RECENT DEVELOPMENTS IN THE THEORY OF MULTIPLE ZETA VALUES IN POSITIVE CHARACTERISTIC

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Abstract. Thakur introduced MZV's in positive characteristic as analogues of the classical multiple zeta values of Euler. This manuscript reports our recent results on these values in [33, 34, 38].

Contents

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1. Classical multiple zeta values

1.1. Multiple zeta values of Euler.

Throughout this text, let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers and $\mathbb{Z}^{\geq 0} = \{0, 1, 2, \dots\}$ be the set of non-negative integers. The multiple zeta values

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(MZV's for short) studied by Euler some centuries ago are the following convergent series

$$\zeta(n_1,\dots,n_r)=\sum_{0<k_1<\dots<k_r}\frac{1}{k_1^{n_1}\dots k_r^{n_r}}$$

where n_i are positive integers with $n_i\geq 1$ and $n_r\geq 2$. Here r is called the depth and $w:=n_1+\dots+n_r$ is called the weight of the presentation $\zeta(n_1,\dots,n_r)$. When $r=1$, we recover the zeta values

$$\zeta(n)=\sum_{k>0}\frac{1}{k^n},\quad \text{where } n\in\mathbb{N} \text{ and } n\geq 2,$$

which were studied well before Riemann studied them as a function $\zeta(s)$ of a complex variable s, and its links with the distribution of primes.

The even zeta values have been extensively studied and are well understood. As early as 1735, Euler proved that when n is even, $\zeta(n)$ is a rational multiple of π^n. Since Lindemann's proof of the transcendence of π, it has been established that all these numbers are transcendental. However, the odd zeta values remain a mystery. A folklore conjecture in the field suggests that:

Conjecture 1.1. *The numbers $\pi,\zeta(3),\zeta(5),\dots$ are all algebraically independent over $\mathbb{Q}$.*

As far as our understanding goes, we currently have no knowledge about the transcendence of odd zeta values. However, with regards to their irrationality, it has been shown that $\zeta(3)$ is irrational by Apéry [5], while Ball-Rivoal [6] showed that there are infinitely many irrational numbers among the remaining odd zeta values (see [42, 43, 56] for related works).

1.2. Ihara-Kaneko-Zagier's conjecture.

1.2.1. *An overview.*

Euler showed that the product of two multiple zeta values can be expressed as a linear combination, with integral coefficients, of multiple zeta values. An example of this is the identity:

$$\zeta(m)\zeta(n)=\zeta(m,n)+\zeta(n,m)+\zeta(m+n)$$

for all integers $m,n\geq 2$. The space of multiple zeta values, denoted by $\mathcal{Z}$, is a $\mathbb{Q}$-vector space that possesses an algebraic structure due to the previous fact. The primary objective of this theory is to comprehend all $\mathbb{Q}$-linear relations that exist among multiple zeta values. Unlike the algebraic structure generated by zeta values, the space $\mathcal{Z}$ has a rich combinatorial structure due to the presence of many linear relations among its elements. One effective approach to generating these linear relations is through the application of extended double shuffle relations introduced by Ihara-Kaneko-Zagier [32]. This process involves defining Hoffman's algebra $\mathfrak{h}$, and its subalgebras $\mathfrak{h}^0$ and $\mathfrak{h}^1$, which are endowed with specific algebraic structures. Two particular cases of quasi-product algebras introduced by Hoffman, namely the stuffle algebra $(\mathfrak{h}^1,*)$ and the shuffle algebra $(\mathfrak{h},⧢)$, are used to construct two algebraic structures. By means of regularization [32, §2], zeta maps can be defined, which are $\mathbb{Q}$-algebra homomorphisms:

$$\zeta_*:(\mathfrak{h}^1,*)\to\mathcal{Z},$$

and

$$\zeta_{ш} : (\mathfrak{h}, ш) \to \mathcal{Z},$$

which give rise to a generalization of the stuffle product and the shuffle product. The extended double shuffle relations arise by comparing the stuffle and shuffle products on $\mathfrak{h}^1$, as detailed in [32, Theorem 2]. Moreover, Ihara, Kaneko, and Zagier put forward a significant conjecture known as Ihara-Kaneko-Zagier's conjecture (see [32, Conjecture 1]), which states that all $\mathbb{Q}$-linear relations among MZV's can be obtained from the extended double shuffle relations. This conjecture implies Goncharov's conjecture, which asserts that all $\mathbb{Q}$-linear relations among MZV's can be deduced from those among MZV's of the same weight.

We now give precise details in the rest of this section.

1.2.2. *Quasi-shuffle algebras.*

The quasi-shuffle product, introduced by Hoffman, is a notion we will review. It involves a field k and a countable set $\mathfrak{X} = \{x_i\}_{i\in\mathbb{N}}$ of letters, each of which is assigned a weight $w(x_i) \in \mathbb{N}$. Tensor products will be taken over k. Additionally, for each $n \in \mathbb{N}$, we assume that the set $\mathfrak{X}_n$ of letters with weight n is finite.

We refer to $\mathfrak{X}$ as an alphabet, and its elements as letters. A word over the alphabet $\mathfrak{X}$ is a finite sequence of letters, and we denote the empty word by 1. The depth of a word $\mathfrak{a}$ is the number of letters in $\mathfrak{a}$, with $\operatorname{depth}(1) = 0$. The weight of a word $\mathfrak{a}$ is the sum of the weights of its letters, denoted by $w(\mathfrak{a})$.

The set of all words over $\mathfrak{X}$ is denoted by $\langle\mathfrak{X}\rangle$. We define the concatenation product on $\langle\mathfrak{X}\rangle$ as follows: for any words $\mathfrak{a} = x_{i_1}\dots x_{i_n}$ and $\mathfrak{b} = x_{j_1}\dots x_{j_m}$, we have

$$\mathfrak{a}\cdot\mathfrak{b} = x_{i_1}\dots x_{i_n} x_{j_1}\dots x_{j_m}.$$

We denote by $k\langle\mathfrak{X}\rangle$ (resp. $k\mathfrak{X}$) the k-vector space with $\langle\mathfrak{X}\rangle$ (resp. $\mathfrak{X}$) as a basis. The concatenation product extends to $k\langle\mathfrak{X}\rangle$ by linearity, so that $k\langle\mathfrak{X}\rangle$ is a graded algebra with respect to weight.

We use the notation $a\mathfrak{u}$ to denote the word obtained by appending a letter $a \in \mathfrak{X}$ to a word $\mathfrak{u} \in \langle\mathfrak{X}\rangle$. For a non-empty word $\mathfrak{a} \in \langle\mathfrak{X}\rangle$, we can write $\mathfrak{a} = x_a\mathfrak{a}_-$, where x_a is the first letter of $\mathfrak{a}$ and $\mathfrak{a}_-$ is the word obtained from $\mathfrak{a}$ by removing x_a.

We define $\overline{\mathfrak{X}} = \mathfrak{X} \cup \{0\}$ and introduce a commutative and associative product $\diamond : \overline{\mathfrak{X}} \times \overline{\mathfrak{X}} \to \overline{\mathfrak{X}}$ which preserves the grading. This means that for any $a, b, c \in \mathfrak{X}$, we have:

- $a \diamond 0 = 0$.
- $a \diamond b = b \diamond a$.
- $(a \diamond b) \diamond c = a \diamond (b \diamond c)$.
- Either $a \diamond b = 0$ or $w(a \diamond b) = w(a) + w(b)$.

We define a new operation denoted by $*_\diamond$ on the vector space $k\langle\mathfrak{X}\rangle$, which is generated by words over the alphabet $\mathfrak{X}$. The definition of $\diamond$ is recursive: we set $1 *_\diamond \mathfrak{u} = \mathfrak{u} *_\diamond 1 = \mathfrak{u}$ for any word $\mathfrak{u}$, and for any letters a, b and any words $\mathfrak{u}, \mathfrak{v}$, we define

$$a\mathfrak{u} *_\diamond b\mathfrak{v} = a(\mathfrak{u} *_\diamond b\mathfrak{v}) + b(a\mathfrak{u} *_\diamond \mathfrak{v}) + (a \diamond b)(\mathfrak{u} *_\diamond \mathfrak{v}),$$

where $\diamond$ is a binary operation on $\mathfrak{X}$. This operation is called the quasi-shuffle product associated to $\diamond$. A theorem due to Hoffman [26, Theorem 2.1] states that the vector space $k\langle\mathfrak{X}\rangle$ equipped with $*_\diamond$ is a commutative k-algebra.

We also define a coproduct map $\Delta : k\langle\mathfrak{X}\rangle \to k\langle\mathfrak{X}\rangle \otimes k\langle\mathfrak{X}\rangle$ and a counit map $\epsilon : k\langle\mathfrak{X}\rangle \to k$. The coproduct is defined by

$$\Delta(\mathfrak{u}) = \sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{u}} \mathfrak{a} \otimes \mathfrak{b},$$

where the sum is taken over all pairs of words $\mathfrak{a}$ and $\mathfrak{b}$ whose concatenation is equal to $\mathfrak{u}$. The counit is defined as follows:

$$\epsilon(\mathfrak{u}) = \begin{cases} 1 & \text{if } \mathfrak{u} = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for all words $\mathfrak{u} \in \langle\mathfrak{X}\rangle$. Hoffman proved in [26, Theorem 3.1] that $k\langle\mathfrak{X}\rangle$ equipped with the multiplication $*_\diamond$ and the comultiplication Δ is a bialgebra. Since both $*_\diamond$ and Δ respect the grading, this implies that the bialgebra structure of $k\langle\mathfrak{X}\rangle$ is graded.

Theorem 1.2. *The algebra $k\langle\mathfrak{X}\rangle$ with the $*_\diamond$-multiplication and Δ-comultiplication is a graded Hopf algebra. Further, it is connected and of finite type.*

Moreover, the antipode $S : k\langle\mathfrak{X}\rangle \to k\langle\mathfrak{X}\rangle$ is given explicitly in [26, Theorem 3.2]: for any word $\mathfrak{u} = x_{i_1} \dots x_{i_n}$ we have

$$S(\mathfrak{u}) = \sum_{(j_1,\dots,j_k)} (-1)^k x_{i_1} \dots x_{i_{j_1}} *_\diamond x_{i_{j_1+1}} \dots x_{i_{j_1+j_2}} *_\diamond \dots *_\diamond x_{i_{j_1+\dots+j_{k-1}+1}} \dots x_{i_{j_1+\dots+j_k}}$$

where the sum runs through the set of all partitions $(j_1, \dots, j_k)$ of n.

For recent developments on quasi-shuffle products, we refer the reader to [27, 29, 30, 31].

1.2.3. *The Hoffman algebra, stuffle product and shuffle product.*

In this section we take $k = \mathbb{Q}$. Let X be the alphabet with two letters x_0, x_1 with weight 1, that means $w(x_0) = w(x_1) = 1$. We denote $\mathfrak{h} = \mathbb{Q}\langle X\rangle$ and call it the Hoffman algebra. A word in the alphabet X is said to be positive if it is of the form $x_1\mathfrak{u}$ and is said to be admissible if it is of the form $x_1\mathfrak{u}x_0$. We denote by $\mathfrak{h}^1$ (resp. $\mathfrak{h}^0$) the subspace of $\mathfrak{h}$ spanned by positive words (resp. admissible words).

For all $i \in \mathbb{N}$ we put $z_i = x_1 x_0^{i-1}$. Then $w(z_i) = i$. Let Z be the alphabet with letters $\{z_i\}_{i\in\mathbb{N}}$. Then $\mathfrak{h}^1 = \mathbb{Q}\langle Z\rangle$. We now equip the alphabet Z with the commutative and associative product $\diamond : Z \times Z \to Z$ given by

$$z_i \diamond z_j = z_{i+j}$$

for all $i, j \in \mathbb{N}$. The associated quasi-product on $\mathfrak{h}^1 = \mathbb{Q}\langle Z\rangle$ will be denoted by $*$ and called the stuffle product. A word in $\mathfrak{h}^1$ is called admissible if it can be expressed as $z_{s_1} \dots z_{s_\ell}$ with $s_\ell > 1$. We note that $\mathfrak{h}^0$ is the subspace generated by admissible words in $\mathfrak{h}^1$ and that $(\mathfrak{h}^0, *)$ is a subalgebra of $(\mathfrak{h}^1, *)$. Further, the harmonic product on MZV's gives rise to a homomorphism of $\mathbb{Q}$-algebras

$$\zeta_* : \mathfrak{h}^0 \to \mathbb{R}$$

which sends an admissible word $z_{s_1} \dots z_{s_\ell}$ to the associated zeta value $\zeta(s_1, \dots, s_r)$, that means

$$\zeta_*(\mathfrak{u} * \mathfrak{v}) = \zeta_*(\mathfrak{u})\zeta_*(\mathfrak{v})$$

for all words $\mathfrak{u}, \mathfrak{v} \in \mathfrak{h}^0$. This map is called the stuffle zeta map.

We now recall the shuffle algebra. We endow X with the trivial product $\diamond : X \times X \to X$ given by

$$a \diamond b = 0$$

for all $a, b \in X$. The associated quasi-product on $\mathfrak{h} = \mathbb{Q}\langle X\rangle$ will be denoted by $\sqcup\!\sqcup$ and called the shuffle product. We see that $(\mathfrak{h}^0, \sqcup\!\sqcup)$ and $(\mathfrak{h}^1, \sqcup\!\sqcup)$ are subalgebras of $(\mathfrak{h}, \sqcup\!\sqcup)$. The shuffle product on MZV's defines a homomorphism of $\mathbb{Q}$-algebras

$$\zeta_{\sqcup\!\sqcup} : \mathfrak{h}^0 \to \mathbb{R}$$

which sends an admissible word $z_{s_1} \dots z_{s_\ell}$ to the associated zeta value $\zeta(s_1, \dots, s_r)$, that means

$$\zeta_{\sqcup\!\sqcup}(\mathfrak{u} \sqcup\!\sqcup \mathfrak{v}) = \zeta_{\sqcup\!\sqcup}(\mathfrak{u})\zeta_{\sqcup\!\sqcup}(\mathfrak{v})$$

for all words $w, v \in \mathfrak{h}^0$. This map is called the shuffle zeta map.

Using these zeta maps yield the so-called double shuffle relations in the convergent case: for all words $\mathfrak{u}, \mathfrak{v} \in \mathfrak{h}^0$,

$$\zeta_*(\mathfrak{u} * \mathfrak{v}) = \zeta_{\sqcup\!\sqcup}(\mathfrak{u} \sqcup\!\sqcup \mathfrak{v}).$$

1.2.4. *Regularized zeta maps.*

Following Ihara-Kaneko-Zagier [32], we note that the homomorphism of $(\mathfrak{h}^0, *)$-algebras $\varphi_* : \mathfrak{h}^0[T] \to \mathfrak{h}^1$ which sends T to z_1 is an isomorphism. Further, the following homomorphisms of $(\mathfrak{h}^0, \sqcup\!\sqcup)$-algebras

$$\varphi_{\sqcup\!\sqcup} : \mathfrak{h}^0[T] \to \mathfrak{h}^1, \quad T \mapsto x_1,$$
$$\varphi_{\sqcup\!\sqcup} : \mathfrak{h}^0[T, U] \to \mathfrak{h}, \quad T \mapsto x_1, U \mapsto x_0,$$

are isomorphisms.

Now we define the stuffle regularized zeta map

$$\zeta_* : \mathfrak{h}^1 \to \mathbb{R} \tag{1.1}$$

as the composition

$$\mathfrak{h}^1 \to \mathfrak{h}^0[T] \to \mathbb{R}[T] \to \mathbb{R}$$

where the first map is φ_*^{-1}, the second map is induced by the stuffle zeta map and the last one is the evaluation at $T = 0$. Similarly, we define the shuffle regularized zeta map

$$\zeta_{\sqcup\!\sqcup} : \mathfrak{h} \to \mathbb{R} \tag{1.2}$$

as the composition

$$\mathfrak{h} \to \mathfrak{h}^0[T, U] \to \mathbb{R}[T, U] \to \mathbb{R}$$

where the first map is $\varphi_{\sqcup\!\sqcup}^{-1}$, the second map is induced by the shuffle zeta map and the last one is the evaluation at $T = U = 0$.

In the study of MZV's, Ihara, Kaneko, and Zagier [32] used the maps discussed earlier to extend the double shuffle relations among MZV's. They formulated the following influential conjecture:

Conjecture 1.3 (Ihara-Kaneko-Zagier's conjecture)**.** *The extended double shuffle relations exhaust all $\mathbb{Q}$-linear relations among MZV's.*

1.2.5. *Stuffle Hopf algebra and shuffle Hopf algebra.*

By the work of Hoffman [26] the above algebras can be endowed with a richer structure, i.e., that of Hopf algebras. In fact, as a direct consequence of Theorem 1.2, we get two Hopf algebras for classical MZV's.

The first graded Hopf algebra

$$H_* = (\mathfrak{h}^1, *)$$

comes from the stuffle product. We note that it is related to the algebra of quasi-symmetric functions over k (see [19, 27]). For some applications of Hopf algebra structure, we refer the reader to [27] (see also [36]).

The second graded Hopf algebra

$$H_{\sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup} = (\mathfrak{h}, \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup)$$

is the shuffle algebra (see [41]). Explicitly,

- $\mathfrak{h} = \mathbb{Q}\langle x_0, x_1 \rangle$.
- The coproduct is given by the shuffle product $\sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup$.
- The unit is given by the empty word 1.
- The coproduct $\Delta : \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$ is given by the deconcatenation
$$\Delta(\mathfrak{u}) = \sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{u}} \mathfrak{a} \otimes \mathfrak{b}$$
for any words $\mathfrak{u} \in \mathfrak{h}$.
- The counit $\epsilon : \mathfrak{h} \to \mathbb{Q}$ is given by
$$\epsilon(\mathfrak{u}) = \begin{cases} 1 & \text{if } \mathfrak{u} = 1, \\ 0 & \text{otherwise.} \end{cases}$$
- The antipode $S : \mathfrak{h} \to \mathfrak{h}$ is given by
$$S(x_{i_1} \dots x_{i_n}) = (-1)^n x_{i_n} \dots x_{i_1}.$$

This Hopf algebra and its motivic version introduced by Goncharov [21] lie in the heart of the works of Brown [7], Deligne-Goncharov [18] and Terosoma [45] (see also [9]).

1.3. **Zagier-Hoffman's conjectures.**

It is remarkable that Zagier [54] and Hoffman [25] were able to guess the dimension and provide a conjectural explicit basis for the $\mathbb{Q}$-vector space $\mathcal{Z}_k$, which is the span of MZV's of weight k for $k \in \mathbb{N}$.

Conjecture 1.4 (Zagier's conjecture)**.** *We define a Fibonacci-like sequence of integers d_k as follows. Letting $d_0 = 1, d_1 = 0$ and $d_2 = 1$ we define $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$. Then for $k \in \mathbb{N}$ we have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k.$$

Conjecture 1.5 (Hoffman's conjecture)**.** *The $\mathbb{Q}$-vector space $\mathcal{Z}_k$ is generated by the basis consisting of MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

The question of determining upper bounds for $\dim_{\mathbb{Q}} \mathcal{Z}_k$ in the conjectures mentioned above, which is an algebraic aspect, was resolved using the theory of mixed Tate motives by Terasoma [45], Deligne-Goncharov [18], and Brown [7].

Theorem 1.6 (Deligne-Goncharov, Terasoma)**.** *For $k \in \mathbb{N}$ we have* $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.

Theorem 1.7 (Brown)**.** *The $\mathbb{Q}$-vector space $\mathcal{Z}_k$ is generated by MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

Indeed, the determination of lower bounds for $\dim_{\mathbb{Q}} \mathcal{Z}_k$ is a major problem in the theory of multiple zeta values. While upper bounds have been obtained using the theory of mixed Tate motives, lower bounds remain completely out of reach.

2. Multiple zeta values in positive characteristic

2.1. Multiple zeta values of Thakur.

There is a well-known analogy between number fields and function fields (see [35, 37, 53]). Inspired by Euler's work on multiple zeta values and that of Carlitz [10] on zeta values in positive characteristic, Thakur [50] introduced multiple zeta values attached to the affine line over a finite field. Multiple zeta values over function fields have been extensively studied in recent years. They share many properties with their classical counterparts, and are closely related to certain algebraic varieties over finite fields, such as Drinfeld modular varieties.

We now need to introduce some notations. Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements of characteristic $p > 0$. We denote by A_+ the set of monic polynomials in A. Let $K = \mathbb{F}_q(\theta)$ be the fraction field of A equipped with the rational point ∞. Let K_∞ be the completion of K at ∞. We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$, and by $|\cdot|_\infty = q^{-v_\infty}$ the associated absolute value on K.

In [10] Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ for $n \in \mathbb{N}$ given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$

which are analogues of classical special zeta values in the function field setting. For any tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, Thakur [46] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1, \dots, s_r)$ by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$. We call r the depth of $\zeta_A(\mathfrak{s})$ and $w(\mathfrak{s}) := s_1 + \dots + s_r$ the weight of $\zeta_A(\mathfrak{s})$. We note that Carlitz zeta values are exactly depth one MZV's. Thakur [47] showed that all the MZV's do not vanish. We refer the reader to [3, 4, 20, 22, 40, 46, 48, 49, 50, 51] for more details about these objects.

Thakur proved that the product of two MZV's is a K-linear combination of MZV's and we call it the shuffle product in positive characteristic. As in the classical setting, the main goal of the theory is to understand all linear relations over K among MZV's.

2.2. Analogues of Zagier-Hoffman's conjectures.

In positive characteristic, Thakur in [50, §8] and Todd in [52] formulated analogues of Zagier-Hoffman's conjectures, which aim to understand all linear relations over K among MZV's.

Recently, the fourth author provided a solution to these conjectures for small weights in [38], using tools from Chen [14], Thakur [49, 50] and Todd [52] as well as the theory of t-motives and dual motives of Anderson [1, 8, 24], and the Anderson-Brownawell-Papanikolas criterion in [2] (see [39, 11, 12] for further development). Later, the authors of [33] developed a new approach and were able to solve these conjectures for all weights. In particular, they proved Theorem B in [33]:

Theorem 2.1 (Zagier's conjecture in positive characteristic)**.** *For $w \in \mathbb{N}$ we denote by $\mathcal{Z}_w$ the K-vector space spanned by the MZV's of weight w. Letting*

$$d(w) = \begin{cases} 1 & \text{if } w = 0, \\ 2^{w-1} & \text{if } 1 \leq w \leq q-1, \\ 2^{w-1} - 1 & \text{if } w = q, \end{cases}$$

we put $d(w) = \sum_{i=1}^{q} d(w-i)$ for $w > q$. Then for any $w \in \mathbb{N}$, we have

$$\dim_K \mathcal{Z}_w = d(w).$$

Theorem 2.2 (Hoffman's conjecture in positive characteristic)**.** *We keep the above notation. A K-basis for $\mathcal{Z}_w$ is given by $\mathcal{T}_w$ consisting of $\zeta_A(s_1, \dots, s_r)$ of weight w with $s_i \leq q$ for $1 \leq i < r$, and $s_r < q$.*

The findings of [33] have been extended to alternating multiple zeta values, which were introduced by Harada [23]. These values have been studied by several mathematicians in the classical setting due to their connections in various contexts. Interested readers can refer to [13, 15, 23, 28, 55] for further details. However, there is currently no knowledge of any algebraic structures of multiple zeta values in positive characteristic (see [38, Remark 2.2, Part 1]). The proofs of the aforementioned theorems use new tools, such as the operations introduced by Todd [52] and the fourth author [38], as well as the Anderson-Brownawell-Papanikolas transcendence criterion [2].

2.3. **Algebraic structures of MZV's.**

In [34] we presented a detailed investigation into the algebraic structures of MZV's in positive characteristic. This work was motivated by a question raised by a referee of [38] and a suggestion made by Deligne [17] in a private letter to Thakur in 2017, proposing the existence of a Hopf algebra structure for MZV's in positive characteristic.

We constructed both the Hopf stuffle algebra and the Hopf shuffle algebra in positive characteristic, thus solving the conjectures posed by Deligne, Thakur, and Shi in their respective works. The results of *loc. cit.* provide a complete solution to the aforementioned questions and conjectures and we hope that it would open new perspectives in the study of MZV's in positive characteristic.

Let us give now more precise statements of our results.

2.3.1. *Composition space.*

We recall a new structure called the composition space $\mathfrak{C}$, which is suggested by Shuji Yamamoto (see [50, §5.2]). The composition space plays a similar role to the Hoffman algebra $\mathfrak{h}$ in the context of MZV's in positive characteristic. We define $\mathfrak{X}$ as a countable set equipped with the weight $w(x_n) = n$ and call it an alphabet. The elements of $\mathfrak{X}$ are called letters. Let $\mathfrak{C} = \mathbb{F}_q\langle \mathfrak{X} \rangle$ be the free $\mathbb{F}_q$-vector space with basis $\langle \mathfrak{X} \rangle$.

2.3.2. *Shuffle algebra and shuffle map.*

We define the unit $u : \mathbb{F}_q \to \mathfrak{C}$ by sending 1 to the empty word 1. Next we define recursively two products on $\mathfrak{C}$ as $\mathbb{F}_q$-bilinear maps

$$\diamond \colon \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C} \quad \text{and} \quad ш \colon \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C}$$

by setting $1 \diamond \mathfrak{a} = \mathfrak{a} \diamond 1 = \mathfrak{a}$, $1 ш \mathfrak{a} = \mathfrak{a} ш 1 = \mathfrak{a}$ and

$$\begin{aligned} \mathfrak{a} \diamond \mathfrak{b} = &\ x_{a+b}(\mathfrak{a}_- ш \mathfrak{b}_-) + \sum_{i+j=a+b} \Delta^j_{a,b} x_i (x_j ш (\mathfrak{a}_- ш \mathfrak{b}_-)), \\ \mathfrak{a} ш \mathfrak{b} = &\ x_a(\mathfrak{a}_- ш \mathfrak{b}) + x_b(\mathfrak{a} ш \mathfrak{b}_-) + \mathfrak{a} \diamond \mathfrak{b}, \end{aligned}$$

for any words $\mathfrak{a}, \mathfrak{b} \in \langle \mathfrak{X} \rangle$, Here the coefficients $\Delta^i_{a,b}$ are given by

$$\Delta^i_{a,b} = \begin{cases} (-1)^{a-1}\binom{i-1}{a-1} + (-1)^{b-1}\binom{i-1}{b-1} & \text{if } (q-1) \mid i \text{ and } 0 < i < a+b, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\diamond$ the diamond product and ш the shuffle product.

Our first result provides a positive solution to the questions posed in [38, Remark 2.2, Part 1] and [44, Conjectures 3.2.2 and 3.2.11]. This result is presented in [34, Theorem A].

Theorem 2.3. *The spaces $(\mathfrak{C}, \diamond)$ and $(\mathfrak{C}, ш)$ are commutative $\mathbb{F}_q$-algebras. Further, for all words $\mathfrak{a}, \mathfrak{b} \in \mathfrak{C}$ we have*

$$\zeta_A(\mathfrak{a} ш \mathfrak{b}) = \zeta_A(\mathfrak{a})\, \zeta_A(\mathfrak{b}).$$

If we denote by $\mathcal{Z}$ the K-vector space spanned by MZV's, then the homomorphism of K-algebras

$$\begin{aligned} Z_ш : \mathfrak{C} \otimes_{\mathbb{F}_q} K &\to \mathcal{Z} \\ \mathfrak{a} &\mapsto \zeta_A(\mathfrak{a}) \end{aligned}$$

is called the shuffle map in positive characteristic.

2.3.3. *Shuffle Hopf algebra.*

We also define recursively a product on $\mathfrak{C}$ as a $\mathbb{F}_q$-bilinear map

$$\triangleright \colon \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C}$$

by setting $1 \triangleright \mathfrak{a} = \mathfrak{a} \triangleright 1 = \mathfrak{a}$ and

$$\mathfrak{a} \triangleright \mathfrak{b} = x_a(\mathfrak{a}_- ш \mathfrak{b})$$

for any words $\mathfrak{a}, \mathfrak{b} \in \langle \mathfrak{X} \rangle$. We call $\triangleright$ the triangle product. Inspired by the work of Shi [44, §3.2.3] we define a coproduct

$$\Delta : \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{C}.$$

using $\triangleright$ rather than the concatenation on recursive steps for words with depth > 1. The counit $\epsilon : \mathfrak{C} \to \mathbb{F}_q$ is defined as follows: $\epsilon(1) = 1$ and $\epsilon(\mathfrak{u}) = 0$ otherwise.

We note that for quasi-shuffle algebras introduced by Hoffman [26] and their generalization, the coproduct is roughly speaking the deconcatenation. The coproduct Δ defined as above is completely different from the deconcatenation and involves complicated combinatorics.

Our second result shows that this construction gives rise to a Hopf algebra structure of the shuffle algebra (see [34, Theorem B]).

Theorem 2.4. *The connected graded bialgebra $(\mathfrak{C}, ш, u, \Delta, \epsilon)$ is a connected graded Hopf algebra of finite type over $\mathbb{F}_q$.*

Next we study the coproduct Δ for letters in detail and prove some key properties. As an immediate consequence, we deduce that the coproduct Δ coincides with the coproduct introduced by Shi in [44, §3.2.3] (see [34, Theorem B]).

2.3.4. *Stuffle algebra and stuffle Hopf algebra.*

The stuffle algebra is easier to define. We introduce the stuffle product in the same way as that of $(\mathfrak{h}^1, *)$ as above. The $*$ product

$$*\colon \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C}$$

is given by setting $1 * \mathfrak{a} = \mathfrak{a} * 1 = \mathfrak{a}$ and

$$\mathfrak{a} * \mathfrak{b} = \ x_a(\mathfrak{a}_- * \mathfrak{b}) + x_b(\mathfrak{a} * \mathfrak{b}_-) + x_{a+b}(\mathfrak{a}_- * \mathfrak{b}_-)$$

for any words $\mathfrak{a}, \mathfrak{b} \in \langle \mathfrak{X} \rangle$. We call $*$ the stuffle product and see that $(\mathfrak{C}, *)$ is a commutative $\mathbb{F}_q$-algebra.

Then we define a coproduct $\Delta_* : \mathfrak{C} \to \mathfrak{C} \otimes \mathfrak{C}$ and a counit $\epsilon : \mathfrak{C} \to \mathbb{F}_q$ by

$$\Delta_*(w) = \sum_{uv=w} u \otimes v$$

and

$$\epsilon(w) = \begin{cases} 1 & \text{if } w = 0, \\ 0 & \text{otherwise}, \end{cases}$$

for any words $w \in \langle \mathfrak{X} \rangle$.

We deduce from the work of Hoffman [26] that the stuffle algebra $(\mathfrak{C}, *, u, \Delta_*, \epsilon)$ is a connected graded Hopf algebra of finite type over $\mathbb{F}_q$. Using our previous works [33, 38] we are able to construct a homomorphism of K-algebras called the stuffle map (see [34, Theorem B]):

Theorem 2.5. *Recall that $\mathcal{Z}$ is the K-vector space spanned by MZV's. Then there exists a homomorphism of K-algebras*

$$Z_* : \mathfrak{C} \otimes_{\mathbb{F}_q} K \to \mathcal{Z}$$

called the stuffle map in positive characteristic.

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