_2 : \mathscr{S}_2 \to \mathbb{C}^4$ be the projection such that

$$\pi_2(x, y, p_x, p_y, u_1, u_2) = (x, y, p_x, p_y)$$

and let

$$\Sigma(\mathscr{S}_2) = \{u_1 = (x-1+\mu)^2 + y^2 = 0\} \cup \{u_2 = (x+\mu)^2 + y^2 = 0\} \subset \mathscr{S}_2.$$

Note that π_2 is singular on $\Sigma(\mathscr{S}_2)$. Applying Theorem 3.1, we prove the following.

Theorem 3.4. *The Hamiltonian system* (3.2) *does not have another first integral that is meromorphic in* $(x, y, p_x, p_y, u_1, u_2)$ *except on* $\Sigma(\mathscr{S}_2)$ *in punctured neighborhoods of*

$$(x, y) = (-\mu, 0) \text{ and } (1-\mu, 0) \quad (\text{resp. } (x, y, z) = (-\mu, 0, 0) \text{ and } (1-\mu, 0, 0))$$

for any $\mu \in (0, 1)$, *as Hamiltonian systems on* $\mathscr{S}_2$.

See [26] for a proof of Theorem 3.4. Note that if the system (3.2) is meromorphically integrable, then so is the system (3.3) such that the first integral is meromorphic on ε near $\varepsilon = 0$.

Remark 3.5. *Using Theorem* 3.1, *we can also prove a result similar to Theorem* 3.3: *The Hamiltonian system* (3.2) *does not have another first integral that is meromorphic on* μ *near* $\mu = 0$ *in a neighborhood of the unperturbed periodic orbit for* $\mu = 0$. *See* [28] *for the details.*

3.3 Perturbations of Hamiltonian systems

We turn to time-periodic perturbations of single-degree-of-freedom Hamiltonian systems,

$$\dot{x} = J\mathrm{D}H(x) + \varepsilon u(x, \nu t), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x \in \mathbb{R}^2, \tag{3.4}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ and $u : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2$ are analytic and J is the 2×2 symplectic matrix. We assume the following on (3.4) with $\varepsilon = 0$:

- **(M1)** For some $\alpha_1 < \alpha_2$ there exists a one-parameter family of periodic orbits $x^\alpha(t)$, $\alpha \in (\alpha_1, \alpha_2)$, with period $T^\alpha > 0$.
- **(M2)** $x^\alpha(t)$ is analytic in $\alpha \in (\alpha_1, \alpha_2)$.

For relatively prime integers $l, m \in \mathbb{N}$, assume that the nondegenerate resonance condition $2\pi/T^\alpha = m\nu/l$ and $\mathrm{d}T^\alpha/\mathrm{d}\alpha \neq 0$ hold at $\alpha = \alpha^{l/m}$. We define the scalar action variable I^α for each periodic orbit $x^\alpha(t) = (x_1^\alpha(t), x_2^\alpha(t))$ as

$$I^\alpha = \frac{1}{2\pi}\int_{x^\alpha} x_2 \mathrm{d}x_1 = \frac{1}{2\pi}\int_0^{T^\alpha} x_2^\alpha(t)\dot{x}_1^\alpha(t)\mathrm{d}t \tag{3.5}$$

in the standard manner (see, e.g., Chapter 10 of [2]). The action variable I can thus be determined only by α. We assume that $\mathrm{d}\alpha/\mathrm{d}I > 0$ without loss of generality, and apply the implicit function theorem to (3.5) to represent α as a function of I: $\alpha = \alpha(I)$. We can show that the symplectic transformation from (I, θ_1) to x is given by

$$x = x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right), \tag{3.6}$$

where

$$\Omega(I) = \frac{2\pi}{T^{\alpha(I)}}.$$

We see that $\mathrm{d}\Omega/\mathrm{d}I \neq 0$ at $I = I^\alpha$ since $\mathrm{d}T^\alpha/\mathrm{d}\alpha \neq 0$. Moreover, we have the relations

$$\mathrm{D}_x I = -J\frac{\partial x}{\partial \theta_1}, \quad \mathrm{D}_x \theta_1 = J\frac{\partial x}{\partial I}. \tag{3.7}$$

Let $\theta_2 = \nu t \mod 2\pi$ in (3.4). Using (3.6) and (3.7), we transform (3.4) into

$$\dot{I} = \varepsilon h(I, \theta_1, \theta_2), \quad \dot{\theta}_1 = \Omega(I) + \varepsilon g_1(I, \theta_1, \theta_2), \quad \dot{\theta}_2 = \nu, \tag{3.8}$$

where

$$h(I, \theta_1, \theta_2) = \frac{1}{\Omega(I)}\mathrm{D}H\left(x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right)\right) \cdot u\left(x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right), \theta_2\right),$$
$$g_1(I, \theta_1, \theta_2) = J\frac{\partial}{\partial I}x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right) \cdot u\left(x^{\alpha(I)}\left(\frac{\theta_1}{\Omega(I)}\right), \theta_2\right).$$

See Section 2 of [23] for the details on these computations. Applying Theorem 3.1 to (3.8), we obtain the following [27].

Theorem 3.6. *Under assumptions* (M1) *and* (M2), *if* $\mathrm{d}T^\alpha/\mathrm{d}\alpha \neq 0$ *at* $\alpha = \alpha^{l/m}$ *and there exists a closed loop* γ_ϕ *for some* $\phi \in \mathbb{S}^1$ *such that* $\gamma_\phi \cap (i\mathbb{R} \cup (T^* + i\mathbb{R})) = \emptyset$ *and*

$$\hat{\mathscr{I}}(\phi) := \int_{\gamma_\phi} \mathrm{D}H(x^\alpha(t)) \cdot g\left(x^\alpha(\tau), \nu\tau + \phi\right) \mathrm{d}\tau \neq 0,$$

then the system (3.4) *is meromorphically nonintegrable near the resonant periodic orbit* $(x^\alpha(t), \nu t + \phi)$ *with* $\alpha = \alpha^{l/n}$ *in the meaning of Theorem* 3.1.

Remark 3.7. *The integrand in the integral* $\hat{\mathscr{I}}(\phi)$ *is the same as in the subharmonic Melnikov function, which enables us to detect the existence of resonant periodic orbits and their stability and bifurcations* (*see, e.g.,* [10, 23, 24]).

Theorem 3.6 was applied to show their nonintegrability near resonant periodic orbits in the meaning of Theorem 3.1 for the Duffing oscillator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = ax_1 - x_1^3 + \varepsilon(\beta\cos\nu t - \delta x_2) \tag{3.9}$$

and the forced pendulum

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + \varepsilon(\beta\cos\nu t - \delta x_2) \tag{3.10}$$

in [26] and [19], respectively, where $a = 0$ or ± 1 and $\delta \geq 0$ and $\beta, \nu > 0$ are constants. The real-analytical nonintegrability of (3.9) with $a = 1$ and (3.10) were also discussed in [18, 19]. Further examples for application of Theorem 3.1 are found in [27]. A result similar to Theorem 3.6 but real-meromorphic integrability near homo- and heteroclinic orbits was proved in [31], in which Theorem 2.1 was used but Theorem 3.1 was not.

4 Degenerate Equilibria

Finally, we review a recent result on nonintegrability of three- and four-dimensional dynamical systems near degenerate equilibria [27]. Here the Morales-Ramis theory is not required to obtain the result.

4.1 Main theorems

Consider systems of the general form (1.1), where $n = 3$ or 4 and $f(x)$ is analytic. We assume that $x = 0$ is an equilibrium, i.e., $f(0) = 0$, and the Jacobian matrix $\mathrm{D}f(0)$ of $f(x)$ at $x = 0$ has (I) a zero and pair of purely imaginary eigenvalues, $\lambda = 0, \pm i\omega$ ($\omega > 0$), for $n = 3$ or (II) two pairs of purely imaginary eigenvalues, $\lambda = \pm i\omega_j$ ($\omega_j > 0$), $j = 1, 2$, with $\omega_1/\omega_2 \notin \mathbb{Q}$ for $n = 4$. Then by polynomial changes of coordinates the system (1.1) is transformed into

$$\begin{aligned}\dot{x}_1 =& -\omega x_2 + \alpha_1 x_1 x_3 - \alpha_2 x_2 x_3, \quad \dot{x}_2 = \omega x_1 + \alpha_2 x_1 x_3 + \alpha_1 x_2 x_3,\\ \dot{x}_3 =& \alpha_3(x_1^2 + x_2^2) + \alpha_4 x_3^2\end{aligned} \tag{4.1}$$

up to $O(|x|^2)$ for case (I), and to

$$\begin{aligned}\dot{x}_1 =& -\omega_1 x_2 + (\alpha_1(x_1^2+x_2^2) + \alpha_2(x_3^2+x_4^2))x_1 - (\beta_1(x_1^2+x_2^2) + \beta_2(x_3^2+x_4^2))x_2,\\ \dot{x}_2 =& \omega_1 x_1 + (\beta_1(x_1^2+x_2^2) + \beta_2(x_3^2+x_4^2))x_1 + (\alpha_1(x_1^2+x_2^2) + \alpha_2(x_3^2+x_4^2))x_2,\\ \dot{x}_3 =& -\omega_2 x_4 + (\alpha_3(x_1^2+x_2^2) + \alpha_4(x_3^2+x_4^2))x_3 - (\beta_3(x_1^2+x_2^2) + \beta_4(x_3^2+x_4^2))x_4,\\ \dot{x}_4 =& \omega_2 x_3 + (\beta_3(x_1^2+x_2^2) + \beta_4(x_3^2+x_4^2))x_3 + (\alpha_3(x_1^2+x_2^2) + \alpha_4(x_3^2+x_4^2))x_4\end{aligned} \tag{4.2}$$

up to $O(|x|^3)$ for case (II), where $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, \ldots, 4$. We state our main results as follows:

Theorem 4.1. *Let $n = 3$ and suppose that the system* (1.1) *is transformed into* (4.1) *up to $O(|x|^2)$. If one of the following conditions holds, then the system* (1.1) *is not real-analytically integrable in the Bogoyavlenskij sense near the origin*:
(i) $\alpha_1\alpha_4 > 0$; (ii) $\alpha_1\alpha_4 < 0$ *and* $\alpha_4/\alpha_1 \notin \mathbb{Q}$.

Theorem 4.2. *Let $n = 4$ and suppose that the system* (1.1) *is transformed into* (4.2) *up to $O(|x|^3)$. If $\alpha_1 \neq \alpha_3$, $\alpha_2 \neq \alpha_4$ and one of the following conditions holds, then the system* (1.1) *is not real-analytically integrable in the Bogoyavlenskij sense near the origin*:
(i) $\alpha_1\alpha_3$ *or* $\alpha_2\alpha_4 > 0$; (ii) $\alpha_1\alpha_3, \alpha_2\alpha_4 < 0$ *and* $\alpha_1/\alpha_3, \alpha_2/\alpha_4 \notin \mathbb{Q}$.

Theorems 4.1 and 4.2 also imply that three- and four-dimensional systems exhibiting fold-Hopf and double-Hopf bifurcations (see, e.g., [11]), respectively, are real-analytically nonintegrable when the two control parameters are zero if their hypotheses are satisfied. The nonintegrability of the normal forms for these bifurcations when the two control parameters are nonzero was previously discussed in [1, 25].

4.2 Key results

We collect key results to prove the main theorems. We first consider the system (1.1) in a more general situation in which $n \neq 3, 4$ is allowed but $x = 0$ is still an equilibrium.

Definition 4.3 (Poincaré-Dulac normal form [33, 34]). *Change the coordinates in* (1.1) *such that* $\mathrm{D}f(0)$ *is in Jordan normal form. The system* (1.1) *is called a* Poincaré-Dulac (PD) normal form *if* $[Sx, f] = 0$, *where* S *is the semisimple part of* $\mathrm{D}f(0)$, *i.e.,* $S = \mathrm{diag}\lambda_j$, *where* λ_j, $j = 1, \dots, n$, *are the eigenvalues of* $\mathrm{D}f(0)$.

We easily see that the full systems for (4.1) and (4.2) without truncation are PD normal forms for (1.1) under our assumptions although their right-hand sides may not be convergent.

Theorem 4.4 (Zung [34]). *Let* $n \geq 1$ *be any integer. If the system* (1.1) *is analytically integrable in the Bogoyavlenskij sense, then there exists an analytic change of coordinates under which it is transformed into a PD normal form.*

Theorem 4.4 also implies that the corresponding PD normal form is convergent and analytically integrable if the system (1.1) is analytically integrable. So we only have to prove the analytic nonintegrability of the full systems for (4.1) and (4.2) to prove Theorems 4.1 and 4.2.

We next recall a result from [1]. Let $m > 0$ be an integer and consider $m + 2$-dimensional systems of the form

$$\dot{x} = f_x(x, y), \quad \dot{y} = f_y(x, y), \quad (x, y) \in D, \tag{4.3}$$

where $D \subset \mathbb{C}^2 \times \mathbb{C}^m$ is a region containing the m-dimensional y-plane $\{(0, y) \in \mathbb{C}^2 \times \mathbb{C}^m \mid y \in \mathbb{C}^m\}$, and $f_x : D \to \mathbb{C}^2$ and $f_y : D \to \mathbb{C}^m$ are analytic. Assume that by the change of coordinates $x = (x_1, x_2) = (r\cos\theta, r\sin\theta)$, Eq. (4.3) is transformed into

$$\dot{r} = R(r, y), \quad \dot{y} = \tilde{f}_y(r, y), \quad \dot{\theta} = \Theta(r, y), \quad (r, y, \theta) \in \tilde{D} \times \mathbb{C}, \tag{4.4}$$

where $\tilde{D} \subset \mathbb{C} \times \mathbb{C}^m$ is a region containing the m-dimensional y-plane, and $R : \tilde{D} \to \mathbb{C}$, $\tilde{f}_y : \tilde{D} \to \mathbb{C}^m$ and $\Theta : \tilde{D} \to \mathbb{R}$ are analytic.

Proposition 4.5. (i) *Suppose that Eq.* (4.3) *has a meromorphic first integral near* $(x_1, x_2) = (0, 0)$. *If* $\tilde{f}_{yj}(0, y) \neq 0$ *for almost all* $y \in \tilde{D}$ *for some* $j = 1, \dots, m$, *then the* (r, y)-*component of* (4.4) *has a meromorphic first integral near* $r = 0$.
(ii) *Suppose that Eq.* (4.3) *has a meromorphic commutative vector field. If* $\Theta(0, y) \neq 0$ *for almost all* $y \in \tilde{D}$, *then the* (r, y)-*component of* (4.4) *has a meromorphic commutative vector field near* $r = 0$.

Since $f(x)$ is analytic near $x = 0$, we have

$$f(x) = \sum_{j=k}^{\infty} f_j(x), \tag{4.5}$$

where $k \in \mathbb{N}$ and the elements of $f_j(x)$ are jth-order homogeneous polynomials of x. We have the following [27].

Theorem 4.6. *Suppose that* $f(x)$ *has the form* (4.5) *for some* $k \in \mathbb{N}$. *If the system* (1.1) *is analytically integrable in the Bogoyavlenskij sense, then so is the truncated system*

$$\dot{y} = f_k(y). \tag{4.6}$$

We finally consider planar vector fields of the form

$$\dot{z} = p(z), \quad z \in \mathbb{C}^2, \tag{4.7}$$

where $p(z)$ is analytic in z. We also have the following [27].

Proposition 4.7. *Let $D \subset \mathbb{C}$ be a region that is covered by nonconstant solutions to* (4.7) *almost everywhere. Suppose that the system* (4.7) *has a first integral $Q(x)$ and commutative vector field $q(x)$ in D. Let*

$$\Delta(x) = \det(p(x), q(x)) = p_1(x)q_2(x) - p_2(x)q_1(x),$$

where $q_j(x)$ and $p_j(x)$ are the jth-elements of $q(x)$ and $p(x)$, respectively. Then there exists a function $\chi : \mathbb{C} \to \mathbb{C}$ such that

$$\Delta(x)\mathrm{D}Q(x) = \chi(Q(x)) \begin{pmatrix} -p_2(x) \\ p_1(x) \end{pmatrix}. \tag{4.8}$$

4.3 Proofs of the main theorems

We sketch the proofs of Theorems 4.1 and 4.2. See [30] for the details.

Using Theorems 4.4 and 4.6 and Proposition 4.5, we obtain the following.

Proposition 4.8. *If the complexification of* (1.1) *in case* (I) *is analytically integrable near the origin $x = 0$, then so is the truncated system*

$$\dot{r} = \alpha_1 r x_3, \quad \dot{x}_3 = \alpha_3 r^2 + \alpha_4 x_3^2 \tag{4.9}$$

near $(r, x_3) = (0, 0)$.

Proposition 4.9. *If the complexification of* (1.1) *in case* (II) *is analytically integrable near the origin $x = 0$, then so is the truncated system*

$$\dot{r}_1 = (\alpha_1 r_1^2 + \alpha_2 r_2^2) r_1, \quad \dot{r}_2 = (\alpha_3 r_1^2 + \alpha_4 r_2^2) r_2 \tag{4.10}$$

near $(r_1, r_2) = (0, 0)$.

We easily see that the systems (4.9) and (4.10) have first integrals

$$Q(r, x_3) = r^{-2\alpha_4/\alpha_1}(\alpha_3 r^2 + (\alpha_4 - \alpha_1)x_3^2)$$

and

$$Q(r_1, r_2) = \left(\frac{r_1^{\alpha_4}}{r_2^{\alpha_2}}\right)^{2(\alpha_1 - \alpha_3)} \left(\frac{(\alpha_1 - \alpha_3)r_1^2}{r_2^2} + \alpha_2 - \alpha_4\right)^{\alpha_2\alpha_3 - \alpha_1\alpha_4},$$

respectively. We can show that if the hypotheses of Theorems 4.1 and 4.2 hold, then the systems (4.9) and (4.10), respectively, have no analytical first integral. When $Q(r, x_3)$ and $Q(r_1, r_2)$ are not analytic, we assume that analytic commutative vector fields $q(r, x_3)$ and $q(r_1, r_2)$ exist and use Proposition 4.7 to obtain

$$\Delta(r, x_3) = Cr(\alpha_3 r^2 + (\alpha_4 - \alpha_1)x_3^2),$$

for (4.9), and

$$\Delta(r_1, r_2) = C r_1 r_2 \left(\frac{r_1^2}{\alpha_2 - \alpha_4} + \frac{r_2^2}{\alpha_1 - \alpha_3} \right)$$

for (4.10), where $C \neq 0$ is some constant. From these expressions we estimate

$$q(r, x_3) = C \begin{pmatrix} r \\ x_3 \end{pmatrix} + O(|r|^2 + |x_3|^2)$$

and

$$q(r_1, r_2) = -\frac{C}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + O(|r|_1^2 + |r_2|^2),$$

which are easily shown to be not commutative vector fields of (4.9) and (4.10), respectively. This contradicts our assumptions. Thus, we obtain the desired results. □

Remark 4.10. *The nonintegrability of Poincaré-Dulac normal forms with higher-order terms was discussed in* [32]*. In particular, it was shown there that they may be meromorphically nonintegrable even if the hypotheses of Theorems* 4.1 *or* 4.2 *do not hold.*

4.4 Examples

As stated in Section 4.1, Theorems 4.1 and 4.2 imply that three- or four-dimensional systems exhibiting fold-Hopf and double-Hopf bifurcations are analytically nonintegrable under the weak conditions. We now give two such examples. See [27] for more details.

We first consider the Rössler system [22] of the form

$$\dot{x}_1 = -(x_2 + x_3), \quad \dot{x}_2 = x_1 + a x_2, \quad \dot{x}_3 = b x_1 + x_3(x_1 - c), \tag{4.11}$$

where a, b, c are constants. We can show that it exhibits fold-Hopf bifurcations when $b = 1$, $c = a \in (-\sqrt{2}, \sqrt{2})$ and $a^2 \notin \mathbb{Q}$ [27].

Proposition 4.11. *When $b = 1$, $c = a \in (-\sqrt{2}, \sqrt{2})$ and $a^2 \notin \mathbb{Q}$, the Rössler system* (4.11) *is not real-analytically integrable near the origin.*

We next consider the coupled van der Pol oscillators [21]

$$\begin{aligned} &\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + (\delta_1 - a_1 x_1^2) x_2 + b_1 x_3, \\ &\dot{x}_3 = x_4, \quad \dot{x}_4 = -c x_3 + (\delta_2 - a_2 x_3^2) x_4 + b_2 x_1, \end{aligned} \tag{4.12}$$

where $a_j, b_j, c > 0$ and $\delta_j \in \mathbb{R}$, $j = 1, 2$, are constants. Let

$$\omega_1 = \sqrt{\frac{(c+1) - \sqrt{(c-1)^2 + 4 b_1 b_2}}{2}}, \quad \omega_2 = \sqrt{\frac{(c+1) + \sqrt{(c-1)^2 + 4 b_1 b_2}}{2}}.$$

We can show that it exhibits double-Hopf bifurcations when $\delta_j = 0$, $j = 1, 2$, $b_1 b_2 < c$, $\omega_1/\omega_2 \notin \mathbb{Q}$, and

$$a_1 b_1((\omega_2^2 - 1)^2 - 2 b_1 b_2) + a_2 b_2((\omega_1^2 - 1)^2 - 2 b_1 b_2) \neq 0. \tag{4.13}$$

See [27] for the details.

Proposition 4.12. *When $\delta_j = 0$, $j = 1, 2$, $b_1 b_2 < c$ and $\omega_1/\omega_2 \notin \mathbb{Q}$, the coupled van der Pol oscillators* (4.12) *are not real-analytically integrable near the origin if condition* (4.13) *holds.*

References

[1] P.B. Acosta-Humánez and K. Yagasaki, Nonintegrability of the unfoldings of codimension-two bifurcations, *Nonlinearity*, **33** (2020), 1366–1387.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer, New York, 1989.

[3] M. Ayoul and N.T. Zung, Galoisian obstructions to non-Hamiltonian integrability, *C. R. Math. Acad. Sci. Paris*, **348** (2010), 1323–1326.

[4] O.I. Bogoyavlenskij, Extended integrability and bi-hamiltonian systems, *Comm. Math. Phys.*, **196** (1998), 19–51.

[5] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Epidemiology*, Springer, New York, 2019.

[6] G. Casale, Morales-Ramis theorems via Malgrange pseudogroup, *Ann. Inst. Fourier (Grenoble)*, **59** (2009), 2593–2610.

[7] T. Combot, A note on algebraic potentials and Morales-Ramis theory, *Celestial Mech. Dynam. Astronom.*, **115** (2013), 397–404.

[8] G. Duval, A. Maciejewski and M. Przybylska, Integrability of Hamiltonian systems with homogeneous potentials of degrees ± 2: An application of higher order variational equations, *Discrete Contin. Dyn. Syst.*, **34** (2014), 4589–4615.

[9] G. Duval, A. Maciejewski and M. Przybylska, Integrability of potentials of degree $k \neq \pm 2$: Second order variational equations between Kolchin solvability and Abelianity, *Discrete Contin. Dyn. Syst.*, **35** (2015), 1969–2009.

[10] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.

[11] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 3rd ed., Springer, New York, 2004.

[12] A. Maciejewski and M. Przybylska, Differential Galois approach to the non-integrability of the heavy top problem, *Ann. Fac. Sci. Toulouse Math. (6)*, **14** (2005), 123–160.

[13] J.J. Morales-Ruiz, *Differential Galois Theory and Non-Integrability of Hamiltonian Systems*, Birkhäuser, Basel, 1999.

[14] J.J. Morales-Ruiz and J.-P. Ramis, Galoisian obstructions to integrability of Hamiltonian systems, *Methods, Appl. Anal.*, **8** (2001), 33–96.

[15] J.J. Morales-Ruiz and J.-P. Ramis, Integrability of dynamical systems through differential Galois theory: A practical guide, in *Differential Algebra, Complex Analysis and Orthogonal Polynomials* (P.B. Acosta-Humánez and F. Marcellán, eds.), Contemp. Math., 509, Amer. Math. Soc., Providence, RI, 2010, pp. 143–220.

[16] J.J. Morales-Ruiz, J.-P. Ramis and C. Simo, Integrability of Hamiltonian systems and differential Galois groups of higher variational equations, *Ann. Sci. École Norm. Suppl.*, **40** (2007), 845–884.

[17] J.J. Morales-Ruiz and S. Simon, On the meromorphic non-integrability of some N-body problems, *Discrete Contin. Dyn. Syst.*, **24** (2009), 1225–1273.

[18] S. Motonaga and K. Yagasaki, Obstructions to integrability of nearly integrable systems near regular level sets, *Arch. Ration. Mech. Anal.*, to appear.

[19] S. Motonaga and K. Yagasaki, Nonintegrability of forced nonlinear oscillators, submitted for publication. `arXiv:2201.05328 [math.DS]`

[20] H. Poincaré, *New Methods of Celestial Mechanics*, Vol. 1, AIP Press, New York, 1992 (original 1892).

[21] R.H. Rand and P.J. Holmes, Bifurcation of periodic motions in two weakly coupled van der Pol oscillators, *Internat. J. Non-Linear Mech.*, **15** (1980), 387–399.

[22] O.E. Rössler, An equation for continuous chaos, *Phys. Lett. A.*, **57** (1976), 397–398.

[23] K. Yagasaki, The Melnikov theory for subharmonics and their bifurcations in forced oscillations, *SIAM J. Appl. Math.*, **56** (1996),1720–1765.

[24] K. Yagasaki, Melnikov's method and codimension-two bifurcations in forced oscillations, *J. Differential Equations*, **185** (2002), 1–24.

[25] K. Yagasaki, Nonintegrability of the unfolding of the fold-Hopf bifurcation, *Nonlinearity*, **31** (2018), 341–350.

[26] K. Yagasaki, Nonintegrability of the restricted three-body problem, submitted for publication. `arXiv:2106.04925 [math.DS]`

[27] K. Yagasaki, Nonintegrability of nearly integrable dynamical systems near resonant periodic orbits, *J. Nonlinear Sci.*, **32** (2022), 43.

[28] K. Yagasaki, A new proof of Poincaré's result on the restricted three-body problem, submitted for publication. `arXiv:2111.11031 [math.DS]`

[29] K. Yagasaki, Nonintegrability of the SEIR epidemic model, submitted for publication. `arXiv: 2203.10513 [math.DS]`

[30] K. Yagasaki, Nonintegrability of dynamical systems near degenerate equilibria, *Comm. Math. Phys.*, **398** (2023), 1129–1152.

[31] K. Yagasaki, Nonintegrability of time-periodic perturbations of single-degree-of-freedom Hamiltonian systems near homo- and heteroclinic orbits, submitted for publication. `arXiv: 2205.04803 [math.DS]`

[32] K. Yagasaki, Nonintegrability of truncated Poincare-Dulac normal forms of resonance degree two, submitted for publication. `arXiv: 2303.12493 [math.DS]`

[33] S. Yamanaka, Local integrability of Poincaré–Dulac normal forms, *Regul. Chaotic Dyn.*, **23** (2018), 933–947.

[34] N.T. Zung, Convergence versus integrability in Poincaré-Dulac normal forms, *Math. Res. Lett.*, **9** (2002), 217–228.

Kazuyuki Yagasaki
Department of Applied Mathematics and Physics,
Graduate School of Informatics, Kyoto University,
Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501 JAPAN.
E-mail address: `yagasaki@amp.i.kyoto-u.ac.jp`