On generalized Fuchs theorem over p-adic polyannuli: an announcement

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1 Introduction

This is an announcement of a recent work [Wan22] of the author on generalized Fuchs theorem over p-adic polyannuli.

Let K be a complete nonarchimedean field of mixed characteristic $(0, p)$. Christol and Mebkhout have given an intrinsic definition of the exponents of a finite free differential module on one dimensional annuli satisfying the Robba condition in [CM97]. They have also shown that if the exponent has p-adic non-Liouville differences ([Ked10] Definition 13.2.1), then there exists a canonical decomposition of this differential module into the ones with exponent identically equal to a single element. This is called the p-adic Fuchs theorem. However, their work was found to be difficult due to the complicated nature of the Frobenius antecedent developed in [CD94], on which their work was built. Dwork gave a simplified proof of p-adic Fuchs theorem on one dimensional annuli, in which Frobenius antecedent no more plays an important role. This method is also written in [Ked10] with a slightly different way. After Dwork's proof on one dimensional annuli, Gachet proved the p-adic Fuchs theorem on higher dimensional polyannuli in [Gac99]. The precise statement of this theorem is as follows:

Theorem 1.1 (Théorème in page 216 of [Gac99])**.** *Let P be a finite free differential module on an open polyannulus over K for the derivations $t_i\partial_{t_i}$, with $1 \leq i \leq n$ satisfying the Robba condition and admitting an exponent on some closed subpolyannulus of positive width with p-adic non-Liouville differences. Then P adimits a basis on which the matrix of action of $\nabla(t_i\partial_{t_i})$ has entries in K and its eigenvalues represent the exponent of P for all $1 \leq i \leq n$. Consequently, P admits a canonical decomposition*

$$P = \bigoplus_{\lambda \in (\mathbb{Z}_p/\mathbb{Z})^n} P_\lambda,$$

in which each P_λ has exponent identically equal to λ.

Meanwhile, Kedlaya proved a generalized version of one dimensional p-adic Fuchs theorem, by loosing the condition on exponents from having p-adic non-Liouville differences to a weaker one, namely, having Liouville partition, and yet still gives a decomposition of such differential module.

Theorem 1.2 (Theorem 3.4.22 in [Ked15]). *Let P be a finite free differential module satisfying the Robba condition over one dimensional annulus over K associated to an open interval I. Let $J \subset I$ be a closed subinterval of positive width, and suppose that P has an exponent A over J admitting a Liouville partition $\mathscr{A}_1, \dots, \mathscr{A}_k$ (for definition see Definition 3.4.4 in[Ked15]). Then there exists a unique direct sum decomposition $P_J = P_1 \oplus \cdots \oplus P_k$ such that for $g = 1, \dots, k$, P_g admits an exponent over J weakly equivalent to $\mathscr{A}_g$.*

Moreover, it is realized that the generalized p-adic Fuchs theorem implies the the original p-adic Fuchs theorem in one dimensional case.

In the article [Wan22], we proved a generalized version of higher dimensional p-adic Fuchs theorem: namely, we defined the notion of exponent A for a finite projective differential module P satisfying the Robba condition on higher dimensional polyannuli over K, and proved a decomposition theorem for P with respect to a Liouville partition of A, which is similar to Theorem 1.2. It is worth mentioning that our result implies Theorem 1.1, and since our generalized p-adic Fuchs theorem works not only for finite free but also for finite projective differential modules, our result is possibly stronger than the result in [Gac99]. Also, though we basically follow the strategy developed by Kedlaya in [Ked15], there are new ingredients applied to get the decomposition from local ones, because of the lack of Quillen-Suslin theorem for arbitrary polyannuli over K.

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2 Preliminaries

In this section, we introduce some basic facts about module theory over p-adic polyannuli and abstract p-adic exponents.

We denote the Berkovich affine n-space $(\mathrm{Spec} K[t_1, \dots, t_n])^{\mathrm{an}}$ over K by $\mathbb{A}^n_K$.

Definition 2.1. For a polysegment $I = \prod_{i=1}^n I_i \subset \mathbb{R}^n_{>0}$, the polyannulus with radius I over K is the subspace of $\mathbb{A}^n_K$ defined by

$$\{x \in \mathbb{A}^n_K : t_i(x) \in I_i,\ 1 \leq i \leq n\},$$

and we call such a subspace an open (resp. closed) polyannulus if I is open (resp. closed). Moreover, we say that it is of positive width (resp. of width 0) if each I_i is not a point (resp. I consists of only one point). The coordinate ring of this polyannulus is

$$\left\{ f = \sum_{i \in \mathbb{Z}^n} f_i t^i \in K[[t, t^{-1}]] : \lim_{|i| \to \infty} |f_i| \rho^i = 0 \quad \forall \rho \in I \right\},$$

and we denote this ring by $K_{I,n}$. Here we put the subscript n in the notation to emphasize the dimension of the associated polyannulus. For $\rho \in I$, we define the ρ-Gauss norm of $f = \sum_{i \in \mathbb{Z}^n} f_i t^i \in K_{I,n}$ to be $|f|_\rho := \max_{i \in \mathbb{Z}^n} |f_i| \rho^i$.

When I is closed, $K_{I,n}$ is a K-affinoid algebra in the sense of Berkovich, and the supremum norm (which is power multiplicative but not necessarily multiplicative) on $K_{I,n}$ is defined by $|f|_I := \max_{\rho\in I}\{|f|_\rho\}$. For polysegments $J \subset I$ in $\mathbb{R}^n_{>0}$ and a $K_{I,n}$-module P, we denote the module $K_{J,n} \otimes_{K_{I,n}} P$ by P_J.

For a finite projective module over a polyannulus of positive width, the following theorem shows that, after properly shrink the inner and outer radius of the polyannulus, the projective module becomes free.

Theorem 2.2 ([Wan22], Theorem 1.18)**.** *Let α, $\beta \in \mathbb{R}^n_{>0}$ with $\alpha < \beta$, and let P be a finite projective $K_{[\alpha,\beta],n}$ module. Then, for any $\rho \in (\alpha, \beta)$, there exist α' and β' with $\alpha < \alpha' < \rho < \beta' < \beta$ such that $P_{[\alpha',\beta']}$ is free.*

For $x \in \mathbb{Q}_p$, we denote by $\langle x\rangle$ the smallest nonnegative rational number a such that one of $x - a$ and $x + a$ is a p-adic integer.

Definition 2.3. We say that $a \in \mathbb{Z}_p$ is a p-adic Liouville number if $a \notin \mathbb{Z}$ and

$$\liminf_{m\to\infty} \frac{p^m}{m} \left\langle \frac{a}{p^m} \right\rangle < \infty.$$

If a is not p-adic Liouville, we say that it is a p-adic non-Liouville number.

In the following, for a multisubset $A = \{A_1, \dots, A_m\}$ of $\mathbb{Z}_p^n$, we denote the i-th entry of A_j by A_j^i, and denote the multisubset $\{A_1^i, \dots, A_m^i\}$ of $\mathbb{Z}_p$ by A^i.

Definition 2.4 ([Wan22], Definition 1.21, Définitions in p.194 of [Gac99], cf. [Ked15], Definition 3.4.2)**.** Let $A = \{A_1, \dots, A_m\}$ be a multisubset of $\mathbb{Z}_p^n$. We say that A is p-adic non-Liouville in the r-th direction if A_j^r is a p-adic non-Liouville number for any $1 \le j \le m$, and we say that A is p-adic non-Liouville if it is p-adic non-Liouville in every direction. We say that A has p-adic non-Liouville differences in the r-th direction if the difference multisubset $A - A := \{A_i - A_j : 1 \le i, j \le m\}$ is p-adic non-Liouville in the r-th direction, and we say that A has p-adic non-Liouville differences if it has p-adic non-Liouville differences in every direction.

Definition 2.5 (Définitions in p.189 of [Gac99], cf.[Ked15] Definition 3.4.3)**.** For two multisubsets $A = (A_1, \dots, A_m)$ and $B = (B_1, \dots, B_m)$ of $\mathbb{Z}_p^n$, we say that A is weakly equivalent to B if there exists a constant $c > 0$ and a sequence of permutations σ_h ($h \in \mathbb{Z}_{>0}$) of $\{1, 2, \dots, m\}$ such that, for all $1 \le i \le n$ and $1 \le j \le m$,

$$p^h \left\langle \frac{A^i_{\sigma_h(j)} - B^i_j}{p^h} \right\rangle \le ch.$$

We say that A is equivalent to B if there exists a permutation σ of $\{1, 2, \dots, m\}$ such that for all $1 \le i \le n$ and $1 \le j \le m$,

$$A^i_{\sigma(j)} - B^i_j \in \mathbb{Z}.$$

Definition 2.6 ([Wan22], Definition 1.24, cf. [Ked15], Definition 3.4.4)**.** Let $A, \mathscr{A}_1, \dots, \mathscr{A}_k$ be multisubsets of $\mathbb{Z}_p^n$ such that $A = \bigcup_{i=1}^k \mathscr{A}_i$ as multisets. We say that $\mathscr{A}_1, \dots, \mathscr{A}_k$ form a Liouville partition of A in the r-th direction if $\mathscr{A}_1^r, \dots, \mathscr{A}_k^r$ is a Liouville partition of A^r, namely, for any $1 \le l < m \le k$ and $a_l \in \mathscr{A}_l^r, a_m \in \mathscr{A}_m^r$, $a_l - a_m$ is a p-adic non-Liouville number which is not an integer.

Proposition 2.7 ([Wan22], Proposition 1.25, cf. [Ked15], Proposition 3.4.5). *Let A be a finite multisubset of $\mathbb{Z}_p^n$ and let $\mathscr{A}_1, \ldots, \mathscr{A}_k$ be a Liouville partition of A in the r-th direction.*

(1) Let $\mathscr{B}_1, \ldots, \mathscr{B}_k$ be multisubsets of $\mathbb{Z}_p^n$ such that $\mathscr{B}_i^r$ is weakly equivalent to $\mathscr{A}_j^r$ for $1 \leq j \leq k$. Then $\mathscr{B}_1, \ldots, \mathscr{B}_k$ form a Liouville partition in the r-th direction of $B = \bigcup_{j=1}^k \mathscr{B}_k$.

(2) Suppose that B is a multisubset of $\mathbb{Z}_p^n$ weakly equivalent to A. Then B admits a Liouville partition $\mathscr{B}_1, \ldots, \mathscr{B}_k$ in the r-th direction such that $\mathscr{B}_j$ is weakly equivalent to $\mathscr{A}_j$ for $1 \leq j \leq k$.

Definition 2.8 ([Wan22], Definition 1.28). Let $k \geq 1$. We define the notion of Liouville partition of a multisubset A of $\mathbb{Z}_p^n$ by k multisubsets $\mathscr{A}_1, \ldots, \mathscr{A}_k$ of $\mathbb{Z}_p^n$ inductively on k as follows:

(1) When $k = 1$, $\mathscr{A}_1$ is a Liouville partition of A if $\mathscr{A}_1 = A$ as multisets.

(2) For general k, $\mathscr{A}_1, \ldots, \mathscr{A}_k$ is a Liouville partition of A if there exists a partition

$$\{1, \ldots, k\} = \bigcup_{i=1}^{l} I_i$$

as sets for some $l \geq 2$ with each I_i nonempty such that $\bigcup_{j \in I_1} \mathscr{A}_j, \ldots, \bigcup_{j \in I_l} \mathscr{A}_j$ is a Liouville partition in the r-th direction of A for some $1 \leq r \leq n$ and that $\mathscr{A}_j$ $(j \in I_i)$ is a Liouville partition of $\bigcup_{j \in I_i} \mathscr{A}_j$, which is defined by the induction hypothesis.

3 The construction of categories $\mathscr{C}_\rho$ and $\mathscr{D}_\rho$

For a polysegment $I \subset \mathbb{R}_{>0}^n$, let $\mathrm{Der}(K_{I,n}/K)$ be the module of continuous K-derivations on $K_{I,n}$, where the topology on $K_{I,n}$ is induced by ρ-Gauss norms for all $\rho \in I$. It is a finite free module generated by derivations with respect to each t_i, which are denoted by ∂_{t_i}, for $1 \leq i \leq n$.

Definition 3.1. Let P be a finite projective $K_{I,n}$-module. A connection over P is a K-linear homomorphism $\nabla : \mathrm{Der}(K_{I,n}/K) \to \mathrm{End}_K(P)$ satisfying the Leibniz rule:

$$\nabla(\partial)(fa) = \partial(f)a + f\nabla(\partial)(a), \text{ for all } \partial \in \mathrm{Der}(K_{I,n}/K),\ f \in K_{I,n},\ a \in P.$$

Moreover, a connection is called integrable if for ∂, $\partial' \in \mathrm{Der}(K_{I,n}/K)$, $\nabla([\partial, \partial']) = [\nabla(\partial), \nabla(\partial')]$, where $[\cdot, \cdot]$ is the Lie bracket.

A (finite projective) differential module over $K_{I,n}$ is a (finite projective) $K_{I,n}$-module P with an integrable connection ∇_P, which we denote simply by ∇ if no ambiguity arises. A horizontal homomorphism between differential modules P and Q over $K_{I,n}$ is a module homomorphism $f : P \to Q$ satisfying $\nabla_Q(\partial)(f(x)) = f(\nabla_P(\partial)(x))$ for all $\partial \in \mathrm{Der}(K_{I,n}/K)$ and for all $x \in P$. In the rest of the paper, we often say a differential module over $K_{I,n}$ a differential module over I, by abuse of terminology.

For any $\rho \in \mathbb{R}_{>0}^n$, there is a direct system of rings $((K_{I,n})_I, (\varphi_{IJ} : K_{I,n} \to K_{J,n})_{I \leq J})$ with the index set being all closed polysegments of $\mathbb{R}_{>0}^n$ containing

ρ in its interior and partially ordered by inverse inclusion, and homomorphisms are given by the canonical inclusion. We denote the direct limit of this direct system by $R_{\rho,n}$ or simply by R_ρ. R_ρ is canonically a differential ring with respect to ∂_{t_i}, for $1 \leq i \leq n$.

Since any finite projective module over R_ρ is extended from a finite projective module over $K_{[\alpha,\beta],n}$ for some $\alpha < \rho < \beta$, it is extended from a finite free $K_{[\alpha,\beta],n}$-module for some $\alpha < \rho < \beta$ by Theorem 2.2. In particular, all finite projective modules over R_ρ are finite free.

Let $\mathrm{Der}(R_\rho/K)$ be the module of K-derivations ∂ on R_ρ such that $\partial|_{K_{I,n}} : K_{I,n} \to R_\rho$ is induced by a continuous K-derivation on $K_{I,n}$ for any closed polysegment I containing ρ in its interior. It is again a finite free module generated by ∂_{t_i}, for $1 \leq i \leq n$. Using $\mathrm{Der}(R_\rho/K)$, we can define the notion of (finite free) differential modules over R_ρ and horizontal homomorphisms between them in the same way as above. It is easy to see that any finite free differential module over R_ρ is extended from some finite free differential module over $K_{I,n}$ for some closed interval I containing ρ in its interior.

Definition 3.2 (cf. [KX10], Definition 1.5.2, [Ked10], Definition 9.4.7, [Ked10], Definition 13.3.1)**.** Let $I \subset \mathbb{R}^n_{>0}$ be a polysegment and let P be a finite projective differential module over $K_{I,n}$. Take $\rho \in I$, let F_ρ be the completion of $K(t_1, \dots, t_n)$ with respect to the ρ-Gauss norm, and put $V_\rho = P \otimes_{K_{I,n}} F_\rho$. The intrinsic radius of P at ρ is defined as

$$IR(V_\rho) = \min_{1\leq i\leq n} IR_{\partial_{t_i}}(V_\rho) = \min_{1\leq i\leq n} \frac{|\partial_{t_i}|_{\mathrm{sp},F_\rho}}{|\nabla(\partial_{t_i})|_{\mathrm{sp},V_\rho}} \in (0,1].$$

We say that P satisfies the Robba condition if $IR(V_\rho) = 1$ for all $\rho \in I$. Also, we say that a finite free differential module over R_ρ satisfies the Robba condition if it is extended from a finite projective differential module over $K_{J,n}$ satisfying the Robba condition for some closed polysegment J containing ρ in its interior.

Let $\mathscr{D}_\rho$ be the category in which objects are finite free differential modules over R_ρ satisfying the Robba condition, and morphisms are horizontal homomorphisms. For a polysegment J containing ρ in its interior, we say that P is an object in $\mathscr{D}_\rho$ defined (by P') over J if P' is a finite free differential module over $K_{J,n}$ and P is extended from P'.

From now on for a positive integer s, Γ_s denotes the group of p^s-th roots of unity in K^{alg}, and $\Gamma = \bigcup_{s>0} \Gamma_s$. Also Γ_s^n and Γ^n denote the product of n copies of Γ_s and Γ, respectively.

Definition 3.3 ([Wan22], Definition 2.3)**.** The category $\mathscr{C}_\rho$ is defined as follows:

The objects are finite free R_ρ-modules P endowed with a semilinear group action of Γ^n on $P \otimes_K K(\Gamma)$ satisfying the following conditions:

(1) P is extended from a finite free $K_{J,n}$-module P' for some closed polysegment J contains ρ in its interior, and the action of Γ^n on $P \otimes_K K(\Gamma)$ is induced from some semilinear group action of Γ^n on $P' \otimes_K K(\Gamma)$.

(2) The action of Γ^n is equivariant with respect to the action of $\mathrm{Gal}(K(\Gamma)/K)$ on both Γ^n and $P \otimes_K K(\Gamma)$. That is, for $\sigma \in \mathrm{Gal}(K(\Gamma)/K)$, $\zeta \in \Gamma^n$ and $x \in P \otimes_K K(\Gamma)$, we have

$$\sigma(\zeta^*(x)) = \sigma(\zeta)^*(\sigma(x)).$$

(3) For some basis $e_1, \dots, e_m$ of P' in (1), there exists $l > 0$ such that, for each positive integer k and $\zeta \in \Gamma_k^n$, the representation matrix $E(\zeta)$ of ζ^* with respect to this basis satisfies the inequality $|E(\zeta)|_J \leq p^{lk}$.

The morphisms $f : P \to Q$ of objects in $\mathscr{C}_\rho$ are module homomorphisms satisfying $\zeta^*((f \otimes \mathrm{id})(x)) = (f \otimes \mathrm{id})(\zeta^*(x))$ for all $\zeta \in \Gamma^n$ and $x \in P \otimes_K K(\Gamma)$.

We summarize properties of these two categories $\mathscr{C}_\rho$ and $\mathscr{D}_\rho$ as follows:

Theorem 3.4 (cf. [Wan22], Lemma 2.6, Remark 2.7, Proposition 2.11)**.** *The following properties are true for $\mathscr{C}_\rho$ and $\mathscr{D}_\rho$:*

(1) Both $\mathscr{C}_\rho$ and $\mathscr{D}_\rho$ are Abelian categories and every object in $\mathscr{C}_\rho$ and $\mathscr{D}_\rho$ is finite free.

(2) Tensor product and dual exist in $\mathscr{C}_\rho$ and $\mathscr{D}_\rho$.

(3) For any $P \in \mathscr{D}_\rho$, $\zeta \in \Gamma^n$ and $x \in P$, the following series converges:

$$\zeta^*(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} (\zeta - 1)^\alpha \binom{tD}{\alpha}(x).$$

This defines a functor $\mathscr{D}_\rho \to \mathscr{C}_\rho$. Here, $1 = (1, \dots, 1)$ and $\binom{tD}{\alpha} = \binom{t_1 D_1}{\alpha_1} \cdots \binom{t_n D_n}{\alpha_n}$ with $D_i = \nabla(\partial_{t_i})$.

(4) The functor defined in (3) is an exact tensor functor of abelian categories.

4 Generalized p-adic Fuchs theorem

In this section, we define p-adic exponents associated to p-adic differential equations over polyannuli satisfying the Robba condition. Moreover, we state some basic facts and our main theorem.

From now on, we use the following conventions. For $\zeta = (\zeta_1, \dots, \zeta_n) \in \Gamma_s^n$, an n-tuple of variables $t = (t_1, \dots, t_n)$ and an n-tuple of $m \times m$ diagonal matrices

$$A = (A^1, \dots, A^n) = (\mathrm{diag}(a_{11}, \dots, a_{1m}), \dots, \mathrm{diag}(a_{n1}, \dots, a_{nm})),$$

set

(1) $\zeta t := (\zeta_1 t_1, \dots, \zeta_n t_n)$.

(2) $\zeta^A := \zeta_1^{A^1} \dots \zeta_n^{A^n}$, with $\zeta_i^{A^i} := \mathrm{diag}(\zeta_i^{a_{i1}}, \dots, \zeta_i^{a_{im}})$.

Firstly, we give the definition of exponent for objects in $\mathscr{C}_\rho$ and show some properties of it.

Definition 4.1 ([Wan22], Definition 3.1, cf. [Ked10], Definition 13.5.1 and [Ked15], Definition 3.4.11)**.** Let P be an object in $\mathscr{C}_\rho$ free of rank m. Take $\alpha < \rho < \beta$ such that P is defined by P' over $[\alpha, \beta]$, and take a basis $e_1, \dots, e_m$ of P'. An exponent of P(admitted by P') is an n-tuple of $m \times m$ diagonal matrices of $A = (A^1, \dots, A^n)$ with entries in $\mathbb{Z}_p$ for which there exists a sequence $\{S_{k,A}\}_{k=1}^\infty$ of $m \times m$ matrices with entries in $K_{[\alpha,\beta],n}$ satisfying the following conditions.

(1) If we put $(v_{k,A,1},\dots,v_{k,A,m})=(e_1,\dots,e_m)S_{k,A}$, then for all $\zeta\in\Gamma_k^n$

$$\zeta^*(v_{k,A,1},\dots,v_{k,A,m})=(v_{k,A,1},\dots,v_{k,A,m})\zeta^A.$$

(2) There exists $l>0$ such that $|S_{k,A}|_{[\alpha,\beta]}\le p^{lk}$ for all k.

(3) We have $|\det(S_{k,A})|_{[\alpha,\beta]}\ge 1$ for all k.

For an object P in $\mathscr{D}_\rho$, we say A is an exponent of P if, when considered as an object in $\mathscr{C}_\rho$, A is an exponent of P.

Theorem 4.2 ([Wan22], Theorem 3.2, cf. [Ked10], Theorem 13.5.5,[Gac99], Théorème in p.173)**.** *Let P be an object in $\mathscr{C}_\rho$. Then there exists an exponent A for P.*

Theorem 4.3 ([Wan22], Theorem 3.3, cf. [Ked10], Theorem 13.5.6)**.** *Let P be an object in $\mathscr{C}_\rho$ defined by P_1 over J_1 and by P_2 over J_2, where J_1,J_2 are closed polysegments containing ρ in its interior. Then the exponents of P defined by P_1 and P_2 are weakly equivalent. In particular, the exponent of P is uniquely determined up to weak equivalence.*

Moreover, exponents are compatible with exact sequence, tensor product and dual:

Lemma 4.4 ([Wan22], Lemma 3.4, cf. [Ked15], Remark 3.4.14, [KS17])**.** *Let P_1, P_2 and P be three objects in $\mathscr{C}_\rho$ with P_i having exponent A_i $(i=1,2)$. Then,*

(1) if there exists a short exact sequence

$$0\to P_1\to P\to P_2\to 0,$$

then P admits the multiset union $A_1\cup A_2$ as an exponent.

(2) the module $P_1\otimes P_2$ is an object in $\mathscr{C}_\rho$, and admits the multiset A_1+A_2 as an exponent.

(3) the module $P_1^\vee$ is an object in $\mathscr{C}_\rho$, and admits the multiset $-A_1$ as an exponent.

Theorem 4.5 ([Wan22], Theorem 3.10, cf.[KS17])**.** *Let P be an object in $\mathscr{D}_\rho$ having an exponent A with Liouville partition $\mathscr{A}_1,\dots,\mathscr{A}_k$ in the r-th direction. Then there exists a unique direct sum decomposition $P=P_1\oplus\cdots\oplus P_k$ in $\mathscr{D}_\rho$ with each P_i having exponent weakly equivalent to $\mathscr{A}_i$ for $1\le i\le k$.*

Let P be a finite projective differential module over an open polysegment I of $\mathbb{R}^n_{>0}$ satisfying the Robba condition. Then, for $\rho\in I$, $P_\rho:=P\otimes_{K_{I,n}}R_\rho$ is an object of $\mathscr{D}_\rho$ and so an exponent A_ρ of P_ρ is defined.

Lemma 4.6 ([Wan22], Lemma 3.12)**.** *Let the notations be as above. Then, for any $\rho,\rho'\in I$, A_ρ and $A_{\rho'}$ are weakly equivalent.*

Then we can define the exponent of a finite projective differential module over an open polysegment as follows:

Definition 4.7 ([Wan22], Definition 3.13)**.** Let P be a finite projective differential module over an open polysegment I in $\mathbb{R}^n_{\geq 0}$ satisfying the Robba condition. We say that A is an exponent of P if it is an exponent of $P_\rho := P \otimes_{K_{I,n}} R_\rho$ for some $\rho \in I$.

By uniqueness of exponent up to weak equivalence, decompositions of P_ρ in $\mathscr{D}_\rho$ for $\rho \in I$ can be glued up to a decomposition of P.

Corollary 4.8 ([Wan22], Corollary 3.15)**.** *Let P be a finite projective differential module over some open polysegment I satisfying the Robba condition, admitting an exponent A with Liouville partition $\mathscr{A}_1, \ldots, \mathscr{A}_k$. Then there exists a unique decomposition $P = P_1 \oplus \cdots \oplus P_k$, where each P_i is a finite projective differential module and admits an exponent weakly equivalent to $\mathscr{A}_i$ for $1 \leq i \leq k$.*

Then a slightly stronger version of Gachet's p-adic Fuchs theorem can be proved using Corollary 4.8:

Corollary 4.9 ([Wan22], Corollary 3.20)**.** *Let P be a finite projective differential module over an open polysegment I in $\mathbb{R}^n_{>0}$ satisfying the Robba condition. Furthermore we assume that P has p-adic non-Liouville exponent differences. Then P admits a basis on which the matrix of action of D_i for $1 \leq i \leq n$ has entries in K whose eigenvalues represents an exponent of P. Consequently, P admits a canonical decomposition*

$$P = \bigoplus_{\lambda \in (\mathbb{Z}_p/\mathbb{Z})^n} P_\lambda,$$

where P_λ is free with exponent identically equal to a representative in $\mathbb{Z}_p^n$ of λ. In particular, P is free, and is extended from some finite differential module over a polydisc for the derivations $t_i \partial_{t_i}$, $1 \leq i \leq n$.

Note that the reason that Corollary 4.9 is possibly slightly stronger than the result of [Gac99] is that we do not know yet if any finite projective differential module on polyannuli is free, and [Gac99] only treated finite free differential modules.

5 Future Prospects

We believe that after appropriate modification, a similar strategy can be applied to study the generalized p-adic Fuchs theorem over relative polyannuli.

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