TOPICS RELATED TO RSZ SHIMURA VARIETIES

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Abstract. This article is concerned with PEL unitary Shimura varieties and their integral models by Rapoport, Smithling and Zhang in relation to the arithmetic Gan–Gross–Prasad conjecture. We call these Shimura varieties RSZ Shimura varieties. We include two results here. One is about the comparison of these integral models with the ones by Kisin and Pappas. The other is about the curve case of the variant of the last conjecture formulated on the above Shimura varieties.

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1. Notations

In this section, we introduce common notations in the two results. We follow [6] here. Let F be a CM number field. Let F_0 be its maximal totally real subfield. Let Φ be a CM type of F. Set W to be a nondegenerate n-dimensional F/F_0-Hermitian space. Here we take n to be positive. For each $\varphi \in \Phi$, write $(r_\varphi, r_{\overline{\varphi}})$ for the signature of $W \otimes_{F,\varphi} \mathbb{C}$. We take a unitary group $G := \mathrm{U}(W)$ over F_0 and a $\mathbb{Q}$-torus $Z^{\mathbb{Q}}$ such that for each $\mathbb{Q}$-algebra R,

$$Z^{\mathbb{Q}}(R) = \{z \in \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_{\mathrm{m}}(R) \mid \mathrm{Nm}_{F/F_0}(z) \in R^\times\}.$$

We note that we also have a $\mathbb{Z}$-model of $Z^{\mathbb{Q}}$. Put $\widetilde{G} := Z^{\mathbb{Q}} \times \mathrm{Res}_{F_0/\mathbb{Q}} G$. This is the group for which we mainly consider Shimura varieties. We refer the reader to [6, Remark 2.6] for the motivation behind introducing this group. We also refer to [6, §2, §3] or [3, §3.2] for a precise definition of the Shimura data $(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$, $(\widetilde{G}, \{h_{\widetilde{G}}\})$ and others. Let E be the reflex field of the Shimura datum for $\widetilde{G}$.

2. The first result: the comparison of integral models

Here we discuss the comparison of the integral models by Rapoport, Smithling and Zhang with the ones by Kisin and Pappas. Let p be an odd prime. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$. Let $\widehat{*}^p$ be the prime-to-p completion.

2.1. Integral models by Rapoport, Smithling and Zhang.

We first describe the former integral models. We refer to [6, Theorem 5.4] and [3, §4, §5] for details.

For the description, we need auxiliary integral models for the Shimura varieties of $Z^{\mathbb{Q}}$. Let ν be the p-adic place of E that the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ induces. Set $\mathfrak{a} \neq 0$ to be an ideal of $\mathcal{O}_{F_0}$ prime to p. The auxiliary integral model $\mathcal{M}_0^{\mathfrak{a}}$ can be constructed as a category fibered in groupoids over the category of locally noetherian $\mathcal{O}_E$-schemes. It carries an object S of the source category to the groupoid of tuples (A_0, i_0, λ_0), where

- A_0 is an abelian scheme over S,

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- $i_0\colon \mathcal{O}_F \to \mathrm{End}_S(A_0)$ satisfies the Kottwitz condition that for every $a \in \mathcal{O}_F$, the characteristic polynomial of $i_0(a)$ acting on $\mathrm{Lie}\, A_0$ is
$$\prod_{\varphi\in\Phi}(T-\varphi(a)),$$
and
- $\lambda_0\colon A_0 \to A_0^\vee$ is an $\mathcal{O}_F$-linear polarization with kernel $A_0[\mathfrak{a}]$.

Morphisms of this groupoid are isomorphisms of abelian schemes preserving the other structures. The moduli problem is represented by a finite etale $\mathcal{O}_E$-scheme by the argument of [1, Proposition 3.1.2]. Also, $\mathcal{M}_0^{\mathfrak{a}}$ is nonempty for some $\mathfrak{a}$.

Let (Λ_0, ψ_0) be a pair of an invertible $\mathcal{O}_F$-module and an alternating $\mathcal{O}_F/\mathcal{O}_{F_0}$-balanced form $\psi_0\colon \Lambda_0\times\Lambda_0 \to \mathbb{Z}$ such that $\psi_0(\sqrt{\Delta}*, *)$ is positive definite and the dual of Λ_0 with respect to $\psi_{0,\mathbb{Q}}$ is $\mathfrak{a}^{-1}\Lambda_0$.

There is a clopen subset $\mathcal{M}_0^{\mathfrak{a},[\Lambda_0]}$ of $\mathcal{M}_0^{\mathfrak{a}}$ with the generic fiber $\mathrm{Sh}_{Z^{\mathbb{Q}}(\widehat{\mathbb{Z}})}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$. Also, we can equip the moduli problem over $\mathrm{Spec}\, \mathcal{O}_{E,(p)}$ with the usual level structure trivializing the prime-to-p Tate module of A_0 with respect to Λ_0. This way, we obtain an integral model $\mathcal{M}_0$ of $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$, where $K_{Z^{\mathbb{Q}}} = Z^{\mathbb{Q}}(\mathbb{Z}_p)K^p_{Z^{\mathbb{Q}}}$ for an open subgroup $K^p_{Z^{\mathbb{Q}}}$ of $Z^{\mathbb{Q}}(\widehat{\mathbb{Z}}^p)$.

We move on to the integral model that we discuss in our result. We need more notations for that. For a p-adic place w of F, fix a uniformizer π_w of F_w. When a place v of F_0 splits into w and $\overline{w}$ in F, we assume that $\pi_w = \pi_{\overline{w}}$. For a p-adic place v of F_0, let $\pi_v := \pi_w$ if w is the unique place of F above v, and let $\pi_v := (\pi_w, \pi_{\overline{w}}) \in F_v = F_w \times F_{\overline{w}}$ if v splits into w and $\overline{w}$ in F.

Set $(*,*)_0\colon \Lambda_0 \times \Lambda_0 \to \mathcal{O}_F$ to be the hermitian form satisfying $\psi_0 = \mathrm{Tr}_{\mathcal{O}_F/\mathbb{Z}} \sqrt{\Delta}^{-1}(*,*)_0$. Similarly, let $\psi = \mathrm{Tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(*,*)$ for the hermitian form $(*,*)$ on W. We take an $\mathcal{O}_F$-lattice Λ of W such that for a p-adic place v of F_0, we have $\Lambda_v \subseteq \Lambda_v^\vee \subseteq \pi_v^{-1}\Lambda_v$ for the symplectic dual $*^\vee$. Let K_G^p be an open compact subgroup of $G(\mathbb{A}_f^p)$ small enough. Also define
$$\begin{aligned} K_{G,v} &:= \mathrm{Stab}_{G(F_{0,v})}(\Lambda_v),\\ K_{G,p} &:= \prod_{v|p} K_{G,v},\\ K_G &:= K_{G,p}K_G^p,\\ K_{\widetilde{G}} &:= K_{Z^{\mathbb{Q}}}K_G \subset \widetilde{G}(\mathbb{A}_f). \end{aligned}$$
Set $\widetilde{\mathcal{G}} := K_{Z^{\mathbb{Q}},p}K_{G,p}$ of $\widetilde{G}(\mathbb{Q}_p)$. In the case of [3], it can be seen as a connected parahoric group scheme associated with a point in the extended Bruhat–Tits building of $\widetilde{G}(\mathbb{Q}_p)$. Put $L := \mathrm{Hom}_{\mathcal{O}_F}(\Lambda_0, \Lambda)$. Also put $V = L_{\mathbb{Q}}$. This admits a hermitian form satisfying
$$(x,x')(m,m')_0 = (xm, x'm')$$
for $m, m' \in \Lambda_0$ and $x, x' \in L$.

We consider a category $\mathcal{M}^{\mathrm{naive}}_{K_{\widetilde{G}}}$ fibered in groupoids over the category of locally noetherian $\mathcal{O}_{E,(p)}$-schemes. The pseudo-functor associates with a test scheme S the groupoid of $(A_0, i_0, \lambda_0, \epsilon^p, A, i, \lambda, \eta^p)$, where

- $(A_0, i_0, \lambda_0, \epsilon^p) \in \mathcal{M}_0(S)$,
- A is an abelian scheme over S,
- $i\colon \mathcal{O}_F \to \mathrm{End}_S(A)$ satisfies the Kottwitz condition that for $a \in \mathcal{O}_F$, the characteristic polynomial of $i(a)$ acting on $\mathrm{Lie}\, A$ is
$$\prod_{\varphi\in\mathrm{Hom}(F,\overline{\mathbb{Q}})}(T-\varphi(a))^{r_\varphi},$$
- $\lambda \in \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a $\mathbb{Q}$-polarization, and
- η^p trivializes $\mathrm{Hom}_{\widehat{\mathcal{O}_F}^p}(T^p(A_0), T^p(A))$ in terms of $(\widehat{L}^p)_S$, preserving the hermitian forms of their rational variants. Here, the hermitian forms on $\mathrm{Hom}_{\mathbb{A}^p_{F,f}}(V^p(A_0), V^p(A))$ is defined by carrying its two sections x, x' to $\lambda_0^{-1}\circ x'^\vee \circ \lambda \circ x$ seen as a section of $\mathit{End}(V^p(A_0)) \simeq (\mathbb{A}^p_{F,f})_S$.

We require the tuple to satisfy the following two conditions. We have

$$A[p^\infty] = \prod_v A[v^\infty],$$

where v is a p-adic place of F_0 and the quotient $\mathcal{O}_{F_{0,v}}$ of $\mathcal{O}_{F_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ acts on $A[v^\infty]$ through i. The first condition is that for each v, the kernel of $\lambda_v \colon A[v^\infty] \to A^\vee[v^\infty]$ induced by λ should be of rank $\#(\Lambda_v^\vee/\Lambda_v)$ and in $A[\pi_v] = \mathrm{Ker}(i(\pi_v) \colon A \to A)$. The second condition is called the sign condition. This condition is about the failure of the Hasse principle. Morphisms of $\mathcal{M}^{\mathrm{naive}}_{K_{\widetilde{G}}}(S)$ are pairs consisting of a morphism in $\mathcal{M}_0$ and an isomorphism of the other abelian schemes preserving the other data.

The model $\mathcal{M}^{\mathrm{naive}}_{K_{\widetilde{G}}}$ is not necessarily flat. Thus, we take its closed subscheme $\mathcal{M}_{K_{\widetilde{G}}}$ with the same generic fiber. The conditions defining it are derived from local models by Rapoport–Zink, Pappas–Rapoport, and so on. We cite [3, §5.4] and [6, Theorem 5.4] for details.

By varying $K^p_{\widetilde{G}} := K^p_{Z^{\mathbb{Q}}} K^p_G$, we obtain a family of integral models $(\mathcal{M}_{K_{\widetilde{G}}})_{K^p_{\widetilde{G}}}$ and morphisms

$$\pi_{K'^p_{\widetilde{G}}, K^p_{\widetilde{G}}} \colon \mathcal{M}_{\widetilde{\mathcal{G}}(\mathbb{Z}_p)K'^p_{\widetilde{G}}} \to \mathcal{M}_{\widetilde{\mathcal{G}}(\mathbb{Z}_p)K^p_{\widetilde{G}}},$$

where $K'^p_{\widetilde{G}} \subseteq K^p_{\widetilde{G}}$.

2.2. **Integral models by Kisin and Pappas.** This subsection is based on [4, §8]. More generally, let (G, X) be a Hodge type Shimura datum with reflex field E such that G splits over a tamely ramified extension of $\mathbb{Q}_p$ and that $p \nmid \pi_1(G_{\mathrm{der}}(\overline{\mathbb{Q}_p}))$. Let $K_p = \mathcal{G}(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p)$ be such that $\mathcal{G}$ is a connected parahoric group scheme associated with a point in the extended Bruhat–Tits building of $G(\mathbb{Q}_p)$. Let $K^p \subseteq G(\mathbb{A}^p_f)$ be a sufficiently small open compact subgroup. Also put $K := K_p K^p$. Let v be the p-adic place of E that the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ induces. Then for some nondegenerate symplectic space V over $\mathbb{Q}$, the associated Siegel double space $S^\pm$ and some $\mathbb{Z}_p$-lattice $\Lambda \subseteq \Lambda^\vee \subset V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, there exists a Hodge embedding $\iota \colon (G, X) \to (\mathrm{GSp}(V), S^\pm)$ and a group scheme homomorphism $\iota \colon \mathcal{G} \to \mathrm{GL}(\Lambda)$ extending $G_{\mathbb{Q}_p} \to \mathrm{GSp}(V)_{\mathbb{Q}_p} \to \mathrm{GL}(V)_{\mathbb{Q}_p}$ such that the latter ι has the scalars $\mathbb{G}_m$ in its image and comes with the corresponding equivariant closed immersion of Pappas–Zhu local models

$$\mathrm{M}^{\mathrm{loc}} \to \mathrm{Gr}\left(\frac{\dim V}{2}, \Lambda\right)_{\mathcal{O}_{E_v}}.$$

Pet $K^\flat_p := \mathrm{GSp}(V_{\mathbb{Q}_p}) \cap \mathrm{GL}(\Lambda)$. There is an open compact subgroup $K^{\flat,p} \subset \mathrm{GSp}(V \otimes_{\mathbb{Q}} \mathbb{A}^p_f)$ such that for $K^\flat := K^\flat_p K^{\flat,p}$, the morphism $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K^\flat}(\mathrm{GSp}(V), S^\pm) \otimes_{\mathbb{Q}} E$ is a closed immersion. Under a choice of a laitice in V, the Shimura variety $\mathrm{Sh}_{K^\flat}(\mathrm{GSp}(V), S^\pm)$ is the generic fiber of a $\mathbb{Z}_{(p)}$-scheme $\mathscr{S}_{K^\flat}$ given by a moduli of polarized abelian schemes. Let $\mathscr{S}_K$ be the normalization of the closure of $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K^\flat}(\mathrm{GSp}(V), S^\pm) \otimes_{\mathbb{Q}} E \to \mathscr{S}_{K^\flat} \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(v)}$.

We can specialize the construction in this subsection to the case $(G, X) = (\widetilde{G}, \{h_{\widetilde{G}}\})$. The author showed the following result in [3].

Theorem 2.1. *The system in §2.1 is canonical with respect to* $(\widetilde{\mathcal{G}}, \mathrm{M}^{\mathrm{loc}})$, *in terms of* [4]*. *Especially, the following holds.*

1. *The morphism* $\pi_{K'^p_{\widetilde{G}}, K^p_{\widetilde{G}}}$ *is finite etale.*
2. *For a discrete valuation ring* R *of mixed characteristic* $(0, p)$, *the map*

$$\varprojlim_{K^p_{\widetilde{G}}} \mathcal{M}_{K_{\widetilde{G}}}(R) \to \varprojlim_{K^p_{\widetilde{G}}} \mathcal{M}_{K_{\widetilde{G}}}\left(R\left[\frac{1}{p}\right]\right) \tag{2.2}$$

is bijective.

3. *For each* $K^p_{\widetilde{G}}$, *there exists on the* p*-adic formal completion of* $\mathcal{M}_{K_{\widetilde{G}}}$ *a locally universal* $(\widetilde{\mathcal{G}}, \mathrm{M}^{\mathrm{loc}})$*-display associated with the pro-etale system on* $\mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}, h_{\widetilde{G}})$ *obtained from the covering*

$$\mathrm{Sh}_{K'_{\widetilde{G},p} K^p_{\widetilde{G}}}(\widetilde{G}, h_{\widetilde{G}}) \to \mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}, h_{\widetilde{G}}),$$

*Our $K^p_{\widetilde{G}}$ is more restrictive than the open compact subgroups considered in [4] because the former is of the form $K^p_{Z^{\mathbb{Q}}} K^p_G$.

where $K'_{\widetilde{G},p}$ runs through open compact subgroups of $\widetilde{\mathcal{G}}(\mathbb{Z}_p)$. These $(\widetilde{\mathcal{G}}, \mathrm{M}^{\mathrm{loc}})$-displays are compatible with respect to the base change by $\pi_{K'^p_{\widetilde{G}}, K^p_{\widetilde{G}}}$.

We cite [4] for unexplained terminology. As a corollary, we have $\mathcal{M}_{K_{\widetilde{G}}} \simeq \mathscr{S}_{K_{\widetilde{G}}}$ as the same characterization is known for $\mathscr{S}_{K_{\widetilde{G}}}$ instead of $\mathcal{M}_{K_{\widetilde{G}}}$.

2.3. **Rough sketch of the proof.** The first and second points of Theorem 2.1 are classical. For the second point, a main tool is Néron–Ogg–Shafarevich criterion. This is useful because for each element of the target of (2.2), its level structures give trivializations of the prime-to-p Tate modules of the abelian schemes involved.

The third point is something not seen in the classical unramified case. This condition is about the comparison of the completion of the strict henselizations of local rings of $\mathcal{M}_{K_{\widetilde{G}}}$ and $\mathrm{M}^{\mathrm{loc}}$. In our case, we can take ι to be the forgetful map $\widetilde{G} \to \mathrm{GSp}(W_0 \oplus W)$ on groups, where the alternating form on $W_0 \oplus W$ is $\psi_0 + \psi$. The corresponding map of Shimura varieties is the generic fiber of $f\colon \mathcal{M}_{K_{\widetilde{G}}} \to \mathscr{S}_{K^\flat}$ carrying a scheme-valued point $(A_0, \cdots, A, \cdots)$ to $(A_0 \times A, \cdots)$. This f also induces a morphism $g\colon \mathcal{M}_{K_{\widetilde{G}}} \to \mathscr{S}_{K_{\widetilde{G}}}$.

Proposition 2.3. *If $x \in \mathcal{M}_{K_{\widetilde{G}}}(\overline{\mathbb{F}_p})$, then the homomorphism*

$$\widehat{\mathcal{O}_{\mathscr{S}_{K^\flat}, f(x)}} \to \widehat{\mathcal{O}_{\mathcal{M}_{K_{\widetilde{G}}}, x}}$$

of the completion of the strict henselizations is surjective.

This proposition is proved by the deformation theory of p-divisible groups.

By this proposition and by counting dimensions, combined with known properties of $\mathcal{M}_{K_{\widetilde{G}}}$, the following induced by g is an isomorphism:

$$\widehat{\mathcal{O}_{\mathscr{S}_{K_{\widetilde{G}}}, g(x)}} \to \widehat{\mathcal{O}_{\mathcal{M}_{K_{\widetilde{G}}}, x}}.$$

This and the known canonicity of $\mathscr{S}_{K_{\widetilde{G}}}$ add up to show the required property.

3. The second result: the curve case of a variant of the arithmetic Gan–Gross–Prasad conjecture

Here we discuss a variant of the arithmetic Gan–Gross–Prasad conjecture on RSZ Shimura curves.

First, we fix notations. Choose one $\varphi_0 \in \Phi$. Assume that $n = 2$. We also assume that $r_\varphi = 0$ if $\varphi_0 \neq \varphi \in \Phi$ and that $r_{\varphi_0} = 1$. Take $u \in W$ such that (u, u) is negative when we embed F to $\mathbb{C}$ in any way. Let $W^\flat$ be the orthogonal complement of Fu.

We can construct H and $\widetilde{H}$ with respect to $W^\flat$ similarly to G and $\widetilde{G}$. Also, put $\widetilde{HG} := \widetilde{H} \times_{Z^{\mathbb{Q}}} \widetilde{G}$. In this section, we investigate the compactified Shimura curve $\overline{\mathrm{Sh}}(\widetilde{HG}, \{h_{\widetilde{HG}}\})$ over E.

Let $\mathscr{A}(\widetilde{HG})$ be the set of irreducible admissible representations of $\widetilde{HG}(\mathbb{A}_{\mathbb{Q},f})$ appearing in

$$H^1(\overline{\mathrm{Sh}}(\widetilde{HG}, \{h_{\widetilde{HG}}\})(\mathbb{C}), \mathbb{C}) := \varinjlim_{\widetilde{K'}} H^1(\overline{\mathrm{Sh}}_{\widetilde{K'}}(\widetilde{HG}, \{h_{\widetilde{HG}}\})(\mathbb{C}), \mathbb{C}).$$

Let π be an irreducible tempered automorphic representation of $\widetilde{HG}(\mathbb{A}_{\mathbb{Q}})$ whose restriction to $Z^{\mathbb{Q}}(\mathbb{A}_{\mathbb{Q}})$ is trivial and $\pi_f \in \mathscr{A}(\widetilde{HG})$. We can think of π as an automorphic representation $\pi_1 \boxtimes \pi_2$ of $H(\mathbb{A}_{F_0}) \times G(\mathbb{A}_{F_0})$ as well.

Let $z_{\widetilde{K'},0}$ be the cycle defined by $\mathrm{Sh}_{\widetilde{K'} \cap \widetilde{H}(\mathbb{A}_{\mathbb{Q},f})}(\widetilde{H}, \{h_{\widetilde{H}}\}) \to \overline{\mathrm{Sh}}_{\widetilde{K'}}(\widetilde{HG}, \{h_{\widetilde{HG}}\})$ modified in a certain way so that its Betti cohomology class is trivial. For details we refer to [2, §2.3]. We see $z_{\widetilde{K'},0}$ as an element of $J(E) \otimes_{\mathbb{Z}} \mathbb{C}$. Let $\mathcal{Z}_{\widetilde{K'},0}$ be the Hecke submodule of $\mathrm{Ch}^1(\overline{\mathrm{Sh}}_{\widetilde{K'}}(\widetilde{HG}, \{h_{\widetilde{HG}}\})) \otimes_{\mathbb{Z}} \mathbb{C}$ generated by $z_{\widetilde{K'},0}$. Below is a theorem from [2] and is a part of [5, Conjecture 6.12].

Theorem 3.1. *Assume that $\pi_{2,\infty}$ is cohomological*†*. The following are equivalent for a sufficiently compact open subgroup $\widetilde{K'} \subset \widetilde{HG}(\mathbb{A}_{\mathbb{Q},f})$.*

- *Néron–Tate height pairing $\langle *, z_{\widetilde{K'},0} \rangle_{\mathrm{NT}}$ is nontrivial on $\mathcal{Z}_{\widetilde{K'},0}[\pi_f^{\widetilde{K'}}]$.*

†This is in terms of the first "$(\mathfrak{g}, K)$"-cohomology.

- *The order of zero of* $L(s, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2))$ *at* $s = 1/2$ *is* 1 *and* $\mathrm{Hom}_{\widetilde{H}(\mathbb{A}_{\mathbb{Q},f})}(\pi_f, \mathbb{C})$ *has dimension* 1 *with a generator nonvanishing on* $\pi_f^{\widetilde{K}'}$.

This is proved by modifying as follows a theorem of Xue in [7, Appendix A], which refines the original arithmetic Gan–Gross–Prasad conjecture. Take decomposable vectors $\phi = \otimes\phi_v \in \pi_f$ and $\phi' = \otimes\phi'_v \in \widetilde{\pi}_f$, the latter representation being the contragradient representation. Choose a locally constant function $\widetilde{t}_{\phi,\phi'} : \widetilde{HG}(\mathbb{A}_{\mathbb{Q},f}) \to \mathbb{C}$ with compact support that induces $\phi \otimes \phi'$ on π_f and 0 on other representations in $\mathscr{A}(\widetilde{HG})$. Here, for the definition and normalization of measures and volumes, we cite [2]. The function $\widetilde{t}_{\phi,\phi'}$ acts on the Jacobian J of $\overline{\mathrm{Sh}}_{\widetilde{K}'}(\widetilde{HG}, \{h_{\widetilde{HG}}\})$ via the Hecke action T.

Theorem 3.2. *When the levels are sufficiently small, we have*

$$\begin{aligned}&\mathrm{vol}\,\mathrm{Sh}_{\widetilde{K}'}(\widetilde{HG})(\mathbb{C})\,\mathrm{vol}'\,\widetilde{K}'\langle T(z_{\widetilde{K}',0}), z_{\widetilde{K}',0}\rangle_{\mathrm{NT}}\\ =&[E : \varphi_0(F)]\frac{\mathrm{vol}\,\widetilde{K}\sharp\,\mathrm{Sh}_{\widetilde{K}}(\widetilde{H})(\mathbb{C})}{2^{\beta-1}}\frac{L'_f(1/2, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2))}{L_f(1, \pi_1, \mathrm{Ad})L_f(1, \pi_2, \mathrm{Ad})}\prod_v \alpha_v^\natural(\phi_v, \phi'_v).\end{aligned}$$

We comment on a few notations. The notation $\mathrm{BC}(\pi_i)$ means the base change of π_i. It is a representation of $\mathrm{GL}_i(\mathbb{A}_F)$. Set β to be the sum of the number of the irreducible cuspidal automorphic representations that appear in the expression of $\mathrm{BC}(\pi_i)$ as isobaric sums for $i = 1, 2$. The L-function $L(s, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2))$ is the Rankin–Selberg convolution. The subscript f of L-functions means that we are only considering their finite part. Finally, $\alpha_v^\natural$ is a local period.

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