A convergence theorem to a solution to an equilibrium problem using CQ projection method
曲率が負の空間における CQ 射影法を用いた均衡問題の解近似定理

東邦大学・大学院理学研究科 板垣陽士
Haruto Itagaki
Department of Information Science
Toho University
東邦大学・理学部 木村泰紀
Yasunori Kimura
Department of Information Science
Toho University

1 Introduction

Equilibrium problems are important problems which include various of nonlinear problems. To analyze it we often use the notion of resolvent operators. It is a very important concept, since the solution of the equilibrium problem coincides with the set of fixed points of the resolvent. In this paper, we prove an approximation theorem for the solution to the equilibrium problem in CAT(-1) space using the resolvent with the CQ projection method.

The CQ projection method for a nonexpansive mapping was firstly proposed by Nakajo and Takahashi.

Theorem 1 (Nakajo and Takahashi [4])**.** *Let H be a Hilbert space. Let $T: H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For given $x = x_1 \in H$, $C_1 = Q_1 = H$, define $\{x_n\}$ by*

$$\begin{aligned}
C_{n+1} &= \{z \in H \mid \|Tx_n - z\| \leq \|x_n - z\|\},\\
Q_{n+1} &= \{z \in H \mid \langle x_n - z, x - x_n\rangle \geq 0\},\\
x_{n+1} &= P_{C_{n+1}\cap Q_{n+1}}x.
\end{aligned}$$

Then $x_n \to P_{F(T)}x$, where $P_K: H \to K$ is the metric projection of H onto a nonempty closed convex subset K of H.

Following this result, Tada and Takahashi proposed an approximation result for equilibrium problems in Hilbert spaces as follows:

Theorem 2 (Tada and Takahashi [5]). *Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (E1)–(E4) and let T be a nonexpansive mapping of C into H such that $F(T) \cap S(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and let*

$$\begin{aligned} u_n &= R_{\lambda_n f} x_n, \\ w_n &= (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n &= \{z \in H \mid \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in H \mid \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x. \end{aligned}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, 1]$ for some $a, b \in (0,1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} \lambda_n > 0$. Then, $\{x_n\}$ converges strongly to $P_{F(T)\cap S(f)} x_n$.

Motivated by this result, the second author introduced a resolvent on CAT(1) spaces [2], and we obtained the following convergence theorem.

Theorem 3 (Itagaki and Kimura [1]). *Let X be an admissible complete CAT(1) space with the convex hull finite property. Suppose that X satisfies the following:*

- *$\{z \in X \mid d(u,z) \leq d(v,z)\}$ is convex for $u, v \in X$;*
- *$\{z \in X \mid \cos d(u,v) \cos d(v,z) \geq \cos d(u,z)\}$ is convex for $u, v \in X$.*

Let K be a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the condition (E1)–(E4). For each $x \in X$, define a subset $R_f x$ of K by

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) - \log\cos d(x,y) + \log\cos d(x,z)) \geq 0 \right\}.$$

Let $\{\lambda_n\} \subset [a, \infty[$ and $0 < a < \infty$. Generate $\{x_n\}$ by $x, x_1 \in X, C_1 = Q_1 = X$, and

$$\begin{aligned} C_{n+1} &= \{z \in X \mid d(R_{\lambda_n f} x_n, z) \leq d(x_n, z)\}, \\ Q_{n+1} &= \{z \in X \mid \cos d(x, x_n) \cos d(x_n, z) \geq \cos d(x,z)\}, \\ x_{n+1} &= P_{C_{n+1}\cap Q_{n+1}} x. \end{aligned}$$

for $n \in \mathbb{N}$. Then $x_n \to P_{S(f)} x$.

We apply the resolvent of the equilibrium problem in CAT(-1) space to the CQ projection method, and prove an approximation theorem of the solution to the equilibrium problem.

2 Preliminaries

Let X be a metric space and $T\colon X \to X$. The set of all fixed points of T is denoted by $F(T)$, that is,

$$F(T) = \{z \in X \mid z = Tz\}.$$

T is said to be quasinonexpansive, if $F(T) \neq \emptyset$ and $d(Tx, z) \leq d(x, z)$ for $x \in X$ and $z \in F(T)$.

Let X be a metric space. For $x, y \in X$, a mapping $c\colon [0, d(x, y)] \to X$ is called a geodesic if c satisfies $c(0) = x, c(d(x, y)) = y$, and $d(c(s), c(t)) = |s - t|$ for every $s, t \in [0, d(x, y)]$. If for any $x, y \in X$, there exists a unique geodesic with endpoints x and y, then X is called a uniquely geodesic space. For a uniquely geodesic space X, the image of the geodesic with endpoints $x, y \in X$ is denoted by $[x, y]$. In this case, there exists a unique $z \in [x, y]$ such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(z, y) = td(x, y).$$

We denote it by $z = tx \oplus (1 - t)y$ and we call it a convex combination of x and y.

Let (X, d) be a uniquely geodesic space. The triangle $\triangle(x, y, z)$ formed by $x, y, z \in X$ is called a geodesic triangle. Consider the two-dimensional hyperbolic space $\mathbb{H}^2$ as a model space of X. Then for a point $x, y, z \in X$, a comparison triangle $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ of $\triangle(x, y, z)$ is defined as a triangle on $\mathbb{H}^2$ such that $d(x, y) = d_{\mathbb{H}^2}(\overline{x}, \overline{y}), d(y, z) = d_{\mathbb{H}^2}(\overline{y}, \overline{z}), d(z, x) = d_{\mathbb{H}^2}(\overline{z}, \overline{x})$. A comparison point of $p = tx \oplus (1 - t)y \in [x, y]$ is defined by $\overline{p} = t\overline{x} \oplus (1 - t)\overline{y} \in [\overline{x}, \overline{y}]$. If X satisfies that

$$d(p, q) \leq d_{\mathbb{H}^2}(\overline{p}, \overline{q})$$

for any $\triangle(x, y, z)$, $p, q \in \triangle(x, y, z)$ and $\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})$, then it is called a CAT(-1) space and this inequality is called the CAT(-1) inequality.

Theorem 2.1. *Let X be a CAT(-1) space. Then*

$$\begin{aligned}\cosh d(tx \oplus (1 - t)y, z) \sinh d(x, y) \\ \leq \cosh d(x, z) \sinh td(x, y) + \cosh d(y, z) \sinh(1 - t)d(x, y)\end{aligned}$$

for $x, y, z \in X$ and $t \in [0, 1]$.

Let X be a complete CAT(-1) space. Let $C \subset X$ be a nonempty closed convex set. Then, there exists a unique $y_x \in C$ satisfying

$$d(x, y_x) = \inf_{y \in C} d(x, y)$$

for $x \in X$. We define $P_C\colon X \to C$ by $P_C x = y_x$ for $x \in X$. We call it the metric projection onto C.

3 Approximation of a solution to an equilibrium problem

Let X be a complete CAT(-1) space. Let $K \subset X$ be a nonempty closed convex set. An equilibrium problem for $f\colon K \times K \to \mathbb{R}$ is the problem of finding $z_0 \in K$ such that $f(z_0, y) \geq 0$ for all $y \in K$. The solution set $S(f)$ is defined by

$$S(f) = \left\{ z \in K \;\middle|\; \inf_{y \in K} f(z, y) \geq 0 \right\}.$$

We suppose the four conditions for f as follows:

(E1) $f(x, x) = 0$ for all $x \in K$;
(E2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$;
(E3) $f(x, \cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex for every $x \in K$;
(E4) For every $u, v, y \in K$, $\limsup_{t \to 1} f(tu \oplus (1-t)v, y) \leq f(u, y)$.

Lemma 3.1 (Kimura and Sasaki [3]). *Suppose that X is a CAT(-1) space with the convex hull finite property. Define a subset $T_f x$ of K by*

$$T_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z, y) + \log\cosh d(x, y) - \log\cosh d(x, z)) \geq 0 \right\}$$

for every $x \in X$. Suppose that $f\colon K^2 \to \mathbb{R}$ satisfies (E1)–(E4) and

(E5) $$\liminf_{\substack{d(v,z) \to \infty \\ z \in K}} \frac{f(v, z)}{d(v, z)} + 1 > 0$$

for all $v \in K$. Then, a mapping T_f is well defined as a single-valued mapping.

Lemma 3.2 (Kimura and Sasaki [3]). *For any $x, y \in X$, $\lambda, \mu > 0$,*

$$(\lambda + \mu) \cdot \cosh d(T_{\lambda f} x, T_{\mu f} y) \leq \mu \cdot \frac{\cosh d(T_{\mu f} y, x)}{\cosh d(T_{\lambda f} x, x)} + \lambda \cdot \frac{\cosh d(T_{\lambda f} x, y)}{\cosh d(T_{\mu f} y, y)}.$$

Theorem 3.1. *Let X be a complete CAT(-1) space with the convex hull finite property. Suppose that X satisfies the following:*

- $\{z \in X \mid d(u, z) \leq d(v, z)\}$ *is convex for $u, v \in X$;*
- $\{z \in X \mid \cosh d(u, v) \cosh d(v, z) \leq \cosh d(u, z)\}$ *is convex for $u, v \in X$.*

Let $K \subset X$ be a nonempty cloded convex set. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies (E1)–(E5) and $S(f) \neq \emptyset$. Define $T_f\colon X \to K$ by

$$T_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z, y) + \log\cosh d(x, y) - \log\cosh d(x, z)) \geq 0 \right\}$$

for every $x \in X$. Let $\{\lambda_n\} \subset [a, \infty[$ and $0 < a < \infty$. Generate $\{x_n\}$ by $x_1 \in X, C_1 = Q_1 = X$, and

$$C_{n+1} = \{z \in X \mid d(T_{\lambda_n f} x_n, z) \leq d(x_n, z)\},$$

$$Q_{n+1} = \{z \in X \mid \cosh d(x, x_n) \cosh d(x_n, z) \le \cosh d(x, z)\},$$
$$x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x$$

for $n \in \mathbb{N}$. *Then* $x_n \to P_{S(f)} x \in K$.

Proof. First, we prove $\{x_n\}$ is well-defined by induction. $C_1 = Q_1 = X$ is a closed convex set and $S(f) \subset C_1 \cap Q_1$. For $k \in \mathbb{N}$, assume that C_k, Q_k are closed convex sets and they satisfy $S(f) \subset C_k \cap Q_k$. Since $\{z \in X \mid d(T_{\lambda_k f} x_k, z) \le d(x_k, z)\}$ is convex by assumption, we know that C_{k+1} is closed and convex. Similarly, since $\{z \in X \mid \cosh d(x, x_k) \cosh d(x_k, z) \le \cosh d(x, z)\}$ is convex by assumption, we also know that Q_{k+1} is closed and convex. Next, we prove $S(f) \subset C_{k+1} \cap Q_{k+1}$. Let $z \in S(f) = F(T_f)$. Since T_f is quasinonexpansive, we have $d(T_{\lambda_k f} x_k, z) \le d(x_k, z)$, and we obtain $z \in C_{k+1}$. This implies $S(f) \subset C_{k+1}$. Moreover, we can show $S(f) \subset Q_{k+1}$. Fix $z \in S(f)$ arbitrarily. Then, $z \in C_k \cap Q_k$ and thus

$$tz \oplus (1-t)x_k = tz \oplus (1-t)P_{C_k \cap Q_k} x \in C_k \cap Q_k$$

for $t \in]0, 1[$. Therefore,

$$\begin{aligned}
&2 \cosh d(x, x_k) \cosh\left(\left(1 - \frac{t}{2}\right) d(x_k, z)\right) \sinh\left(\frac{t}{2} d(x_k, z)\right) \\
&\quad = \cosh d(x, x_k)(\sinh d(x_k, z) - \sinh((1-t)d(x_k, z))) \\
&\quad = \cosh d(x, P_{C_k \cap Q_k} x) \sinh d(x_k, z) - \cosh d(x, x_k) \sinh((1-t)d(x_k, z)) \\
&\quad \le \cosh d(x, tz \oplus (1-t)x_k) \sinh d(x_k, z) - \cosh d(x, x_k) \sinh((1-t)d(x_k, z)) \\
&\quad \le \cosh d(x, z) \sinh(td(x_k, z)) \\
&\quad = 2 \cosh d(x, z) \cosh\left(\frac{t}{2} d(x_k, z)\right) \sinh\left(\frac{t}{2} d(x_k, z)\right).
\end{aligned}$$

When $z \ne x_k$, dividing by $2 \sinh(td(x_k, z)/2)$ and letting $t \to 0$, we have

$$\cosh d(x, x_k) \cosh d(x_k, z) \le \cosh d(x, z).$$

From the definition of Q_{k+1}, we have $z \in Q_{k+1}$. If $z = x_k$, then obviously $z \in Q_{k+1}$. Therefore, we get $S(f) \subset Q_{k+1}$. Hence we have C_{k+1} and Q_{k+1} are closed convex sets and $S(f) \subset C_{k+1} \cap Q_{k+1}$. Since the intersection of closed convex sets is a closed convex set, there exists the metric projection to $C_{k+1} \cap Q_{k+1}$ and $x_{k+1} = P_{C_{k+1} \cap Q_{k+1}} x$ can be defined. Therefore $\{x_n\}$ is well-defined. It is also shown that $P_{S(f)} x \in S(f) \subset C_n \cap Q_n$ and $C_n \cap Q_n \subset Q_{n+1}$, for arbitrary $n \in \mathbb{N}$.

Next, we prove $d(T_{\lambda_n f} x_n, x_n) \to 0$. For arbitrary $n \in \mathbb{N}$, since $P_{S(f)} x \in S(f) \subset C_n \cap Q_n$, from the definition of the metric projection, we get

$$d(x, x_n) = d(x, P_{C_n \cap Q_n} x) \le d(x, P_{S(f)} x) < \infty.$$

Therefore, $\{x_n\}$ is bounded. Fix $z \in Q_{n+1}$ arbitrarily. From the definition of Q_{n+1}, we have

$$\cosh d(x, x_n) \cosh d(x_n, z) \le \cosh d(x, z)$$

and then,

$$\cosh d(x, x_n) \leq \cosh d(x, z).$$

It follows that

$$\inf_{y \in Q_{n+1}} d(x, y) \leq d(x, x_n) \leq d(x, z).$$

It implies that $d(x, x_n) = \inf_{y \in Q_{n+1}} d(x, y)$. Therefore, we have $P_{Q_{n+1}} x = x_n = P_{C_n \cap Q_n} x \in C_n \cap Q_n \subset Q_{n+1}$. Thus, we obtain

$$\begin{aligned} d(x, x_n) = d(x, P_{C_n \cap Q_n} x) &= d(x, P_{Q_{n+1}} x) \\ &\leq d(x, P_{C_{n+1} \cap Q_{n+1}} x) = d(x, x_{n+1}), \end{aligned}$$

for $n \in \mathbb{N}$. Hence $\{d(x, x_n)\}$ is a nondecreasing sequence. Thus, $\{\cosh d(x, x_n)\}$ is nondecreasing and bounded above, so we get

$$\lim_{n \to \infty} \cosh d(x, x_n) = c > 0.$$

Also, since $x_{n+1} \in C_{n+1} \cap Q_{n+1} \subset Q_{n+1}$, we have

$$\cosh d(x, x_n) \cosh d(x_n, x_{n+1}) \leq \cosh d(x, x_{n+1})$$

for $n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$c \limsup_{n \to \infty} \cosh d(x_n, x_{n+1}) \leq c.$$

Thus, dividing by $c > 0$, we get

$$\limsup_{n \to \infty} \cosh d(x_n, x_{n+1}) \leq 1,$$

and since

$$1 \leq \liminf_{n \to \infty} \cosh d(x_n, x_{n+1}) \leq \limsup_{n \to \infty} \cosh d(x_n, x_{n+1}) \leq 1,$$

we get $\lim_{n \to \infty} \cosh d(x_n, x_{n+1}) = 1$. This implies $\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$. Furthermore, since $x_{n+1} \in C_{n+1} \cap Q_{n+1} \subset C_{n+1}$, we have $d(T_{\lambda_n f} x_n, x_{n+1}) \leq d(x_n, x_{n+1})$ for $n \in \mathbb{N}$. Thus we get,

$$0 \leq d(T_{\lambda_n f} x_n, x_n) \leq d(T_{\lambda_n f} x_n, x_{n+1}) + d(x_{n+1}, x_n) \leq 2d(x_n, x_{n+1}) \to 0.$$

Finally, we show $x_n \to P_{S(f)} x$. Since $\sup_{n \in \mathbb{N}} d(x, x_n) < \infty$, $\{x_n\}$ is bounded. Fix $\{x_{n_i}\} \subset \{x_n\}$ arbitrarily. Then there exist $\{\lambda_{n_{i_j}}\} \subset \{\lambda_{n_i}\}$ and $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that $\lambda_{n_{i_j}} \to \lambda_0 \in [a, \infty]$ and $x_{n_{i_j}} \xrightarrow{\Delta} w_0$. Suppose $\lambda_{n_{i_j}} \to \infty$. For any $y \in X$, we have

$$d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, y) \leq d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, x_{n_{i_j}}) + d(x_{n_{i_j}}, y)$$

$$\leq 2d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, x_{n_{i_j}}) + d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, y).$$

Then,

$$\limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, y) = \limsup_{j\to\infty} d(x_{n_{i_j}}, y).$$

We also have

$$(\lambda_{n_{i_j}} + 1)\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, T_f w_0) \\ \leq \frac{\cosh d(T_f w_0, x_{n_{i_j}})}{\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, x_{n_{i_j}})} + \lambda_{n_{i_j}} \frac{\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, w_0)}{\cosh d(T_f w_0, w_0)}.$$

Thus,

$$\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, T_f w_0) \\ \leq \frac{1}{\lambda_{n_{i_j}} + 1} \cdot \frac{\cosh d(T_f w_0, x_{n_{i_j}})}{\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, x_{n_{i_j}})} + \frac{\lambda_{n_{i_j}}}{\lambda_{n_{i_j}} + 1} \cdot \frac{\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, w_0)}{\cosh d(T_{fw_0}, w_0)}.$$

It follows that

$$\limsup_{j\to\infty} \cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, T_f w_0) \leq \limsup_{j\to\infty} \frac{\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, w_0)}{\cosh d(T_f w_0, w_0)},$$

and

$$\cosh \limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, T_f w_0) \leq \frac{\cosh \limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, w_0)}{\cosh d(T_f w_0, f w_0)}.$$

Therefore

$$\cosh \limsup_{j\to\infty} d(x_{n_{i_j}}, T_f w_0) \leq \frac{\cosh \limsup_{j\to\infty} d(x_{n_{i_j}}, w_0)}{\cosh d(T_f w_0, w_0)}.$$

Hence we get $w_0 \in F(T_f) = S(f)$.

Next, suppose $\lambda_{n_{i_j}} \to \lambda_0 \in [a, \infty[$. We also have

$$(\lambda_{n_{i_j}} + 1)\cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, T_f w_0) \\ \leq \lambda_{n_{i_j}} \cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, w_0) + \cosh d(x_{n_{i_j}}, T_f w_0).$$

Then,

$$\lambda_{n_{i_j}} \cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, T_f w_0) + \cosh d(T_{\lambda_{n_{i_j}}f}x_{n_{i_j}}, T_f w_0)$$

$$\leq \lambda_{n_{i_j}} \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0) + \cosh d(x_{n_{i_j}}, T_f w_0).$$

It follows that

$$\begin{aligned}
&\lambda_{n_{i_j}} \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \\
&\quad \leq \lambda_{n_{i_j}} \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0) + \cosh d(x_{n_{i_j}}, T_f w_0) - \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0),
\end{aligned}$$

and

$$\begin{aligned}
&\cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \\
&\qquad \leq \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0) + \frac{\cosh d(x_{n_{i_j}}, T_f w_0) - \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0)}{\lambda_{n_{i_j}}} \\
&\qquad \leq \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0) + \frac{|\cosh d(x_{n_{i_j}}, T_f w_0) - \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0)|}{a}.
\end{aligned}$$

It follows that

$$\limsup_{j\to\infty} \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \leq \limsup_{j\to\infty} \cosh d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0),$$

which implies

$$\limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, T_f w_0) \leq \limsup_{j\to\infty} d(T_{\lambda_{n_{i_j}} f} x_{n_{i_j}}, w_0).$$

Therefore we have

$$\limsup_{j\to\infty} d(x_{n_{i_j}}, T_f w_0) \leq \limsup_{j\to\infty} d(x_{n_{i_j}}, w_0).$$

Thus $T_f w_0 = w_0$. Then we have

$$\begin{aligned}
d(x, P_{S(f)}x) \leq d(x, w_0) &\leq \liminf_{j\to\infty} d(x, x_{n_{i_j}}) \\
&\leq \limsup_{j\to\infty} d(x, x_{n_{i_j}}) \\
&\leq \sup_{n\in\mathbb{N}} d(x, x_n) \\
&\leq d(x, P_{S(f)}x).
\end{aligned}$$

Thus, $d(x, P_{S(f)}x) = d(x, w_0)$, and hence $w_0 = P_{S(f)}x$. From the inequality above, we also have $\lim_{j\to\infty} d(x, x_{n_{i_j}}) = d(x, P_{S(f)}x)$, and then $x_{n_{i_j}} \to P_{S(f)}x$. Consequently, we have

$$x_n \to P_{S(f)}x,$$

which is the desired result. $\square$

Acknowledgement. This work was partially supported by JSPS KAKENHI Grant Number JP21K03316.

References

[1] H. Itagaki and Y. Kimura, *Approximation of a solution to equilibrium problems in geodesic spaces using projection methods*, Study on nonlinear analysis and convex analysis, RIMS Kôkyûroku, vol. 2240, Kyoto University, Kyoto, 2023, 104–112.

[2] Y. Kimura, *Resolvents of equilibrium problems on a complete geodesic space with curvature bounded above*, Carpathian J. Math. **37** (2021), 463–476.

[3] Y. Kimura and K. Sasaki, *Sufficient conditions for perturbations to define the resolvent of the equilibrium problem on complete geodesic spaces*, J. Fixed Point Theory Appl. **25** (2023), 24pp.

[4] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[5] A. Tada and W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mapping and equilibrium problem*, J. Optim. Theory. Appl. **133** (2007), 359–370.