Non-admissibility in the cohomology of the unramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

J.Muller
LAGA, Université Sorbonne Paris Nord

This manuscript is a summary of the author's papers [Mul22b] and [Mul22a].

1 Introduction

One commonly accepted strategy in order to build Langlands' conjectural conrrespondences is to look into the cohomology (here ℓ-adic) of some geometric spaces, which are equipped with an action of the reductive group under consideration. In the global setting, one is generally interested in the cohomology of Shimura varieties, whereas in the local setting one may consider the cohomology of Rapoport-Zink spaces, as introduced in [RZ96], or of their generalizations as local Shimura varieties, which were conjectured in [RV14], and later built as moduli of local shtukas (see [SW20]).
In this manuscript, we will stick to the context of Rapoport-Zink spaces. Given a choice of local EL or PEL datum $\mathcal{D}$ as in [RZ96] Definition 3.18, one may define a moduli problem classifying p-divisible groups with additional structures whose reduction mod p are quasi-isogeneous to a fixed p-divisible group $\mathbb{X}$ with additional structures, called the framing object, whose isocristal is determined by the local datum. This problem is represented by a formal scheme $\mathcal{M} := \mathcal{M}_{\mathcal{D}}$ over the formal spectrum of $\mathcal{O}_{\breve{E}}$, where $\breve{E}$ is the completion of the maximal unramified extension of the local reflex field E (a finite extension of $\mathbb{Q}_p$). The local datum determines a p-adic group G over $\mathbb{Q}_p$ and an open compact subgroup $K_0 \subset G(\mathbb{Q}_p)$, defined as the stabilizer of a certain chain lattice. Let $\mathcal{M}^{\mathrm{an}}$ denote the generic fiber of $\mathcal{M}$ as an analytic space over $\breve{E}$. For $K \subset K_0$ an open compact subgroup, one may define a finite étale covering $\mathcal{M}_K \to \mathcal{M}^{\mathrm{an}}$ by parametrizing the trivializations mod K of the Tate module of the universal p-divisible group over $\mathcal{M}$ (see [RZ96] 5.32 for the precise constructions). We have $\mathcal{M}_{K_0} = \mathcal{M}^{\mathrm{an}}$, and these spaces fit together inside a projective system $\mathcal{M}_\infty$ which we refer to as the Rapoport-Zink tower. The group $G(\mathbb{Q}_p)$ acts on $\mathcal{M}_\infty$ by Hecke correspondences. Besides, the group $\mathrm{Aut}(\mathbb{X})$ can be seen as the group

of $\mathbb{Q}_p$-rational points of a certain inner form J of a Levi complement in G. The natural action of $J(\mathbb{Q}_p)$ on $\mathcal{M}$ extends to an action of each individual space $\mathcal{M}_K$, compatible with the transition maps and commuting with the $G(\mathbb{Q}_p)$-action. Taking cohomology, the groups

$$\mathrm{H}_c^\bullet(\mathcal{M}_\infty \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) := \varinjlim_{K \subset K_0} \varinjlim_{U \subset \mathcal{M}_K} \varprojlim_{k} \mathrm{H}_c^\bullet(U \otimes \mathbb{C}_p, \mathbb{Z}/\ell^k\mathbb{Z}) \otimes \overline{\mathbb{Q}_\ell},$$

where U runs over the relatively compact open subspaces of $\mathcal{M}_K$, are naturally representations of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_E$, where W_E is the absolute Weil group of E. The W_E-action is induced by the (usually non effective) Weil descent datum on $\mathcal{M}$ as defined in [RZ96] 3.48. This cohomology is expected to realize a local Langlands correspondence for G, and as such it is a natural object to study.
The Kottwitz' conjecture (see [Rap95]) describes the $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$-supercuspidal part in the alternate sum of the cohomology of $\mathcal{M}_\infty$, in terms of the local Langlands correspondence. This conjecture has been proved in a variety of cases. The case of the Lubin-Tate tower is done in [Boy99] and [HT01], the Drinfeld case has been dealt with in [Hau05], but can also be deduced from the Lubin-Tate case by duality (see [Fal02] and [FGL08]). Both the Lubin-Tate and the Drinfeld cases correspond to Rapoport-Zink spaces of EL type, and the Kottwitz' conjecture was proved for all such basic unramified EL Rapoport-Zink spaces in [Far04] and [Shi12]. Eventually, the case of the basic unramified unitary Rapoport-Zink space of signature $(r, n-r)$ with n odd was also done in [Ngu19] and [BMN21]. A generalization of Kottwitz' conjecture to moduli spaces of local shtuka is also proved in [HKW22].
Outside of the supercuspidal part, very little seems to be known. To our knowledge, the entire cohomology of $\mathcal{M}_\infty$ is only known in the Lubin-Tate case by [Boy09]. One deduces the Drinfeld case by duality, but in this case the cohomology at Iwahori level was already described in [SS91]. A derived description of the cohomology of the Drinfeld tower is also obtained in [Dat06].

2 Bruhat-Tits stratifications on Rapoport-Zink spaces

In general, understanding the non-supercuspidal part of the cohomology of any Rapoport-Zink space seems to be a target which is out of reach. There exists however a certain small family of Rapoport-Zink spaces whose special fiber exhibits some very nice geometric properties. Such spaces are said to be "fully Hodge-Newton decomposable" and they have been fully classified in [GHN19] using a group theoretic approach. The special fiber of a fully Hodge-Newton decomposable Rapoport-Zink space admits a stratification by Deligne-Lusztig varieties, and the incidence relations of the stratification is closely related to the combinatorics of the Bruhat-Tits building of an underlying p-adic group. Consequently, this stratification is known as the Bruhat-Tits stratification. The Rapoport-Zink space is said to be "of Coxeter type" if it is fully Hodge-Newton decomposable, and if the Deligne-Lusztig varieties occuring in the Bruhat-Tits stratification are of Coxeter type. This subfamily of Rapoport-Zink space has also been entirely classified in [GHN22].

To our knowledge, the first time that Deligne-Lusztig varieties were explicitely mentioned in the context of the Langlands program was in [Yos10], dealing with the Lubin-Tate tower. However, it is the pioneering work of [Vol10] and [VW11] which coined the notion of Bruhat-Tits stratification. The authors used an approach based on Dieudonné theory and the combinatorics of vertex lattices in a hermitian space. The corresponding space was the $\mathrm{GU}(1, n-1)$ PEL Rapoport-Zink space at inert p and hyperspecial level. This paved the way to the study of the geometry of the special fiber on a case-by-case basis by many authors, using either a similar Dieudonné theoretic approach or a group theoretic approach.

Deligne-Lusztig varieties naturally arise in Deligne-Lusztig theory, a field of mathematics whose aim is the classification of all irreducible complex representations of finite groups of Lie type, ie. reductive groups over finite fields. Let $\mathbf{G}$ be a connected reductive group over an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. Let q be a power of p and assume that $\mathbf{G}$ has an $\mathbb{F}_q$-structure, induced by a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$. Let $G := \mathbf{G}(\mathbb{F}_q) \simeq \mathbf{G}^F$ be the associated finite group of Lie type. A Levi complement $L \subset G$ is the group of $\mathbb{F}_q$-points of some rational Levi complement $\mathbf{L}$ of $\mathbf{G}$. Such a Levi complement L is said to be split if $\mathbf{L}$ is the Levi complement of a rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. One way of building irreducible representations of G is to decompose representations parabolically induced from proper split Levi complements L of G. However, this process fails to recover the cuspidal representations. To remedy this issue, Deligne and Lusztig defined in their innovative work [DL76] new induction functors from any (not necessarily split) Levi L of G, generalizing the usual parabolic induction. They did so by associating a certain variety $Y_{\mathbf{L}\subset\mathbf{P}}$ to any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with *rational* Levi complement $\mathbf{L}$, which is naturally equipped with commuting actions of G and of $L = \mathbf{L}^F$. The alternate sum of the cohomology of $Y_{\mathbf{L}\subset\mathbf{P}}$ provides a virtual G−bimodule−L, which is used to define the Deligne-Lusztig induction functor R_L^G between the categories of representations of L and of G. Reducing to the case where $L = T$ is a maximal torus in G and computing explicitely the decompositions of the induced representations $\mathrm{R}_T^G\theta$ for all characters θ of T, Lusztig managed in [Lus84] to give a complete classification of all irreducible representations of all simple finite groups of Lie type.

To sum up, the geometry of certain Rapoport-Zink spaces can be described in terms of Deligne-Lusztig varieties, and cohomology plays a crucial role in both the Langlands program and Deligne-Lusztig theory. This observation is the starting point of our work. We brought our attention to the case that has been chronologically first considered, that is the unramified unitary PEL Rapoport-Zink space of signature $(1, n-1)$ as in [Vol10] and [VW11].

3 Non-admissibility in the cohomology of the unramified $\mathrm{GU}(1, n-1)$ Rapoport-Zink spaces

Given a smooth admissible (complex) representation ρ of $J(\mathbb{Q}_p)$ (meaning that $\dim(\rho^K) < \infty$ for all open compact subgroups $K \subset J(\mathbb{Q}_p)$), it is well-known that the $G(\mathbb{Q}_p)$-modules

$$\mathrm{Ext}^a_J(\mathrm{H}^b_c(\mathcal{M}_\infty \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}), \rho)$$

are admissible. It is proved in [Far04], and it is a simple consequence of the fact that the cohomology groups $\mathrm{H}^b_c(\mathcal{M}_K \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ are finitely generated $J(\mathbb{Q}_p)$-modules. We note that this admissibility statement has been largely generalized in [FS21], in so that the authors prove a derived version with integral coefficients regarding local Shimura varieties.
However, it is also known that the cohomology groups $\mathrm{H}^b_c(\mathcal{M}_K \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})_\chi$ are not necessarily of finite length as $J(\mathbb{Q}_p)$-modules. Here, for a continuous character χ of the center $\mathrm{Z}(J(\mathbb{Q}_p))$ and V a smooth representation of $J(\mathbb{Q}_p)$, we denote by V_χ the largest quotient of V on which the center acts through χ. Thus, the cohomology is somewhere in between finitely generated and finite length. What about admissibility? In the Lubin-Tate and Drinfeld cases, the groups $\mathrm{H}^b_c(\mathcal{M}_K \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})_\chi$ are admissible as $J(\mathbb{Q}_p)$-modules. However, it turns out that it is not the case in general, our study led to a counter-example.
From now on, let $\mathcal{M}$ denote the basic unramified unitary PEL Rapoport-Zink space of signature $(1, n-1)$. The prime p is odd, the local reflex field $E = \mathbb{Q}_{p^2}$ is the quadratic unramified extension of $\mathbb{Q}_p$, the group $G = \mathrm{GU}_n$ is the quasi-split group of unitary similitudes of a certain n-dimensional $\mathbb{Q}_{p^2}/\mathbb{Q}_p$-hermitian space, and $K_0 \subset G(\mathbb{Q}_p)$ is a maximal hyperspecial parahoric subgroup, corresponding to the stabilizer of a self-dual lattice. The group J is isomorphic to G when n is odd, and it is the non quasi-split inner form of G when n is even. The framing object $\mathbb{X}$ is the supersingular unitary p-divisible group of signature $(1, n-1)$ as defined in [VW11]. We observe that in this case, the Rapoport-Zink space can actually be defined over $\mathbb{Z}_{p^2}$ instead of the ring of integers of $\breve{\mathbb{Q}}_p$. However, this rational structure does not stem from the Weil descent datum defined in [RZ96], which is not effective. They are however related up to the action of an element in the center of $J(\mathbb{Q}_p)$. We define an integer $m \geqslant 0$ by the equality

$$n = \begin{cases} 2m+1 & \text{if } n \text{ is odd,} \\ 2(m+1) & \text{if } n \text{ is even.} \end{cases}$$

Theorem. *Let χ be an unramified character of $\mathrm{Z}(J(\mathbb{Q}_p)) \simeq \mathbb{Q}_{p^2}^\times$. If $n \geqslant 3$ then the cohomology group $\mathrm{H}^{2(n-1-m)}_c(\mathcal{M}^{\mathrm{an}} \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})_\chi$ is not $J(\mathbb{Q}_p)$-admissible. If $n \geqslant 5$, the same holds for $\mathrm{H}^{2(n-1-m)+1}_c(\mathcal{M}^{\mathrm{an}} \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})_\chi$.*

We emphasize that the statement is about the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}} = \mathcal{M}_{K_0}$, that is the maximal level of the Rapoport-Zink tower. In regard to the cohomology of $\mathcal{M}_\infty$,

it amounts to taking the $G(\mathbb{Q}_p)$-invariants.
The key ingredient in the proof is the Bruhat-Tits stratification on the reduced special fiber $\mathcal{M}_{\mathrm{red}}$ of the Rapoport-Zink space, and the cohomology of the closed Bruhat-Tits strata as generalized Deligne-Lusztig varieties.

4 Cohomology of the closed Bruhat-Tits strata

The stratification can be written as

$$\mathcal{M}_{\mathrm{red}} = \bigsqcup_{\Lambda \in \mathrm{BT}} \mathcal{M}_\Lambda^\circ,$$

where each stratum $\mathcal{M}_\Lambda^\circ$ is a locally closed subvariety over $\mathbb{F}_{p^2}$, and Λ runs over the set of vertices $\mathrm{BT} = \mathrm{BT}(J, \mathbb{Q}_p)$ of the Bruhat-Tits building of J. For all $\Lambda \in \mathrm{BT}$ let $\mathcal{M}_\Lambda$ denote the closure of the stratum $\mathcal{M}_\Lambda^\circ$, which is a smooth projective variety of dimension $\theta \geqslant 0$. The J-action on the special fiber $\mathcal{M}_{\mathrm{red}}$ is compatible with the Bruhat-Tits stratification, in the sense that any $g \in J$ induces an isomorphism $g : \mathcal{M}_\Lambda^\circ \xrightarrow{\sim} \mathcal{M}_{g(\Lambda)}^\circ$, and thus an isomorphism between the closed strata $\mathcal{M}_\Lambda \xrightarrow{\sim} \mathcal{M}_{g(\Lambda)}$ as well. Let $J_\Lambda := \mathrm{Fix}_J(\Lambda)$ be the fixator of Λ, that is a maximal parahoric subgroup of $J(\mathbb{Q}_p)$. If J_Λ^+ denotes the pro-p radical, the quotient $\mathcal{J}_\Lambda := J_\Lambda / J_\Lambda^+$ is a finite group of Lie type, isomorphic to $\mathrm{G}(\mathrm{U}_{2\theta+1}(\mathbb{F}_p) \times \mathrm{U}_{n-2\theta-1}(\mathbb{F}_p))$. We have $0 \leqslant \theta \leqslant m$, so that the integer $t(\Lambda) := 2\theta + 1$, called the type of Λ, is not greater than n. It turns out that the induced action of J_Λ on $\mathcal{M}_\Lambda^\circ$ and $\mathcal{M}_\Lambda$ factors through an action of $\mathcal{J}_\Lambda$, which is trivial on second unitary component. Thus, each Bruhat-Tits stratum comes equipped with an action of the finite group of Lie type $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$. With respect to this action, $\mathcal{M}_\Lambda^\circ$ is isomorphic to the Coxeter variety for $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$, and the closure $\mathcal{M}_\Lambda$ is isomorphic to a generalized parabolic Deligne-Lusztig variety. Let us explain what we mean by this.
In general, let $\mathbf{G}$ be a connected reductive group over $\overline{\mathbb{F}_p}$ equipped with a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$ inducing an $\mathbb{F}_q$-structure. Let $G := \mathbf{G}^F$ be the associated finite group of Lie type. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$. The associated generalized Deligne-Lusztig variety is

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

It is defined over $\mathbb{F}_{q^\delta}$ where $\delta \geqslant 1$ is the smallest integer such that $F^\delta(\mathbf{P}) = \mathbf{P}$, and it is equipped with an action $G \curvearrowright X_{\mathbf{P}}$ by left translations. We say that a generalized Deligne-Lusztig variety $X_{\mathbf{P}}$ is classical if in addition, there exists a rational Levi complement $\mathbf{L} \subset \mathbf{P}$. When this condition is satisfied, the Deligne-Lusztig variety inherits an action $X_{\mathbf{P}} \curvearrowleft L := \mathbf{L}^F$ by right translations, which commutes with the action of G. In this case, the cohomology of $X_{\mathbf{P}}$ is a G-bimodule-L, and can be used to defined the Deligne-Lusztig induction functor between the categories of representations of L and of G. We note that

the varieties denoted in the introduction by $Y_{\mathbf{L}\subset\mathbf{P}}$ are in fact some L-torsor of $X_{\mathbf{P}}$. Thus, in the context of Deligne-Lusztig theory which focuses on the study of the induction functors afforded by the varieties $Y_{\mathbf{L}\subset\mathbf{P}}$, one is only interested in classical Deligne-Lusztig varieties. For this reason, to our knowledge their generalized versions have not been systematically studied in the literature, except in [BR06] where a criterion for the irreducibility of $X_{\mathbf{P}}$ is proved.
Since $\mathcal{M}_\Lambda$ is a generalized (non classical) Deligne-Luztig variety, computing its cohomology may be difficult. Fortunately, [VW11] provides us with enough geometric understanding so that the computation is doable. If $t(\Lambda) = 2\theta + 1$, the Ekedahl-Oort stratification on the closed Bruhat-Tits stratum is

$$\mathcal{M}_\Lambda = \bigsqcup_{0\leqslant\theta'\leqslant\theta} \mathcal{M}_\Lambda(\theta'),$$

where each locally closed subvariety $\mathcal{M}_\Lambda(\theta')$ is isomorphic to a parabolic induction of the Coxeter variety for $\mathrm{GU}_{2\theta'+1}(\mathbb{F}_p)$. In [Lus76], Lusztig has computed the cohomology of the Coxeter varieties for all finite classical groups in terms of unipotent representations. The unipotent representations of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ are classified in [LS77] by the integer partitions λ of $2\theta+1$ and we denote them ρ_Λ. From the Howlett-Lehrer comparison theorem proved in [HL83] and the Pieri rule for Coxeter groups of type B (see [GM20] and [GP00]), one derives the combinatorical rules to compute parabolic induction of unipotent representations. It allows us to entirely determine the cohomology of the Ekedahl-Oort strata $\mathcal{M}_\Lambda(\theta')$, which we can plug into the spectral sequence

$$E_1^{a,b} = \mathrm{H}_c^{a+b}(\mathcal{M}_\Lambda(a) \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}).$$

The distribution of the weights of the Frobenius implies that the sequence degenerates on the second page, and one may compute all the E_2 terms leading to the following statement.

Theorem. *Let* $\Lambda \in \mathrm{BT}$ *with* $t(\Lambda) = 2\theta + 1$.

(1) *The cohomology group* $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ *is zero unless* $0 \leqslant j \leqslant 2\theta$.
(2) *The Frobenius* τ *acts like multiplication by the scalar* $(-p)^j$ *on* $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$.
(3) *For* $0 \leqslant j \leqslant \theta$ *we have*

$$\mathrm{H}_c^{2j}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-j)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant j \leqslant \theta - 1$ *we have*

$$\mathrm{H}_c^{2j+1}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-1-j)} \rho_{(2\theta-2s,2s+1)}.$$

In particular, all irreducible representations in the cohomology groups of even index belong to the unipotent principal series, whereas all the ones in the groups of odd index have cuspidal support determined by the unique cuspidal unipotent representation ρ_{Δ_2} of $\mathrm{GU}_3(\mathbb{F}_p)$, where $\Delta_2 = (2,1)$ is a partition of 3. The cohomology group $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ contains no cuspidal representation of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ unless $\theta = j = 0$ or $\theta = j = 1$. If $\theta = 0$ then H_c^0 is the trivial representation of $\mathrm{GU}_1(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times$, and if $\theta = 1$ then H_c^1 is the representation ρ_{Δ_2} of $\mathrm{GU}_3(\mathbb{F}_p)$.

5 The Čech spectral sequence on the generic fiber

Let red : $\mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$ denote the reduction map. It is anticontinuous, so that the analytical tubes $U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda)$ are open subspaces of $\mathcal{M}^{\mathrm{an}}$. Let $\mathrm{BT}^{\max}$ denote the subset of all vertex lattices $\Lambda \in \mathrm{BT}$ having maximal orbit type $t(\Lambda) = t_{\max} = 2m+1$. Then $\{U_\Lambda\}_{\Lambda \in \mathrm{BT}^{\max}}$ forms an open cover of the generic fiber $\mathcal{M}^{\mathrm{an}}$ to which one can associate the following $J(\mathbb{Q}_p) \times W$-equivariant Čech spectral sequence, concentrated in degrees $a \leqslant 0$ and $0 \leqslant b \leqslant 2(n-1)$,

$$E_1^{a,b} : \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U(\gamma) \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}} \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}).$$

Here, for $a \leqslant 0$ the index set I_{-a+1} consists of all tuples $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1})$ such that the $\Lambda^i \in \mathrm{BT}^{\max}$ satisfy $U(\gamma) := \bigcap_{j=1}^{-a+1} U_{\Lambda^j} \neq \varnothing$. By the properties of the Bruhat-Tits stratification, there exists a unique vertex $\Lambda(\gamma) \in \mathrm{BT}$ such that $U(\gamma) = U_{\Lambda(\gamma)}$. Each cohomology group $\mathrm{H}_c^b(U(\gamma) \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ is naturally a representation of $(J_\Lambda \times I)\tau^{\mathbb{Z}}$ where $I \subset W$ is the inertia subgroup, and $\tau := (p^{-1} \cdot \mathrm{id}, \mathrm{Frob}) \in J \times W$ is called the rational Frobenius element. Here, $\mathrm{Frob} \in W$ is a lift of the geometric Frobenius, and $p^{-1} \cdot \mathrm{id}$ is seen as an element of the center $\mathrm{Z}(J(\mathbb{Q}_p)) \simeq \mathbb{Q}_{p^2}^\times$. The cohomology of the analytical tubes U_Λ and of its special fiber $\mathcal{M}_\Lambda$ are related by the following statement.

Proposition. *Let $\Lambda \in \mathrm{BT}$ and let $0 \leqslant b \leqslant 2(n-1)$. Write $t(\Lambda) = 2\theta + 1$. There is a natural $(J_\Lambda \times I)\tau^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}_c^b(U_\Lambda \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})(n-1-\theta).$$

On the right-hand side the inertia I acts trivially and the rational Frobenius τ corresponds to the geometric Frobenius.

The key point is that $K_0 \subset G(\mathbb{Q}_p)$ in the local PEL datum is hyperspecial, so that the local model associated to the Rapoport-Zink space is smooth. Thus, nearby cycles are trivial and we may identify the cohomology (without support) of U_Λ and of $\mathcal{M}_\Lambda$. Poincaré duality then induces a shift in degrees and the Tate twist.
It follows that τ acts like multiplication by the scalar $(-p)^b$ on any term $E_1^{a,b}$. Thus, the spectral sequence degenerates on the second page and the filtration on the abutment splits, ie. the k-th cohomology group of $\mathcal{M}^{\mathrm{an}}$ is the direct sum of the $E_2^{a,b}$ terms on the diagonal

$a + b = k$. The Frobenius τ then acts in a semi-simple manner on the abutment. Besides, by considering the alternate version of the Čech spectral sequence, it is apparent that two specific terms in the first page are not connected to any non-zero differential, so that they remain untouched in the abutment. More precisely, we have $E_\infty^{0,2(n-1-m)} \simeq E_1^{0,2(n-1-m)}$, and if $n \geqslant 3$ we also have $E_\infty^{0,2(n-1-m)+1} \simeq E_1^{0,2(n-1-m)+1}$. The reason is that the terms in the column $a = 0$ correspond to the cohomology of a single closed Bruhat-Tits stratum of dimension m, however the terms in the other columns for $a < 0$ correspond to the cohomology of the intersection of $|a|$ different strata. Such an intersection has dimension $< m$, so that the range of non-zero cohomology groups drops by 2 from to $a = 0$ column to the $a < 0$ columns.

In order to study the $J(\mathbb{Q}_p)$-action, one may rewrite all the terms $E_1^{a,b}$ in terms of compact inductions. Let us fix a vertex $\Lambda_m \in \mathrm{BT}^{\max}$ and write $J_m = J_{\Lambda_m}$. The two terms identified above can be rewritten as

$$\begin{aligned} E_1^{0,2(n-1-m)} &\simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \mathrm{H}_c^{2(n-1-m)}(U_{\Lambda_m} \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \mathrm{H}_c^{0}(\mathcal{M}_{\Lambda_m} \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}), \\ E_1^{0,2(n-1-m)+1} &\simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \mathrm{H}_c^{2(n-1-m)+1}(U_{\Lambda_m} \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \mathrm{H}_c^{1}(\mathcal{M}_{\Lambda_m} \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}). \end{aligned}$$

Recall that $\mathrm{H}_c^0(\mathcal{M}_{\Lambda_m} \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) \simeq \rho_{(2m+1)} = \mathbf{1}$ is the trivial representation of $\mathrm{GU}_{2m+1}(\mathbb{F}_p)$, and $\mathrm{H}_c^1(\mathcal{M}_{\Lambda_m} \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) \simeq \rho_{(2m,1)}$. The non-admissibility of the cohomology groups of $\mathcal{M}^{\mathrm{an}}$ of degrees $2(n-1-m)$ and $2(n-1-m)+1$ is now a consequence of the following general theorem.

Theorem ([Bus90])**.** *Let G be a p-adic group and let $L \subset G$ be an open subgroup containing $\mathrm{Z}(G)$ and compact modulo $\mathrm{Z}(G)$. Let (σ, V) be a smooth irreducible representation of L. There is a canonical decomposition*

$$\mathrm{c}-\mathrm{Ind}_L^G \sigma \simeq V_0 \oplus V_\infty,$$

where V_0 is the (finite) sum of all supercuspidal representations of $\mathrm{c}-\mathrm{Ind}_L^G \sigma$, and V_∞ contains no non-zero admissible subrepresentation.

In particular, V_∞ does not contain any irreducible subrepresentation, however it may still have many irreducible quotients and subquotients.

If χ is an unramified character of $\mathrm{Z} := \mathrm{Z}(J(\mathbb{Q}_p))$ and ρ is any unipotent representation of $\mathcal{J}_m$, inflated to J_m, Frobenius reciprocity gives a natural isomorphism

$$(\mathrm{c}-\mathrm{Ind}_{J_m}^{J} \rho)_\chi \simeq \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}J_m}^{J} \chi \otimes \rho \simeq V_{0,\rho,\chi} \oplus V_{\infty,\rho,\chi},$$

where $V_{0,\rho,\chi}$ and $V_{\infty,\rho,\chi}$ are given by the previous theorem. What remains is to prove that if $\rho = \rho_{(2m+1)}$ and $m \geqslant 1$, or if $\rho_{(2m,1)}$ and $m \geqslant 2$, then $V_{0,\rho,\chi} = 0$. Namely, it is enough to justify that the compactly induced representation $\mathrm{c}-\mathrm{Ind}_{J_m}^{J} \rho$ does not have any supercuspidal subquotient. This is true as soon as ρ is not a cuspidal representation of $\mathcal{J}_m$, as follows from type theory and the notion of level-0 types developped in [Mor99].

6 The cohomology of the supersingular locus of the associated Shimura variety at an inert prime for $n = 3, 4$

The Rapoport-Zink space $\mathcal{M}$ is related to the supersingular locus of a certain PEL Shimura variety via the p-adic uniformization theorem, and a spectral sequence relates the cohomology of both spaces. In particular, for small values of n, our computations so far allow us to describe the cohomology of the supersingular locus at an inert place. Let $\mathbb{E}$ be an imaginary quadratic field, and let $\mathbb{V}$ be an n-dimensional non-degenerate $\mathbb{E}/\mathbb{Q}$-hermitian space of signature $(1, n-1)$ at infinity, and such that $\mathbb{V}\otimes\mathbb{Q}_p$ is isomorphic to the hermitian space defining the group of unitary similitudes G. In particular $\mathbb{E}_p \simeq \mathbb{Q}_{p^2}$, so that p is inert in $\mathbb{E}$. Let $\mathbb{G}$ be the group of unitary similitudes of $\mathbb{V}$, seen as a reductive group over $\mathbb{Q}$. Then $\mathbb{G}_{\mathbb{Q}_p} = G$ and $\mathbb{G}_{\mathbb{R}} = \mathrm{GU}(1, n-1)$. Assume that there exists a self-dual $\mathcal{O}_{\mathbb{E}}$-lattice Γ in $\mathbb{V}$, and let $\mathrm{Stab}(\Gamma)$ denote the compact subgroup of $\mathbb{G}(\mathbb{A}_f)$ of elements g such that $g(\Gamma \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) = \Gamma \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Here $\mathbb{A}_f$ denotes the ring of finite adèles. For any open compact subgroup $K^p \subset \mathrm{Stab}(\Gamma) \cap \mathbb{G}(\mathbb{A}_f^p)$ which is small enough, there is an integral model S_{K^p} of the associated PEL Shimura variety which is defined over $\mathcal{O}_E$. Since we have hyperspecial level structure at p, the integral model S_{K^p} is smooth and quasi-projective. Let $\overline{S}_{K^p}$ denote the special fiber of S_{K^p}, and let $\overline{S}^{\mathrm{ss}}_{K^p}$ denote the supersingular locus. Let I be the inner form of $\mathbb{G}$ such that $I(\mathbb{Q}_p) = J$, $I_{\mathbb{A}_f^p} = \mathbb{G}_{\mathbb{A}_f^p}$ and $I_{\mathbb{R}} = \mathrm{GU}(0, n)$. The p-adic uniformization theorem of [RZ96] gives natural isomorphisms of analytic spaces

$$I(\mathbb{Q})\backslash(\mathcal{M}^{\mathrm{an}} \times \mathbb{G}(\mathbb{A}_f^p)/K^p) \xrightarrow{\sim} \widehat{S}^{\mathrm{ss,an}}_{K^p},$$

which are compatible as the level K^p varies. Here $\widehat{S}^{\mathrm{ss,an}}_{K^p}$ denotes the analytical tube of the supersingular locus inside the analytification of the generic fiber of S_{K^p}. Associated to this geometric identity, a spectral sequence computing the cohomology of $\widehat{S}^{\mathrm{ss,an}}_{K^p}$ is built in [Far04]. Since S_{K^p} is smooth, it amounts to the cohomology of the supersingular locus $\overline{S}^{\mathrm{ss}}_{K^p}$ itself. The $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant spectral sequence takes the following shape

$$F_2^{a,b} = \bigoplus_{\Pi\in\mathcal{A}_\xi(I)} \mathrm{Ext}^a_J\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}} \otimes \mathbb{C}_p, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \implies \mathrm{H}_c^{a+b}(\overline{S}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \mathcal{L}_\xi),$$

where ξ is a finite dimensional irreducible algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbb{G}$ of weight $w(\xi) \in \mathbb{Z}$, $\mathcal{L}_\xi$ is the associated local system on the Shimura variety, $\mathcal{A}_\xi(I)$ is the space of all automorphic representations of $I(\mathbb{A})$ of type $\check{\xi}$ at infinity, and $\mathrm{H}_c^\bullet(\overline{S}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \mathcal{L}_\xi) := \varinjlim_{K^p} \mathrm{H}_c^\bullet(\overline{S}^{\mathrm{ss}}_{K^p} \otimes \overline{\mathbb{F}_p}, \mathcal{L}_\xi)$.

By [Far04] Lemme 4.4.12, we have $F_2^{a,b} = 0$ as soon as a is strictly bigger than the semisimple rank of J, which is equal to m. In particular, if $m \leqslant 1$ then all the differentials are zero and the spectral sequence is already degenerated, allowing us to compute the abutment

entirely. Since the case $m = 0$ is kind of trivial, we now assume $m = 1$ (ie. $n = 3$ or 4). In particular, the supersingular locus $\overline{S}_{K^p}^{\mathrm{ss}}$ has dimension $m = 1$. Let $X^{\mathrm{un}}(J(\mathbb{Q}_p))$ denote the set of unramified characters of $J(\mathbb{Q}_p)$. Let St_J denote the Steinberg representation of $J(\mathbb{Q}_p)$. If $x \in \overline{\mathbb{Q}_\ell}^\times$, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of the Weil group W where the inertia acts trivially and Frob acts like multiplication by the scalar x.
Let $\tau_1 := \mathrm{c} - \mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J} \widetilde{\rho_{\Delta_2}}$ where $\mathrm{N}_J(J_1)$ is the normalizer of J_1, and $\widetilde{\rho_{\Delta_2}}$ is an extension to $\mathrm{N}_J(J_1)$ of the cuspidal representation ρ_{Δ_2} of J_1. Then τ_1 is an irreducible supercuspidal representation of $J(\mathbb{Q}_p)$. If $\Pi \in \mathcal{A}_\xi(I)$, we define $\delta_{\Pi_p} := \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id})p^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times$ where ω_{Π_p} is the central character of Π_p, and $p^{-1} \cdot \mathrm{id}$ lies in the center of J. For any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ we have $|\iota(\delta_{\Pi_p})| = 1$.

Theorem. *There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathrm{BT}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

$$\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathrm{BT}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}],$$

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathrm{BT}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

In order to compute the Ext groups involved in the spectral sequence, two different techniques have been used. The first is actually an indirect argument, exploiting the weights of the Frobenius to show that some terms can not be nonzero. The second tool is a generalization of a duality theorem of Schneider and Stuhler, which is proved in [NP20].

7 Some perspectives

The strategy which we explained in this manuscript essentially consists of three steps:

1. compute the cohomology of the closed Bruhat-Tits strata via Deligne-Lusztig theory,
2. introduce the Čech spectral sequence on the generic fiber, and relate its terms to Step 1. via the nearby cycles,
3. plug the results of Step 2. into the Hochschild-Serre spectral sequence, and find out what it tells about the supersingular locus of the Shimura variety.

In principle, this strategy could be applied to any Rapoport-Zink space that is fully Hodge-Newton decomposable, however some additional technical difficulties may show up.
First, the success of Step 1. depends on how much is known on the Deligne-Lusztig varieties involved. Provided that they are of Coxeter type, it is reasonable to expect that explicit

computations can be done. However, even in the case of Coxeter type, the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$ are not necessarily smooth, eg. see [RTW14]. In such cases, we can not rely on the purity of the Frobenius to simplify the computations in the spectral sequence induced by the Ekedahl-Oort stratification.

Then, the feasibility of Step 2. entirely relies on whether the nearby cycles can be computed or not. In the case of good reduction such as the unramified $\mathrm{GU}(1, n-1)$, this is not a problem. More generally, as long as we have semistable reduction, it should be possible to carry out explicit computations. Similar statements regarding non-admissibility of the cohomology groups of certain degrees should be expected in more general cases.

Eventually, Step 3. is doable as long as n is small enough so that the semisimple rank of J is not too big. In particular if it is $\leqslant 1$, the Hochschild-Serre spectral sequence already degenerates on the second page. It may be interesting to keep track of the contribution of the supersingular locus (at maximal level) to the supercuspidal part of the cohomology. It amounts to checking when do cuspidal representations occur in the cohomology of the Deligne-Lusztig varieties $\mathcal{M}_\Lambda$.

Another possibility would be to adapt this approach to mod ℓ coefficients. The current hot topics in Deligne-Lusztig theory seem to be widely focused on the modular theory, and many results are known on the modular cohomology of Coxeter varieties at primes ℓ which correspond to "the Coxeter case", cf. [Dud12], [Dud14] and [DR14]. We could thus reach statements regarding the existence of torsion or not in the cohomology of the Rapoport-Zink spaces.

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