SURVEY ON THE GEOMETRIC BOGOMOLOV CONJECTURE

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Introduction

In Diophantine geometry, there is a notion of heights of points of algebraic varieties. The heights estimate a kind of arithmetic complexity of points, and points of "small" height are considered as the "arithmetically simple" points.

Focusing on the points of small height, one can formulate conjectures of "Bogomolov type" for classes of algebraic varieties in various settings. Those conjectures predict that closed subvarieties with many points of small height should be very special kinds of subvarieties.

The geometric Bogomolov conjecture, which is our main topic in this note, is one of such conjectures of Bogomolov type for abelian varieties over function fields. It was formulated in 2013 by the author inspired by Gubler's theorem. Then the author gave several partial answers to the conjecture, and Xie and Yaun gave a final answer by solving the remaining special case.

In this note, we formulate the geometric Bogomolov conjecture and explain its background. Further, we outline how it was solved.

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Notation and conventions.

- Throughout the note, we fix an algebraically closed field k and a smooth projective curve $\mathfrak{B}$ over k. When we say a "function field", this means the function field of $\mathfrak{B}$.
- Let K always denote a number field or a function field. We fix an algebraic closure $\overline{K}$ of K. A finite extension of K will be always taken in $\overline{K}$.
- Assume that K is a function field. For a finite extension K' of K, there exists a Cartesian product
$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & \mathfrak{B}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathfrak{B}, \end{array}$$
where $\mathfrak{B}' \to \mathfrak{B}$ is a finite covering of smooth projective curves over k and $\mathrm{Spec}(K') \to \mathrm{Spec}(K)$ is a morphism induced from the canonical inclusion $K \hookrightarrow K'$. Such a diagram is unique up to canonical isomorphism. We call this diagram, or simply $\mathfrak{B}'$, the *normalization* of $\mathfrak{B}$ in K'.
- Let K be a number field or a function field. For a finite extension K' of K, we define the set of *places* $M_{K'}$ of K' as follows.

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– When K is the function field of $\mathfrak{B}$, we take the normalization $\mathfrak{B}'$ of $\mathfrak{B}$ in K' and set $M_{K'} := \mathfrak{B}'(k)$, the set of closed points of $\mathfrak{B}'$.
– When K is a number field, we set

$$M_{K'} := (\operatorname{Spec}(O_{K'}) \setminus \{(0)\}) \coprod K'(\mathbb{C}),$$

where $O_{K'}$ is the ring of integers of K' and $K'(\mathbb{C})$ is the set of embeddings of K' into $\mathbb{C}$. An element in $\operatorname{Spec}(O_{K'}) \setminus \{(0)\}$ is called a finite place, and an element in $K'(\mathbb{C})$ is called an infinite place.

- For each $v' \in M_{K'}$, we define an absolute value $|\cdot|_{v'}$, which is called the *normalized absolute value* on K' associated to v', as follows.
  – Assume that K is a function field. Then the local ring $\mathcal{O}_{\mathfrak{B}',v'}$ is a discrete valuation ring, and hence we have the order function $\operatorname{ord}_{v'} : K' \to \mathbb{Z} \cup \{+\infty\}$ arising from $\mathcal{O}_{\mathfrak{B}',v'}$. We set $|a|_{v'} := e^{-\operatorname{ord}_{v'}(a)}$ for $a \in K'$.
  – Assume that K is a number field. When v' is a finite place, then the local ring $O_{K',v'}$ is a discrete valuation ring, and we set $|a|_{v'} = |O_{K'}/\mathfrak{p}_{v'}|^{-\operatorname{ord}_{v'}(a)}$, where $\mathfrak{p}_{v'}$ is the prime ideal of $O_{K'}$ corresponding to v' and $\operatorname{ord}_{v'} : K' \to \mathbb{Z} \cup \{+\infty\}$ is the order function. When v' is an infinite place $K' \hookrightarrow \mathbb{C}$, we set $|v'(a)|$, where $|\cdot|$ is the usual absolute value on $\mathbb{C}$.

1. Heights

In this section, we quickly review the notion of heights and the canonical height functions on abelian varieties. See [2, 14] for details on height theory.

1.1. **Heights.** Let K be a function field or a number field. First, we recall the Weil height function on the projective space. It is classically known that there exists a unique function $h^W : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$ that has the following property: for any $p \in \mathbb{P}^n(\overline{K})$ and for any finite extension K'/K with $p \in \mathbb{P}^n(K')$, if we write $p = (a_0 : \cdots : a_n)$ with $a_0, \ldots, a_n \in K'$, then

$$h^W(p) := \frac{1}{[K' : K]} \sum_{v' \in M_{K'}} \log \max\{|a_i|_{v'} \mid i = 0, \ldots, n\}.$$

This function h^W is called the *Weil height function* on $\mathbb{P}^n_{\overline{K}}$. It depends on the choice of the homogeneous coordinates on the projective space.

To define the notion of heights, we fix a notation. For a set S, let $F(S)$ denote the set of real valued functions on S and let $BF(S)$ denote the subset of $F(S)$ consisting of bounded functions. For an $h \in F(S)$, let $[h]$ denote the class in $F(S)/BF(S)$ to which h belongs.

Let X be a projective variety over $\overline{K}$. Then it is classically known that there exists a group homomorphism $\mathfrak{h}_X : \operatorname{Pic}(X) \to F(X(\overline{K}))/BF(X(\overline{K}))$ that has the following property: if L is a very ample line bundle and if $\phi : X \to \mathbb{P}^n_{\overline{K}}$ is a closed embedding with $\phi^*(\mathcal{O}_{\mathbb{P}^n_{\overline{K}}}(1)) \cong L$, then $\mathfrak{h}_X(L) = [h^W_{\mathbb{P}^n}]$. We call $\mathfrak{h}_X(L)$ the *height* associated to L. A function $h_L \in F(X(\overline{K}))$ that represents $\mathfrak{h}_X(L)$ is called a *height function* associated to L.

Remark 1.1. The Weil height function is a nonnegative function. It follows that if L is an ample line bundle on a projective variety X over $\overline{K}$, a height function associated to L is bounded below.

For a set S and for $h_1, h_2 \in F(S)$, we write $h_1 \sim h_2$ if $[h_1] = [h_2]$ in $F(S)/BF(S)$. Let h_L and h_M be height functions associated to line bundles L and M on X, respectively. By the definition of height functions,

$$h_L + h_M \sim h_{L\otimes M}. \tag{1.2}$$

As for the relationship between the morphisms and line bundles, we have the functoriality in the following sense.

Proposition 1.3. *Let $f : Y \to X$ be a morphism of projective varieties over $\overline{K}$ and let L be a line bundle on X. Let h_L and $h_{f^*(L)}$ be height functions on X and Y associated to L and $f^*(L)$, respectively. Then $h_{f^*(L)} \sim f^*(h_L)$.*

From here to the end of this subsection, assume that K is a function field, i.e., the function field of $\mathfrak{B}$. We recall that a height function can be constructed from a model.

Let X be a projective variety over $\overline{K}$. Let K' be a finite extension of K and let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'. A *model (over $\mathfrak{B}'$) of X* means a flat morphism $\mathcal{X} \to \mathfrak{B}'$ equipped with an isomorphism $\mathcal{X} \times_{\mathfrak{B}'} \operatorname{Spec}(\overline{K}) \cong X$. We sometimes simply write $\mathcal{X}$ for a model $\mathcal{X} \to \mathfrak{B}'$. We say that a model $\mathcal{X} \to \mathfrak{B}'$ is *proper* if it is a proper morphism. Let L be a line bundle on X. A *model* $(\mathcal{X}, \mathcal{L})$ over $\mathfrak{B}'$ of (X, L) means a pair of a model $\mathcal{X} \to \mathfrak{B}'$ of X and a line bundle $\mathcal{L}$ on $\mathcal{X}$ equipped with an isomorphism $\mathcal{L}|_X \cong L$.

Suppose that we are given a proper model $(\pi : \mathcal{X} \to \mathfrak{B}', \mathcal{L})$ of (X, L). We define a function $h_{(\mathcal{X},\mathcal{L})} : X(\overline{K}) \to \mathbb{R}$ as follows. For any $x \in X(\overline{K})$, let Δ_x denote the Zariski closure of the point x in the mordel $\mathcal{X}$; the natural morphism $\Delta_x \to \mathfrak{B}$ is a finite morphism, and let $[\Delta_x : \mathfrak{B}]$ denote the degree of this finite morphism. Then we set

$$h_{(\mathcal{X},\mathcal{L})}(x) := \frac{\deg(c_1(\mathcal{L}) \cdot [\Delta_x])}{[\Delta_x : \mathfrak{B}]}.$$

Proposition 1.4. *Let X and L be as above. Let $(\mathcal{X}, \mathcal{L})$ be a proper model of (X, L). Then $h_{(\mathcal{X},\mathcal{L})}$ is a height function on X associated to L.*

We call $h_{(\mathcal{X},\mathcal{L})}$ the *model height function* induced from $(\mathcal{X}, \mathcal{L})$[1].

1.2. **Canonical height functions on abelian varieties.** In general, height functions are determined only up to bounded function. However, it is classically known that on abelian varieties, one has a canonical choice of a height function associated to each line bundle, and such height functions are called the canonical height functions.

In this section, we will recall the canonical height functions associated to "even" line bundles. Let A be an abelian variety. For an $m \in \mathbb{Z}$, let $[m]$ denote the m-times endomorphism on A. A line bundle L is said to be *even*[2] if $[-1]^*(L) \cong L$. If L is an even line bundle on A, then for any $m \in \mathbb{Z}$, $[m]^*(L) \cong L^{\otimes m^2}$ holds by the theorem of cube ([20]).

Theorem 1.5. *Let A be an abelian variety over $\overline{K}$ and let L be a line bundle on A.*

[1]We can show that, for any X and L as above, there exists a proper model of (X, L). It follows that a height function associated to L can be constructed as a model height function. Also over a number field, we have a notion of models over the ring of integers of a number field. Further, using the arithmetic intersection theory in the sense of Arakelov geometry, we can construct height functions in a similar maner.

[2]One sometimes says that it is *symmetric*.

(1) *Fix an* $m \in \mathbb{Z}_{\geq 2}$. *Then there exists a unique height function* $\widehat{h}_L$ *on* A *associated to* L *such that* $[m]^*(\widehat{h}_L) = m^2 \widehat{h}_L$.
(2) *Let* $\widehat{h}_L$ *be as in (1). Then it is a quadratic form on the* $\mathbb{Z}$*-module* $A(\overline{K})$ *in the sense that there exists a bilinear form* $b : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$ *such that* $\widehat{h}_L(x) = \frac{1}{2}b(x,x)$ *for all* $x \in A(\overline{K})$.

We call $\widehat{h}_L$ in Theorem 1.5 the *canonical height function* associated to L. We saw several equalities and inequalities modulo bounded functions between height functions (cf. Remark 1.1, (1.2), and Proposition 1.3). For the canonical height functions, they are equalities and inequalities as functions.

Proposition 1.6. *Let* A *be an abelian variety over* $\overline{K}$.

(1) *Let* L_1 *and* L_2 *be even line bundles on* A. *Then* $\widehat{h}_{L_1 \otimes L_2} = \widehat{h}_{L_1} + \widehat{h}_{L_2}$.
(2) *If* L *is an ample even line bundle, then* $\widehat{h}_L \geq 0$.
(3) *Let* $f : B \to A$ *be a homomorphism of abelian varieties over* $\overline{K}$. *Then for any even line bundle* L *on* A, $f^*(L)$ *is an even line bundle on* B, *and* $f^*(\widehat{h}_L) = \widehat{h}_{f^*(L)}$.

From here to the end of this subsection, we assume that K is a function field. Let A be an abelian variety over $\overline{K}$ and let L be an even line bundle on A. In general, the canonical height function associated to L is not a model height function. However, as we are going to see, if A is "nowhere degenerate", then it turns out to be a model height function.

Let K' be a finite extension of K and let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'. An *abelian scheme model (over* $\mathfrak{B}'$*) of* A is an abelian scheme $\pi : \mathcal{A} \to \mathfrak{B}'$ of A equipped with an isomorphism $\mathcal{A} \times_{\mathfrak{B}'} \operatorname{Spec}(\overline{K}) \cong A$ of abelian varieties. We say that A is *nowhere degenerate* if it has an abelian scheme model for some finite extension K'/K.

Proposition 1.7. *Let* A *be an abelian variety over* $\overline{K}$ *and let* L *be an even line bundle on* A. *Suppose that* A *is nowhere degenerate. Then the following holds.*

(1) *There exist a finite extension* K'/K *and a proper model* $(\pi : \mathcal{A} \to \mathfrak{B}', \mathcal{L})$, *where* $\mathfrak{B}'$ *is the normalization of* $\mathfrak{B}$ *in* K', *such that the following hold:*
   (i) $\pi : \mathcal{A} \to \mathfrak{B}'$ *is an abelian scheme model of* A;
   (ii) $0_\pi^*(\mathcal{L}) \cong \mathcal{O}_{\mathfrak{B}'}$, *where* 0_π *is the zero section of the abelian scheme* π.
(2) *Let* $(\pi : \mathcal{A} \to \mathfrak{B}', \mathcal{L})$ *be as in (1) above. Then for any* $m \in \mathbb{Z}$, *we have* $[m]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes m^2}$, *where* $[m] : \mathcal{A} \to \mathcal{A}$ *is the* m*-times endomorphism of the abelian scheme.*

Let $(\pi : \mathcal{A} \to \mathfrak{B}', \mathcal{L})$ be a model as in Proposition 1.7 (1). By (2) in this proposition and the projection formula, we see that $[m]^*\left(h_{(\mathcal{A},\mathcal{L})}\right) = m^2 h_{(\mathcal{A},\mathcal{L})}$. Since $h_{(\mathcal{A},\mathcal{L})}$ is a height function associated to L (Proposition 1.4), it follows from Theorem 1.5 that $h_{(\mathcal{A},\mathcal{L})} = \widehat{h}_L$.

1.3. **Height 0 points.** Let K be a function field or a number field. Let A be an abelian variety over $\overline{K}$. We define the notion of height 0 points of $A(\overline{K})$. Let L be an ample even line bundle. For an $a \in A(\overline{K})$, we can show that the property "$\widehat{h}_L(a) = 0$" does not depend on the choice of ample even L. Thus it makes sense to say that a point $a \in A(\overline{K})$ has *heihgt* 0, which means that $\widehat{h}_L(a) = 0$ for some and hence any ample even line bundle L on A.

We set $A(\overline{K})_{h=0} := \{a \in A(\overline{K}) \mid a \text{ has height } 0\}$. Let $A(\overline{K})_{tor}$ denote the set of torsion points of the additive group $A(\overline{K})$. Since $\widehat{h}_L$ for an even line bundle L is a quadratic form, we have $A(\overline{K})_{tor} \subset A(\overline{K})_{h=0}$.

One may ask if any height 0 point is a torsion point or not. When K is a number field, the answer is affirmative.

Theorem 1.8. *Assume that K is a number field. Let A be an abelian variety over $\overline{K}$. Then $A(\overline{K})_{h=0} = A(\overline{K})_{tor}$.*

It is natural to ask about the case of function fields. In fact, it does not hold in general. Assume that K is a function field. We say that an abelian variety A over $\overline{K}$ is *constant* if there exists an abelian variety $\widetilde{A}$ over the constant field k such that $A \cong \widetilde{A} \otimes_k \overline{K}$. Note that, via this isomorphism, we regard $\widetilde{A}(k) \subset A(\overline{K})$. We claim that $\widetilde{A}(k) \subset A(\overline{K})_{h=0}$; thus this implies that $A(\overline{K})_{h=0}$ has non-torsion points in general. Indeed, the canonical projection $\widetilde{A} \times_{\mathrm{Spec}(k)} \mathfrak{B} \to \mathfrak{B}$ is an abelian scheme model of A; we take an ample even line bundle $\widetilde{L}$ on $\widetilde{A}$ and let $\mathcal{L}$ and L be the pullbacks of $\widetilde{L}$ by the first projections $\widetilde{A} \times_{\mathrm{Spec}(k)} \mathfrak{B} \to \widetilde{A}$ and $\widetilde{A} \otimes_k \overline{K} \to \widetilde{A}$, respectively. Then $(\widetilde{A} \times_{\mathrm{Spec}(k)} \mathfrak{B} \to \mathfrak{B}, \mathcal{L})$ is a model of (A, L) that satisfies the conditions in Proposition 1.7 (1). We take any $\widetilde{x} \in \widetilde{A}(k)$. Regarding this point as a point in $A(\overline{K})$, we take its Zariski closure $\Delta_{\widetilde{x}}$ in the model. Then $\Delta_{\widetilde{x}} = \{\widetilde{x}\} \times \mathfrak{B}$, and we see that $\deg(c_1(\mathcal{L}) \cdot \Delta_{\widetilde{x}}) = 0$. By Proposition 1.7 (1), this shows that $\widetilde{x} \in A(\overline{K})_{h=0}$.

We will see later how the height 0 points are characterized over function fields.

2. Background

The geometric Bogomolov conjecture has a long history in its background, which we are going to brief in this section.

2.1. **History.** We start with the Manin–Mumford conjecture. Let F be an algebraically closed field. Let C be a smooth projective curve over F of genus $g \geq 2$. Fix a divisor D on C of degree 1, and let $\jmath : C \hookrightarrow \mathrm{Jac}_C$ be the embedding of C into its Jacobian given by $x \mapsto [x - D]$.

Conjecture 2.1 (Manin–Mumford conjecture)**.** Assume that $\mathrm{char}(F) = 0$. Then $\jmath^{-1}\left(\mathrm{Jac}_C(\overline{K})_{tor}\right)$ is a finite subset of $C(\overline{K})$.

In 1980, Bogomolov "arithmetized" the Manin–Mumford conjecture. With the above notation, assume that $F := \overline{K}$, where K is a function field or a number field (not necessarily $\mathrm{char}(\overline{K}) = 0$). Let $h : \mathrm{Jac}_C(\overline{K}) \to \mathbb{R}_{\geq 0}$ be the Néron–Tate height function, which is the canonical height function associated to a certain theta divisor. For an $\epsilon \in \mathbb{R}$, we set

$$C(\epsilon) := \{x \in C(\overline{K}) \mid h(\jmath(x)) \leq \epsilon\}.$$

Conjecture 2.2 (Bogomolov conjecture for curves, [1])**.** Assume that, when K is a function field, C cannot be defined over k. Then there exists an $\epsilon > 0$ such that $C(\epsilon)$ is finite.

For $\epsilon \geq 0$, we have $\jmath^{-1}\left(\mathrm{Jac}_C(\overline{K})_{tor}\right) \subset C(\epsilon)$. Thus the Bogomolov conjecture implies the Manin–Mumford conjecture when $F = \overline{\mathbb{Q}}$.

In 1983, Raynaud proved that the Manin–Mumford conjecture holds, and moreover, he established a generalization. To describe this, let A be an abelian variety over an algebraically closed field F. A subvariety T of A is said to be *torsion* if it is the translate of an abelian subvariety by a torsion point, i.e., if there exist an abelian subvariety A' of A and a $t \in A(\overline{K})_{tor}$ such that $T = A' + t$. We say that a closed subvariety *has dense torsion points* if it has a dense subset of torsion points. Since an abelian variety over an algebraically closed field has dense torsion points, so does a torsion subvariety. Raynaud proved that the converse also holds if $\mathrm{char}(F) = 0$.

Theorem 2.3 (Raynaud's theorem, [22] when $\dim(X) = 1$ and [23] in general)**.** *In the above setting, assume that* $\mathrm{char}(F) = 0$. *Then for any closed subvariety X of A, if X has dense torsion points, then it is a torsion subvariety.*

Raynaud's theorem implies the Manin–Mumford conjecture. Indeed, under the setting of the conjecture, since $\jmath(C)$ is not a torsion subvariety, it follows from Raynaud's theorem that it does not have dense torsion points. Since $\dim(C) = 1$, this means that $\jmath(C)$ has only a finite number of torsion points.

In 1995, S. Zhang proposed an "arithmetization" of Raynaud's theorem over number fields. To state that, we define the notion of density of small points. Let A be an abelian variety over $\overline{K}$. For an ample even line bundle L on A, a closed subvariety $X \subset A$, and an $\epsilon \in \mathbb{R}$, set $X(\epsilon; L) := \{x \in X(\overline{K}) \mid h_L(x) \leq \epsilon\}$. It is easy to see that the property that $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$ does not depend on the choice of an ample even line bundle L. We say that X *has dense small points* if $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$ and for some (hence any) ample even line bundle L on A.

Since a torsion subvariety has dense torsion points, it has dense small points. Zhang conjectured that, when K is a number field, the converse should also hold.

Conjecture 2.4 (Generalized Bogomolov conjecture, [36])**.** Assume that K is a number field. Let A be an abelian variety over $\overline{K}$. Then for any closed subvariety X of A, if X has dense small points, then it is a torsion subvariety.

Note that Conjecture 2.4 generalizes the Bogomolov conjecture for curves (Conjecture 2.2) over a number field and Raynaud's theorem when $F = \overline{\mathbb{Q}}$.

In 1998, Ullmo proved that the Bogomolov conjecture for curves over number fields holds.

Theorem 2.5 (Ullmo's theorem, [26])**.** *When K is a number field, Conjecture 2.2 holds.*

The key ingredient of the proof is the "equidistribution theorem of small points" by Szpiro–Ullmo–Zhang ([25]), which will be explained in the next subsection.

Inspired by Ullmo's idea, Zhang proved the generalized Bogomolov conjecture by using the equidistribution theorem in the same year.

Theorem 2.6 (Zhang's theorem, [37])**.** *Conjecture 2.4 holds.*

The idea that uses the equidistribution theorem of small points, which was the key in the proofs of Ullmo's theorem and Zhang's theorem, will play a crucial role in the discussion in the sequel. To figure out the idea, we will outline the proof of Zhang's theorem in the following subsections.

2.2. **Canonical measures and the equidistribution of small points.** In this subsection, assume that K is a number field. For a projective variety X over $\overline{K}$ and for any embedding $v : \overline{K} \hookrightarrow \mathbb{C}$ (called an *archimedean place* of $\overline{K}$, here), let X_v denote the basechange of X by v. This is a projective variety over $\mathbb{C}$, and let X_v^{an} denote the associated complex analytic space. For a line bundle L on X, let L_v^{an} denote the induced line bundle on X_v^{an}.

Let A be an abelian variety over $\overline{K}$. A *rigidification* of a line bundle L on A is an isomorphism $L(0) \cong \overline{K}$, where $L(0)$ is the fiber of L over the point $0 \in A(\overline{K})$. A line bundle with a rigidification is called a *rigidified line bundle*. An isomorphism of rigidified line bundles means an isomorphism of line bundles that respects the rigidifications. Note that an isomorphism between rigidified line bundles is unique. If the rigidified line bundle L on A is even, then for any $m \in \mathbb{Z}$, there exists a unique isomorphism $[m]^*(L) \cong L^{\otimes m^2}$ of rigidified line bundles.

Theorem 2.7. *Let L be a rigidified even line bundle on an abelian variety A over $\overline{K}$. Let $v : \overline{K} \hookrightarrow \mathbb{C}$ be an archimedean place. Then there exists a unique metric $\|\cdot\|_{can}$ on L_v^{an} such that for any $m \in \mathbb{Z}$, we have $[m]^*(\|\cdot\|_{can}) = \|\cdot\|_{can}^{m^2}$ via the isomorphism $[m]^*(L_v^{\mathrm{an}}) \cong (L_v^{\mathrm{an}})^{\otimes m^2}$ of rigidified line bundles. This metric is smooth, i.e., C^∞-class. Further, if L is ample, then the curvature form $c_1(L, \|\cdot\|_{can})$ is positive.*

We call the metric $\|\cdot\|_{can}$ the *canonical metric* on L_v^{an}.

Let X be a closed subvariety of A of dimension d. Then $c_1(L_v^{\mathrm{an}}, \|\cdot\|^{can})^{\wedge d}$ is a smooth d-form, which naturally gives a (signed) Borel measure $c_1(L_v^{\mathrm{an}}, \|\cdot\|^{can})^{\wedge d}\big|_{X_v^{\mathrm{an}}}$ on X_v^{an} by restriction. The total volume of this measure equals $\deg_L(X)$. Suppose that L is ample. Then $\deg_L(X) > 0$, and we have a probability measure

$$\mu_{X_v^{\mathrm{an}},L} := \frac{1}{\deg_L(X)}\, c_1(L_v^{\mathrm{an}}, \|\cdot\|^{can})^{\wedge d}\big|_{X_v^{\mathrm{an}}}.$$

We call this measure the *canonical measure* on X_v^{an} associated to L. By Theorem 2.7, this measure is positive.

Next, we explain the equidistribution theorem of small points. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points in $X(\overline{K})$. We say that $(x_i)_{i\in\mathbb{N}}$ is *generic* if for any $i_0 \in \mathbb{N}$ and for any proper closed subset Y of X, there exists an $i_1 \in \mathbb{N}$ such that $i_1 \geq i_0$ and $x_{i_1} \notin Y$. We say that it is *small* if $\lim_{i\to\infty} \widehat{h}(x_i) = 0$, where $\widehat{h}$ is the canonical height function associated to an ample even line bundle; note that this property is well-defined and not depend on the choice of the ample even line bundle.

Since $\overline{K}$ has countable cardinality, the set of closed subsets of X has countable cardinality, and hence we see the following.

Lemma 2.8. *If X has dense small points, then there exists a generic and small sequence on $X(\overline{K})$.*

We take a finite extension K'/K over which the above A and X can be defined. We fix a model A' and X' over K' of A and X, respectively. Then $\mathrm{Gal}(\overline{K}/K')$ acts on $X(\overline{K})$. For an $x \in X(\overline{K})$, let $O_{K'}(x)$ denote the $\mathrm{Gal}(\overline{K}/K')$-orbit of x in $X(\overline{K})$. It is a nonempty finite set.

Theorem 2.9 (Equidistribution theorem, [25, 37])**.** *Let A, X, and v be as above. Let L be an ample even line bundle on A. Let K' be a finite extension of K over which A and X can* *be defined, and we fix models of them over K'. Let $(x_i)_{i\in\mathbb{N}}$ be a generic and small sequence on $X(\overline{K})$. Then on X_v^{an}, we have a weak convergence of measures*

$$\frac{1}{|O_{K'}(x_i)|}\sum_{z\in O_{K'}(x_i)}\delta_z \to \mu_{X_v^{\mathrm{an}},L}$$

as $i\to\infty$, where δ_z denotes the dirac measure with support z and $|O_{K'}(x_i)|$ is the number of elements of $O_{K'}(x_i)$.

2.3. **Idea of the proof of Zhang's theorem.** We outline the proof of Zhang's theorem (Theorem 2.6). We note the following lemma, which is easily deduced from Proposition 1.6.

Lemma 2.10. *Let X be a closed subvariety of an abelian variety A over $\overline{K}$. Let B be an abelian variety.*

1. *Let Y be a closed subvariety of B. If X and Y have dense small points, then so does the closed subvariety $X\times Y$ of $A\times B$.*
2. *Let $\phi: A\to B$ be a homomorphism. If X has dense small points, then so does $\phi(X)$.*

The proof of Theorem 2.6 is given by contradiction. Assume that there exists a counterexample to the generalized Bogomolov conjecture; then there exist an abelian variety A over $\overline{K}$ and a closed subvariety $X\subset A$ such that X is not a torsion subvariety but has dense small points. We call $\{a\in A \mid X+a\subset X\}$ the *stabilizer* of X. Taking the quotient by the stabilizer, we may assume that there exists a counterexample with trivial stabilizer. By Theorem 1.8, we note that $\dim(X)\geq 1$.

For a natural number n, we define the difference homomorphism

$$A^n\to A^{n-1};\quad (a_1,\ldots,a_n)\mapsto(a_1-a_2,\ldots,a_{n-1}-a_n).$$

Let $\alpha: X^n\to A^{n-1}$ be the restriction of this homomorphism to X^n. If n is large enough, it follows from the assumption that X has trivial stabilizer that α is generically injective; in particular, it is generically finite. We take such an n. Let Y denote the image of α and set $Z:=X^n$. Thus α induces a surjective generically finite morphism $Z\to Y$.

Since X has dense small points, so does $Z:=X^n$ by Lemma 2.10 (1). Then by Lemma 2.8, we can take a generic and small sequence $(z_i)_{i\in\mathbb{N}}$ on $Z(\overline{K})$. We set $y_i:=\alpha(z_i)$ for all i. Since the morphism $Z\to Y$ is surjective, the sequence $(y_i)_{i\in\mathbb{N}}$ is generic. Since α is a restriction of a homomorphism of abelian varieties, it follows from Lemma 2.10 (2) that $(y_i)_{i\in\mathbb{N}}$ is also small. Let K' be a finite extension over which A and X can be defined. We note that

$$\alpha_*\left(\frac{1}{|O_{K'}(z_i)|}\sum_{z\in O_{K'}(z_i)}\delta_z\right)=\frac{1}{|O_{K'}(y_i)|}\sum_{y\in O(y_i)}\delta_y,$$

Then by the equidistribution theorem (Theorem 2.9), we get

$$\alpha_*(\mu_{Z_v^{\mathrm{an}},M})=\mu_{Y_v^{\mathrm{an}},L}, \tag{2.11}$$

where L and M are ample even line bundles on A^{n-1} and A^n, respectively.

The right-hand side of equality (2.11) is a smooth measure (in the sense that it comes from a smooth form). On the other hand, since $\mu_{Z_v^{\mathrm{an}},M}$ is a strictly positive smooth measure and the diagonal $\Delta\subset X^n=Z$ is contracted to a point by α, it follows that the right-hand side

$\alpha(\mu_{Z_v^{\mathrm{an}},M})$ should be "too dense" to be a smooth measure at $\alpha(\Delta)$. That is a contradiction, which completes the proof.

3. Geometric Bogomolov conjecture

In this section, we assume that K is a function field. It is natural to ask if the same statement as Zhang's theorem holds also over function fields. In general, this does not hold because there exists an abelian variety A over $\overline{K}$ that has a non-torsion height zero point $a \in A(\overline{K})$; see the argument just below Theorem 1.8. Indeed, $\{a\}$ has dense small points and is not a torsion subvariety. More generally, if A can be defined over the constant field k of the function field, then a closed subvariety that can be defined over k has dense height 0 points.

Thus, there are two possible options over function fields.

Options 3.1. (1) *We restrict ourselves to a certain class of abelian varieties and establish the same assertion as Zhang's;*
(2) *We define a suitable counterpart of torsion subvarieties and formulate a conjecture for all abelian varieties.*

Gubler proved a theorem in the direction of Options 3.1 (1). He considered abelian varieties that are totally degenerate at some place. More details will be explained in the first subsection below. On the other hand, the geometric Bogomolov conjecture was an attempt to establish a theorem in the direction of Options 3.1 (1). The statement will be given in the last subsection of this section.

3.1. Gubler's theorem.

If K' is a finite extension of K and K'' is a finite extension of K', then there is a natural map $M_{K''} \to M_{K'}$. Thus the family $(M_{K'})_{K'}$, where K' runs through the finite extensions (in $\overline{K}$) of K, is an inverse system. We set $M_{\overline{K}} := \varprojlim_{K'} M_{K'}$ and call an element of this set a *place* of $\overline{K}$. Note that the canonical map $M_{\overline{K}} \to M_{K'}$ is surjective for any finite extension K'/K. For each $v = (v_{K'})_{K'} \in M_{\overline{K}}$, we can put a unique absolute value $|\cdot|_v$ on $\overline{K}$ such that: its restriction to K' *is equivalent to* the normalized absolute value on K' corresponding to the place $v_{K'}$; its restriction to K *coincides with* the normalized absolute value on K. Further, the completion of $\overline{K}_v$ of $\overline{K}$ with respect to $|\cdot|_v$ is an algebraically closed valuation field with nonarchimedean absolute value $|\cdot|_v$. Let $\overline{K}_v^{\circ}$ denote the valuation ring of $\overline{K}_v$. For an algebraic variety X over $\overline{K}$ and for a $v \in M_{\overline{K}}$, let X_v denote the basechange of X by the field extension $\overline{K} \to \overline{K}_v$.

Let A be an abelian variety over $\overline{K}$. By the Grothendieck semistable reduction theorem, there exists a semiabelian scheme over $\overline{K}_v^{\circ}$ with generic fiber A, which we call a *semiabelian scheme model* of A_v. We say that A is *totally degenerate* at $v \in M_{\overline{K}}$ if A has a semiabelian scheme model over the valuation ring $\overline{K}_v^{\circ}$ that has torus reduction.

Remark 3.2. Let $v \in M_{\overline{K}}$. The product of two totally degenerate abelian varieties at v is again totally degenerate at v. A quotient abelian variety of an abelian variety that is totally degenerate at v is again totally degenerate at v.

It is classically known that if A is totally degenerate at some place, then $A(\overline{K})_{h=0} = A(\overline{K})_{tor}$. Note that this is the same equality that holds when K is a number field, and in particular, there is no non-torsion closed subvariety of A of dimension 0 that has dense small points. Thus, one may expect that the same statement as Zhang's theorem should hold for an abelian variety that is totally degenerate at some place. Indeed, this expectation was realized by Gubler in 2007.

Theorem 3.3 (Gubler's theorem, [10]). *Assume that K is a function field. Let A be an abelian variety over $\overline{K}$. Assume that A is totally degenerate at some place. Then if a closed subvariety $X \subset A$ has dense small points, then X is a torsion subvariety.*

The basic strategy of the proof of Gubler's theorem is the same as Zhang's in the sense that the key is the equidistribution theorem. The difference is that: in Gubler's proof, the equidistribution theorem is applied on the tropical analytic varieties over the place at which the abelian variety is totally degenerate, while in Zhang's proof, it is done on the complex analytic spaces.

We will later outline the proof of Gubler's theorem, but we notice that this proof will be slightly different from Gubler's original one. The proof that we are going to explain will use canonical measures and the equidistribution theorem on the Berkovich analytic spaces as the counterparts of those on the complex analytic spaces in Zhang's proof. On the other hand, Gubler's original proof uses those on tropical varieties ([9, 10]). When Gubler proved the theorem, the equidistribution theorem was not established on the Berkovich analytic spaces. Then he proved the equidistribution in the tropical geometric setting and applied them to prove his theorem. Now, necessary techniques on Berkovich analytic spaces have been developed enough, and we can argue in terms of Berkovich analytic geometry.

Let us recall the canonical measures and the equidistribution theorem on the Berkovich analytic spaces. For an algebraic variety X over $\overline{K}$ and for a $v \in M_{\overline{K}}$, let X_v^{an} denote the Berkovich analytic space over $\overline{K}_v$ associated to X_v. Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Let L be an ample even line bundle on A. Let $v \in M_{\overline{K}}$. It is known that the canonical measure $\mu_{X_v^{\mathrm{an}},L}$ on X_v^{an} is defined (cf. [4, 12]).

Let $(x_i)_{i \in I}$ be a net on $X(\overline{K})$, where I is a directed set. Then we can define the notions of genericness and smallness of $(x_i)_{i\in I}$ in the same way as we did in Subsection 2.2 when K is a number field. Assume that A and X can be defined over a finite extension K'/K. By fixing models over K' of A and X, respectively, $\mathrm{Aut}_{K'}(\overline{K})$ acts on $X(\overline{K})$. Thus we can consider $\mathrm{Aut}_{K'}(\overline{K})$-orbit of a point in $X(\overline{K})$. For an $x \in X(\overline{K})$, let $O_{K'}(x)$ denote the $\mathrm{Aut}_{K'}(\overline{K})$-orbit of x in $X(\overline{K})$. We regard $X(\overline{K}) \subset X_v^{\mathrm{an}}$ naturally.

Theorem 3.4 ([10, 11]). *Under the setting above, let $(x_i)_i$ be a generic and small net on $X(\overline{K})$. Then, we have a weak convergence of measures on X_v^{an}*

$$\frac{1}{|O_{K'}(x_i)|} \sum_{z \in O_{K'}(x_i)} \delta_z \to \mu_{X_v^{\mathrm{an}},L}$$

as $i \to \infty$.

Remark 3.5. One may think that we have enough ingredients to prove Gubler's theorem in a similar way, as we have the canonical measures and the equidistribution theorem. However, we notice that more properties on the canonical measures should be needed. In fact, it was important that the canonical measures are smooth positive measure on the positive dimensional spaces. If the canonical measures in (2.11) were the dirac measures of points, for example, we could not find any contradiction from this equality.

Therefore, in the proof, some "smoothness" and "positivity" properties on the canonical measures will be required. Gubler proved that the canonical measures on a d-dimensional subvariety are d-dimensional Lebesgue measures, as follows.

Theorem 3.6. *Assume that an abelian variety A is totally degenerate at v. Let X be a closed subvariety of dimension d. Then there exists a subspace $S_{X_v^{\mathrm{an}}}$ of X_v^{an} with the following properties:*

(i) *$S_{X_v^{\mathrm{an}}}$ has a canonical structure of polyhedral complex of pure dimension d;*
(ii) *there exist a finite polyhedral decomposition $\bigcup_\sigma \sigma$ of $S_{X_v^{\mathrm{an}}}$ and a positive real number r_σ for any σ of dimension d such that the canonical measure $\mu_{X_v^{\mathrm{an}}}$ (associated to an ample even line bundle on A) equals*

$$\sum_\sigma r_\sigma \delta_\sigma,$$

where σ runs through the d-dimensional polytopes in the polyhedral decomposition and δ_σ is a Lebesgue measure on the polytope σ.

We outline the proof of Gubler's theorem. Suppose that there exists a counterexample to the theorem. Then one constructs an abelian variety A that is totally degenerate at some place and a closed subvariety X of positive dimension such that X has trivial stabilizer (Remark 3.2). By the same construction as Zhang, we have a generically finite morphism $\alpha : Z \to A^{N-1}$, where $Z := X^N$, and Z has dense small points. Set $Y := \alpha(Z)$. The same argument using the equidistribution of small points, we get

$$\alpha_* \left(\mu_{Z_v^{\mathrm{an}}}\right) = \mu_{Y_v^{\mathrm{an}}}, \tag{3.7}$$

where $\mu_{Z_v^{\mathrm{an}}}$ and $\mu_{Y_v^{\mathrm{an}}}$ are the canonical measures associated to some ample even line bundles on A^N and A^{N-1}, respectively.

Since A^N and A^{N-1} are totally degenerate at v (Remark 3.2), it follows from Theorem 3.6 that there exist closed subsets $S_{Z_v^{\mathrm{an}}} \subset Z_v^{\mathrm{an}}$ and $S_{Y_v^{\mathrm{an}}} \subset Y_v^{\mathrm{an}}$ that have structures of polyhedral complex of pure dimension $d = N \dim(X)$. Further, $\mu_{Z_v^{\mathrm{an}}}$ and $\mu_{Z_v^{\mathrm{an}}}$ are d-dimensional Lebesgue measures on $S_{Z_v^{\mathrm{an}}}$ and $S_{Y_v^{\mathrm{an}}}$, respectively.

Then the same argument as Zhang's leads us to a contradiction. Indeed, since α contracts the diagonal to a point and $\mu_{Z_v^{\mathrm{an}}}$ is a Lebesgue measure around the diagonal, $\alpha_* \left(\mu_{Z_v^{\mathrm{an}}}\right)$ must be too dense to be a d dimensional Lebesgue measure around the image of the diagonal. On the other hand, $\mu_{Y_v^{\mathrm{an}}}$ is a d-dimensional Lebesgue measure. That is a contradiction.

3.2. **Special subvarieties.** Next, we want to think of Options 3.1 (2). Then we have to define a counterpart of the notion of torsion subvarieties. To do that, we will use the "$\overline{K}/k$-trace" of an abelian variety.

In this subsection, let F/k be any field extension with k algebraically closed. We recall the notion of F/k-trace. For an abelian variety A over F, a *model* over k is an abelian variety over k with an isomorphism $A \cong \widetilde{A} \otimes_k F$. An abelian variety A over F is said to be *F/k-constant*, or simply *constant*, if A has a model over k.

Theorem 3.8 (Chow's theorem, [13]). *Let A and B be abelian varieties over F and let $\phi : A \to B$ be a homomorphism. Assume that A and B are F/k-constant and let $\widetilde{A}$ and $\widetilde{B}$ be models over k of A and B, respectively. Then there exists a unique homomorphism $\widetilde{\phi} : \widetilde{A} \to \widetilde{B}$ such that the basechange $\widetilde{\phi} \otimes_k F : A \to B$ coincides with ϕ.*

Let A be an F/k-constant abelian variety. Let $\widetilde{A}_1$ and $\widetilde{A}_2$ be models over k of A. Applying Chow's theorem to $\mathrm{id}_A : A \to A$, we get a unique isomorphism $\widetilde{A}_1 \to \widetilde{A}_2$ whose basechange to F equals id_A. Thus, a model of A is unique up to canonical isomorphism. Further, $\widetilde{A}_1(k)$ and $\widetilde{A}_2(k)$ are naturally regarded as subgroups of $A(\overline{K})$, and they coincide with each other. We write $A(k)$ for this subgroup and call a point of it a *constant point*.

An *F/k-trace* of A is a pair $\left(A^{F/k}, \mathrm{Tr}_A\right)$ of an F/k-constant abelian variety $A^{F/k}$ and a homomorphism $\mathrm{Tr}_A : A^{F/k} \to A$ that has the following universal property: for any F/k-constant abelian variety B and for any homomorphism $\phi : B \to A$, there exists a unique homomorphism $\phi' : B \to A^{F/k}$ such that $\mathrm{Tr}_A \circ (\phi' \otimes_k F) = \phi$. For an abelian variety over F, its F/k-trace is unique up to canonical isomorphism by the universal property. Further, it is classically known that any abelian variety over F has an F/k-trace (cf. [13]).

Let B be an F/k-constant abelian variety. A closed subvariety Y of B is said to be *F/k-constant* if there exist a model $\widetilde{B}$ over k of B and a closed subvariety $\widetilde{Y} \subset \widetilde{B}$ such that $Y = \widetilde{Y} \otimes_k F$. By Chow's theorem, this notion does not depend on the choice of a model $\widetilde{B}$. For an F/k-constant abelian variety B, we set $Y(k) := Y(\overline{K}) \cap B(k)$

Definition 3.9 (F/k-special subvariety, [30] when $F = \overline{K}$). Let A be an abelian variety over F. A closed subvariety $X \subset A$ is said to be *F/k-special* if

$$X = T + \mathrm{Tr}_A(Y)$$

for some torsion subvariety $T \subset A$ and some F/k-constant closed subvariety $Y \subset A^{F/k}$.

Remark 3.10. Let $X = T + \mathrm{Tr}_A(Y)$ be an F/k-special subvariety, where T is a torsion subvariety and $Y \subset A^{F/k}$ is an F/k-constant subvariety. Then if F is algebraically closed, then $T(F) + \mathrm{Tr}_A(Y(k))$ is dense in X.

We should mention that, using the notion of F/k-special subvarieties, we have a positive characteristic version of the Manin–Mumford conjecture.

Theorem 3.11 (Manin–Mumford conjecture in positive characteristic, [21, 24]). *Let F be an algebraically closed field and let k_0 be the algebraic closure in F of the prime field. Assume that* $\mathrm{char}(F) > 0$. *Let A be an abelian variety over F. Then for any closed subvariety $X \subset A$, if X has dense small points, then it is F/k_0-special.*

Let k be an algebraically closed field with $F \supset k \supset k_0$. If an abelian variety A over F has trivial F/k-trace, then it has trivial F/k_0-trace. Thus, we have the following.

Corollary 3.12. *In the setting of Theorem 3.11, assume further that A has trivial F/k-trace. Then for any closed subvariety $X \subset A$, if X has dense small points, then it is a torsion subvariety.*

3.3. **Geometric Bogomolov conjecture.** We use what we explained in the previous subsection with $F = \overline{K}$. Then the set of height 0 points is illustrated as follows.

Theorem 3.13 ([14]). *Let $\left(A^{\overline{K}/k}, \mathrm{Tr}_A\right)$ be the $\overline{K}/k$-trace of an abelian variety A over $\overline{K}$. Then*

$$A(\overline{K})_{h=0} = A(\overline{K}) + \mathrm{Tr}_A(A^{\overline{K}/k}(k)).$$

By Remark 3.10 and Theorem 3.13, we see that a $\overline{K}/k$-special subvariety has dense small points. In 2013, the author formulated the geometric Bogomolov conjecture, which asserts that the converse should also hold.

Conjecture 3.14 (Geometric Bogomolov conjecture, [30]). Let A be an abelian variety over $\overline{K}$. Then for any closed subvariety X of A, if X has dense small points, then it is $\overline{K}/k$-special.

Since the notion of $\overline{K}/k$-special subvarieties and that of torsion subvarieties are the same when the $\overline{K}/k$-trace is trivial, the geometric Bogomolov conjecture should generalize Gubler's theorem. The author proved in [30] that the geometric Bogomolov conjecture holds for abelian varieties that satisfy some milder condition of degeneracy than the total degeneracy, which generalized Gubler's theorem.

4. Reduction theorems

In this section, we explain the reduction theorems of the geometric Bogomolov conjecture by the author. First, we reduce the geometric Bogomolov conjecture in full generality to that for nowhere degenerate abelian varieties. Next, we reduce that for nowhere degenerate abelian varieties to that for nowhere degenerate abelian varieties with trivial $\overline{K}/k$-trace. Those results will play a crucial role in the final solution of the conjecture.

4.1. **Reduction to the nowhere degenerate case.** One might wish to prove the geometric Bogomolov conjecture along the same line as Zhang and Gubler. However, the equidistribution method seems helpless to prove the geometric Bogomolov conjecture for nowhere degenerate abelian varieties. As we mentioned in Remark 3.5, if the support of the canonical measure is a finite number of points, then no contradictions appear from (3.7). In fact, if an abelian variety has good reduction at a place $v \in M_{\overline{K}}$, the support of the canonical measure on any closed subvariety is a finite number of points over v. Therefore, the possible maximal contribution of the equidistribution method to the conjecture should be the reduction of the conjecture in full generality to that for nowhere degenerate abelian varieties.

In 2016, the author established such a reduction theorem, as we are going to explain. Let A be an abelian variety over $\overline{K}$. Then we can show that A has a unique maximal nowhere degenerate abelian subvariety $\mathfrak{m}$.

Theorem 4.1 ([31]). *With the notation above, the geometric Bogomolov conjecture holds for A if and only if that holds for $\mathfrak{m}$.*

Remark 4.2. In general, if two abelian varieties A_1 and A_2 are isogeneous to each other, then the geometric Bogomolov conjecture holds for A_1 if and only that holds for A_2.

Note that A is isogeneous to $A/\mathfrak{m} \times \mathfrak{m}$. By Remark 4.2, we can show that Theorem 4.1 is deduced from the following theorem.

Theorem 4.3 ([31]). *Let A and $\mathfrak{m}$ be as above. Assume that $\mathfrak{m} = 0$. Then the geometric Bogomolov conjecture holds for A.*

If one tries to prove Theorem 4.3 by deducing a contradiction from equality (3.7), then there arises a nontrivial task. In Gubler's setting, the canonical measures are smooth in the sense that they are Lebesgue measures on positive dimensional space, and hence a contradiction comes out from (3.7) automatically. Does the same hold, in general? Here, we recall Gubler's structure theorem of the canonical measures.

Theorem 4.4 ([12]). *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A of dimension d. Let $v \in M_{\overline{K}}$. Then there exists a subset $S_{X_v^{\mathrm{an}}} \subset X_v^{\mathrm{an}}$ that satisfies the following conditions:*

(i) *$S_{X_v^{\mathrm{an}}}$ has a canonical structure of polyhedral set;*
(ii) *there exists a polyhedral decomposition Σ of $S_{X_v^{\mathrm{an}}}$ such that for any ample even line bundle L on A, $\mu_{X_v^{\mathrm{an}},L}$ is the pushout by the canonical injection $S_{X_v^{\mathrm{an}}} \hookrightarrow X_v^{\mathrm{an}}$ of a measure of form*
$$\sum_{\sigma \in \Sigma} r_\sigma \delta_\sigma$$
with $r_\sigma \geq 0$.

We notice that $\sigma \in \Sigma$ with $r_\sigma > 0$ may have various dimensions. If the canonical measure on the right-hand side in (3.7) is not a sum of non-equidimensional Lebesgue measures, we cannot get an immediate contradiction form (3.7).

To overcome such a difficulty, we analyzed the canonical measures in detail. The key was the following result on the canonical measures.

Proposition 4.5 ([31]). *Let $\phi : A \to B$ be a homomorphism of abelian varieties. Let X be a closed subvariety of A. Assume that $\phi|_X$ is a generically finite morphism. Let v be a place of $\overline{K}$. Let $\mu_{X_v^{\mathrm{an}}}$ be the canonical measure on X_v^{an} associated to an ample even line bundle on A. We write $\sum_{\sigma\in\Sigma} r_\sigma \delta_\sigma$, where Σ is a polyhedral decomposition of the canonical subset of X_v^{an}. Then, for any $\sigma \in \Sigma$ with $r_\sigma > 0$, ϕ_v^{an} induces a homeomorphism $\sigma \to \phi_v^{\mathrm{an}}(\sigma)$.*

Assume that the maximal nowhere degenerate abelian subvariety of A is trivial and that $\dim(A) \neq 0$. Then we note that A has trivial $\overline{K}/k$-trace. Further, we can show the following.

(1) There exists a place v at which A is degenerate.
(2) Let X be a closed subvariety of A of positive dimension. Let v be a place as in (1). Then the canonical measure on X_v^{an} associated to an ample even line bundle on A has positive dimensional support.
(3) The canonical measure on $Z_v^{\mathrm{an}} = (X^N)_v^{\mathrm{an}}$ a certain ample even line bundle can be regarded as the product of the N copies of a canonical measure on X_v^{an}.

We outline the proof of Theorem 4.3. To prove this by contradiction, assume that there exists a counterexample to the theorem. Then there exist an abelian variety A and a non-special closed subvariety X of A that has dense small points and has trivial stabilizer. We take a place v as in (1) above. By the same argument as Gubler's, we obtain the same equality as (3.7). By (2) above, the canonical measures in this equality have positive dimensional support. We write

$$\mu_{Z_v^{\mathrm{an}}} = \sum_{\sigma \in \Sigma} r_\sigma \delta_\sigma$$

as in Theorem 4.4. Since the diagonal contracts to a point and we have (3) above, we can show that there exists $\sigma \in \Sigma$ such that $r_\sigma > 0$ and the map $\sigma \to \alpha_v^{\mathrm{an}}(\sigma)$ induced by α_v^{an} is not injective. However, this contradicts Proposition 4.5.

4.2. **Reduction to the nowhere degenerate and with trivial trace case.** Recall that $\left(A^{\overline{K}/k}, \mathrm{Tr}_A\right)$ denotes the $\overline{K}/k$-trace of an abelian variety A over $\overline{K}$. In 2018, the author proved the following reduction theorem.

Theorem 4.6 ([33]). *Assume that A is nowhere degenerate. Then the geometric Bogomolov conjecture holds for A if and only if that holds for* $\mathrm{Coker}(\mathrm{Tr}_A)$.

Let A be any abelian variety over $\overline{K}$ and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety. Then the trace homomorphism $\mathrm{Tr}_A : A^{\overline{K}/k} \to A$ factors through $\mathfrak{m}$, and $\mathfrak{m}/\mathrm{Im}(\mathrm{Tr}_A)$ is nowhere degenerate and has trivial $\overline{K}/k$-trace. It follows that the above theorem together with Theorem 4.1 implies the following corollary.

Corollary 4.7 ([33]). *The geometric Bogomolov conjecture holds for A if and only if that holds for* $\mathfrak{m}/\mathrm{Im}(\mathrm{Tr}_A)$. *In particular, the geometric Bogomolov conjecture is reduced to that for nowhere degenerate abelian varieties with trivial $\overline{K}/k$-trace.*

To outline the proof of Theorem 4.6, we recall some properties on the canonical height of a closed subvariety.

Remark 4.8. Let X be a closed subvariety of A. Then we have a notion of canonical height $\widehat{h}_L(X)$ of X, where L is an ample even line bundle on A. It is known that X has dense small points if and only if X has canonical height 0, i.e., $\widehat{h}_L(X) = 0$ (cf. [11]).

If A is nowhere degenerate, the canonical height of a closed subvariety can be described as an intersection number on a model. More precisely, the following is known to be true.

Proposition 4.9 ([11]). *Let K' be a finite extension of K over which A, X, and L can be defined. Let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'. Let $(\pi : \mathcal{A} \to \mathfrak{B}', \mathcal{L})$ be a model of (A, L) such that $\pi : \mathcal{A} \to \mathfrak{B}'$ is an abelian scheme model of A and $\mathcal{L}$ satisfies condition Proposition 1.7 (1) (ii). Let $\mathcal{X}$ be the Zariski closure of X in $\mathcal{A}$. Then*

$$\widehat{h}_L(X) = \frac{1}{[K' : K]} \deg_{\mathcal{L}}(\mathcal{X}).$$

Let us outline the proof of Theorem 4.6. Let A be a nowhere degenerate abelian variety. Then it is easy to see that A is isogeneous to $\mathrm{Coker}(\mathrm{Tr}_A) \times A^{\overline{K}/k}$. It suffices to prove the theorem for this abelian variety (cf. Remark 4.2). Thus we may and do assume that $A = B \times C$, where B is a nowhere degenerate abelian variety that has trivial $\overline{K}/k$-trace and C is a constant abelian variety with model $\widetilde{C}$ over k. Set $\mathcal{C} := \widetilde{C} \times_{\mathrm{Spec}(k)} \mathfrak{B}'$. Then the second projection $\mathcal{C} \to \mathfrak{B}'$ is an abelian scheme model of C. Let $\widetilde{N}$ be an ample even line bundle on $\widetilde{C}$ and let $\mathcal{N}$ and N be the pullbacks of $\widetilde{N}$ by the first projections $\mathcal{C} \to \widetilde{C}$

and $\widetilde{C} \otimes_k \overline{K} \to \widetilde{C}$, respectively. Then $(\mathcal{C} \to \mathfrak{B}', \mathcal{N})$ is a model of (C, N) that satisfies the conditions in Proposition 1.7 (1). Let $Z \subset C$ be a closed subvariety and let $\mathcal{Z}$ be the Zariski closure of Z in $\mathcal{C}$.

Suppose that Z has dense small points. Then Z has canonical height 0 (Remark 4.8). By Proposition 4.9, $\deg_{\mathcal{N}}(\mathcal{Z}) = 0$. From this, we can prove that there exists a closed subvariety $\widetilde{Z}$ of $\widetilde{C}$ such that $Z = \widetilde{Z} \otimes_k \overline{K}$. This is a special subvariety of C, and thus the geometric Bogomolov conjecture holds for C.

Next, let X be a closed subvariety of $B \times C$ and suppose that X has dense small points. Suppose that $Z \subset C$ is the image of X by the canonical projection $B \times C \to C$. Let $f : X \to Z$ be the morphism given by restricting the second projection $B \times C \to C$. Then, since X has dense small points, so does Z. It follows from what we saw above that $Z = \widetilde{Z} \otimes_k \overline{K}$ for some $\widetilde{Z} \subset \widetilde{C}$.

Let $Z(k)$ denote the image of $\widetilde{Z}(k)$ by the canonical map $\widetilde{Z}(k) \to Z(\overline{K})$. Then, for any $z \in Z(k)$, we can show that any irreducible component Y of $f^{-1}(z)$ has canonical height 0; we use Proposition 4.9 here. Note that Y is a closed subvariety of B. Since the geometric Bogomolov conjecture holds for B by the assumption, Y is a torsion subvariety. Furthermore, we can show that $Y = f^{-1}(z)$ and the family $\{f^{-1}(z)\}_{z \in Z(k)}$ is constant in the sense that there exists a torsion subvariety $T \subset B$ such that $f^{-1}(z) = T \times \{z\} = T$; here we use the assumption that B has trivial $\overline{K}/k$-trace. This proves that $X = T \times Z$, and this is a special subvariety of $B \times C$.

4.3. **Bogomolov conjecture for curves over function fields.** As a benefit of the results in the previous subsections, the author proved the following theorem in 2017.

Theorem 4.10 ([32]). *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Assume that* $\operatorname{codim}(X, A) = 1$. *Then if X has dense small points, then it is special.*

The proof of Theorem 4.10 is based on Corollary 4.7. We may assume that A is nowhere degenerate and has trivial $\overline{K}/k$-trace. Let X be a closed subvariety of A of codimension 1. Then we can prove that X has positive canonical height by using Proposition 4.9. When we compute the intersection number on a model, the assumption on the codimension is crucial.

It is interesting that from Theorem 4.10, which is for *codimension* 1 subvarieties, we can deduce that the conjecture holds for the closed subvariety of *dimension* 1.

Theorem 4.11 ([32]). *Assume that* $\dim(X) = 1$. *Then if X has dense small points, then X is special.*

As an immediate consequence, we got the final answer to the Bogomolov conjecture for curves over function fields without any restriction.

Corollary 4.12 ([32]). *The Bogomolov conjecture for curves over function fields holds.*

The idea to deduce Theorem 4.11 from Theorem 4.10 is as follows. Let X be a 1-dimensional closed subvariety of A. We set $Y_0 := \{0\}$, and for each positive integer m, let Y_m be the sum of m copies of $X - X := \{x - x' \mid x, x' \in X\}$. Then Y_m is an abelian subvariety of A for large m, and let N be the smallest positive integer among such m. Suppose that X has dense small points. Then we can show that there exists a $\tau \in A(\overline{K})_{tor}$ such that Y_{N-1} or $Y_{N-1} + (X - \tau)$ is a closed subvariety of the abelian variety Y_N of codimension 1, for which we write D. Note that D also has dense small points. By considering these D and Y_N, we can deduce Theorem 4.11 from Theorem 4.10.

Using the above technique, the author proved a partial result for 5-dimensional nowhere degenerate abelian varieties with trivial $\overline{K}/k$-trace ([34]).

5. Final solution

As we saw in Corollary 4.7, it now suffices to show the geometric Bogomolov conjecture only for nowhere degenerate abelian varieties with trivial $\overline{K}/k$-trace. The conjecture for this kind of abelian varieties was solved by Xie and Yuan, as we describe in the first subsection. In the last subsection, we will give some comments on the related topics.

5.1. **Xie–Yuan's theorem.** Assume that K be a function field. In 2022, Xie and Yuan proved the following theorem.

Theorem 5.1 ([27]). *Let A be an abelian variety over $\overline{K}$. Assume that A is nowhere degenerate and has $\overline{K}/k$-trivial trace. Then for a closed subvariety $X \subset A$, if X has dense small points, then it is a torsion subvariety.*

By Theorem 5.1 together with Corollary 4.7, the geometric Bogomolov conjecture was completely solved.

We outline the proof of Theorem 5.1. Let A be as in the theorem and let X be a closed subvariety of A. We define a sequence $(X_m)_{m\in\mathbb{Z}_{\geq 0}}$ of closed subvarieties inductively, as follows: set $X_0 = \{0\} \subset A$; for $m \in \mathbb{Z}_{\geq 1}$, let $f_{m-1} : X_{m-1} \times X \to A$ be the morphism induced from the addition and let X_m be the image of f_{m-1}.

Lemma 5.2 ([27]). *Suppose that X has dense small points. Then there exists an $r \in \mathbb{Z}_{\geq 1}$ such that* $\dim(X_{r-1}) < \dim(X_r)$ *and X_r is a torsion subvariety.*

Let r be as in Lemma 5.2. There exist an $x_0 \in A(\overline{K})_{tor}$ and an abelian subvariety A' such that $X_r = A' + rx_0$.

Proposition 5.3 ([27]). *With the above notation, assume that X_r is an abelian subvariety, i.e., $rx_0 = 0$ and $X_r = A'$ with the above notation. We write f for f_{r-1}. Set* $e := \dim(X_{r-1}) + \dim(X) - \dim(A')$ *and*

$$A'_{e+1} := \{y \in A' \mid \dim(f^{-1}(y)) \geq e+1\}.$$

Let $t \in A'(\overline{K})_{tor} \setminus A'_{e+1}(\overline{K})$ and assume that the order of t is not divisible by $\mathrm{char}(k)$. *Then any irreducible component of $f^{-1}(t)$ has canonical height* 0.

We outline the proof. Replacing X by $X - x_0$, we may assume that $X_r = A'$ is an abelian subvariety of A. Since the morphism $f : X_{r-1} \times X \to A'$ induced from the addition is surjective, $A'_{e+1} \subsetneq A'$, and hence

$$T := \{t \in A'(\overline{K})_{tor} \setminus A'_{e+1}(\overline{K}) \mid \mathrm{char}(k) \nmid \mathrm{ord}(t)\}$$

is dense in A'.

We argue by induction on $\dim(X)$. We note that since $\dim(X_{r-1}) < \dim(A')$, $e < \dim(X)$. Since the case of $\dim(X) = 0$ is trivial by Theorem 3.13, assume that $\dim(X) \geq 1$. We take any $t \in T$. By Proposition 5.3 with Remark 4.8, any irreducible component $Z \subset f^{-1}(t)$ has dense small points, and since $\dim(Z) \leq e < \dim(X)$, it follows from the induction hypothesis that Z is a torsion subvariety. Thus, $f^{-1}(t)$ has dense torsion points. Since T is dense in A', this shows that $X_{r-1} \times X$ has dense torsion points. By the Manin–Mumford conjecture in positive characteristic (Corollary 3.12), it follows that $X_{r-1} \times X$ is a torsion subvariety. Taking the second projection $X_{r-1} \times X \to X$ concludes that X is a torsion subvariety. This completes the proof of Theorem 5.1.

We remark that the proof of Theorem 5.1 uses the Manin–Mumford conjecture in positive characteristic, so that this conjecture has not yet been a corollary of the geometric Bogomolov conjecture. It should be interesting to find a proof of Theorem 5.1 without using the Manin–Mumford.

5.2. **Comments.** We give some comments on topics related with the geometric Bogomolov conjecture.

5.2.1. *Transcendence degree of the function fields.* In this article, we only consider the function fields of transcendence degree 1 over k. However, the geometric Bogomolov conjecture is formulated over function fields of any transcendence degree. In fact, over function fields of any transcendence degree, Gubler proved his theorem, the geometric Bogomolov conjecture formulated, and every theorem on the geometric Bogomolov conjecture in this note holds.

5.2.2. *Bogomolov conjecture for curves over function fields (of transcendence degree* 1*).* Several partial results on the Bogomolov conjecture for curves over function fields were proved before Corollary 4.12. In 1993, Zhang made a theory of admissible pairing on curves in [35]. Using this theory, Moriwaki and Yamaki established several partial answers to the Bogomolov conjecture for curves over function fields ([15, 16, 17, 18, 28, 29]); some of them need the assumption that $\mathrm{char}(k) = 0$. Later, Zhang developed the theory of height paring of the Gross–Schoen cycles in [38]. Using this theory, Faber proved a partial result in 2009 ([6]). Further, Cinkir proved in 2011 that when $\mathrm{char}(k) = 0$, the Bogomolov conjecture for curves over function fields holds ([5]). Cinkir's proof uses the positivity of the height of the Gross–Schoen cycles, and to see this positivity, one needs a kind of Hodge index theorem that is established only when $\mathrm{char}(k) = 0$. That is the main reason why the assumption of $\mathrm{char}(k) = 0$ is needed. We remark that the those results on the Bogomolov conjecture for curves over a function field deals only with the case where K has transcendence degree 1 over k, while the author proved Corollary 4.12 in any characteristic and without any assumption on the transcendence degree (cf. Subsubsection 5.2.1).

5.2.3. *Moriwaki's arithmetic height.* In 2000, Moriwaki defined in [19] a notion of height over finitely generated fields. More precisely, let K be a finitely generated field. Then Moriwaki defined a notion of polarizations on K, and after a choice of a polarization, he constructed a height theory for projective varieties over K. If the polarization is "big", then the height theory has the same property as the height theory over number fields. Moriwaki proved that for an abelian variety over K, the same assertion as Zhang's theorem holds.

Recall that Zhang's theorem can recover Raynaud's theorem when $F = \overline{\mathbb{Q}}$. We note that Moriwaki's theorem recovers Raynaud's theorem (Theorem 2.3) for an arbitrary F (not necessarily $F = \overline{\mathbb{Q}}$).

5.2.4. *Geometric Bogomolov conjecture in characteristic zero.* The geometric Bogomolov conjecture under the assumption that $\mathrm{char}(k) = 0$ was proved slightly earlier. In 2019, Gao–Habegger [8] proved the conjecture under the assumption that the function field has transcendence degree one. Then, in 2021, Cantat–Gao–Habegger–Xie [3] proved it without the assumption on the transcendence degree. Their proofs heavily depend on real analytic methods, such as Betti-maps.

References

[1] F. A. Bogomolov, Points of Finite Order on Abelian Varieties, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no.4, 782–804, 973.
[2] E. Bombieri and W. Gubler, Heights in Diophantine Geometry, New Mathematical Monographs, Series No. 4, Cambridge (2006).
[3] S. Cantat, Z. Gao, P. Habegger, and J. Xie, The geometric Bogomolov conjecture, Duke Math. J. 170 (2021), no.2, 247–277.
[4] A. Chambert-Loir, Mesure et équidistribution sur les espaces de Berkovich, J. Reine Angew. Math. 595 (2006), 215–235.
[5] Z. Cinkir, Zhang's conjecture and the effective Bogomolov conjecture over function fields, Invent. Math. 183, No.3 (2011), 517–562.
[6] X.W.C. Faber, The geometric Bogomolov conjecture for curves of small genus, Experiment. Math. 18 (2009), no. 3, 347–367.
[7] X.W.C. Faber, Equidistribution of dynamically small subvarieties over the function field of a curve, Acta Arith. 137 (2009), no. 4, 345–389.
[8] Z. Gao and P. Habegger, Heights in families of abelian varieties and the Geometric Bogomolov Conjecture, Annals of Mathematics, Vol. 189, No. 2 (2019), 527–604.
[9] W. Gubler, Tropical varieties for non-archimedean analytic spaces, Invent. Math. 169 (2007), 321–376.
[10] W. Gubler, The Bogomolov conjecture for totally degenerate abelian varieties, Invent. Math. 169 (2007), 377–400.
[11] W. Gubler, Equidistribution over function fields, manuscripta math. 127 (2008), 485–510.
[12] W. Gubler, Non-archimedean canonical measures on abelian varieties, Compos. Math. 146 (2010), 683–730.
[13] S. Lang, Abelian varieties, Springer-Verlag (1983).
[14] S. Lang, Fundamentals of Diophantine Geometry, Springer-Verlag (1983).
[15] A. Moriwaki, Bogomolov conjecture for curves of genus 2 over function fields, J. Math. Kyoto Univ. 36 (1996), 687–695.
[16] A. Moriwaki, Bogomolov conjecture over function fields for stable curves with only irreducible fibers, Compositio Math. 105 (1997), 125–140.
[17] A. Moriwaki, A sharp slope inequality for general stable fibrations of curves, J. Reine Angew. Math. 480 (1996), 177–195.
[18] A. Moriwaki, Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves, J. of AMS, 11 (1998), 569–600.
[19] A. Moriwaki, Arithmetic height functions over finitely generated fields, Invent. Math. 140 (2000), 101–142.
[20] D. Mumford, Abelian varieties, Tata institute of fundamental research, Bombay (1970).
[21] R. Pink and D. Roessler, On ψ-invariant subvarieties of semiabelian varieties and the Manin–Mumford conjecture, J. Algebraic Geom. 13 (2004), 771–798.
[22] M. Raynaud, Courbes sur une variété abélienne et points de torsion, Invent. math. 71 (1983), 207–233.
[23] M. Raynaud, Sous-variétés dúne variété abélienne et points de torsion, in *Arithmetic and Geometry*, vol.I, Progress in Mathematics, vol. 35 (Birkhäuser, Boston, MA, 1983), 327–352.
[24] T. Scanlon, A positive characteristic Manin–Mumford theorem, Compos. Math. 141 (2005), 1351–1364.
[25] L. Szpiro, E. Ullmo, and S. Zhang, Equirépartition des petits points, Invent. Math. 127 (1997), 337–347.

[26] E. Ullmo, Positivité et discrétion des points algébriques des courbes, Ann. of Math. 147 (1998), 167–179.
[27] Junyi Xie, and Xinyi Yuan, Geometric Bogomolov conjecture in arbitrary characteristics, Invent. Math. 229 (2022), no. 2, 607–637.
[28] K. Yamaki, Geometric Bogomolov's conjecture for curves of genus 3 over function fields, J. Math. Kyoto Univ. 42 (2002), 57–81.
[29] K. Yamaki, Effective calculation of the geometric height and the Bogomolov conjecture for hyperelliptic curves over function fields, J. Math. Kyoto. Univ. 48 (2008), 401–443.
[30] K. Yamaki, Geometric Bogomolov conjecture for abelian varieties and some results for those with some degeneration (with an appendix by Walter Gubler: The minimal dimension of a canonical measure), Manuscr. Math. 142 (2013), 273–306.
[31] K. Yamaki, Strict support of canonical measures and applications to the geometric Bogomolov conjecture, Compos. Math. 152 (2016), no.5, 997–1040.
[32] K. Yamaki, Non-density of small points on divisors on abelian varieties and the Bogomolov conjecture, J. Amer. Math. Soc. 30 (2017), no.4, 1133–1163.
[33] K. Yamaki, Trace of an abelian varieties over function fields and the geometric Bogomolov conjecture, J. Reine Angew. Math. 741 (2018), 133–159.
[34] K. Yamaki, Geometric Bogomolov conjecture for nowhere degenerate abelian varieties of dimension 5 with trivial trace, Math. Res. Lett. 24 (2017), no.5, 1555–1563.
[35] S. Zhang, Admissible pairing on a curve, Invent. Math. 112 (1993), 171–193.
[36] S. Zhang, Small points and adelic metrics, J. Alg. Geometry 4 (1995), 281–300.
[37] S. Zhang, Equidistribution of small points on abelian varieties, Ann. of Math. (2) 147 (1998), no. 1, 159–165.
[38] S. Zhang, Gross-Schoen cycles and dualising sheaves, Invent. Math. 179 (2010), 1–73.

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