Approximation of a common fixed point using a balanced mapping in Hadamard spaces and its application
均衡写像を用いた共通不動点近似定理とその応用

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Abstract

In this paper, we introduce a Mann iteration of a balanced mapping of a countable family of nonexpansive mappings in Hadamard spaces. Further, we prove a strong convergence theorem with the combining projection method of balanced type using a finite family of mappings in a real Hilbert ball.

1 Introduction

Approximation of a fixed point is studied by many researchers in various spaces. As a most famous method of approximation technique, we know the projection method. In 2003, Nakajo and Takahashi introduced *Nakajo–Takahashi projection method* in Hilbert spaces [9]. In 2005, Takahashi et al. introduced *the shrinking projection method* in Hilbert spaces [11].

In 2011, Kimura et al. introduced another projection method, which is called *the combining projection method*.

Theorem 1.1 (Kimura et al. [7]). *Let C be a nonempty closed convex subset C of a Hilbert space. Let $I_N = \{1, 2, \dots, N\}$ and T_j a nonexpansive mapping of C into itself for $j \in I_N$ such that $\bigcap_{j=1}^N \operatorname{Fix} T_j \neq \varnothing$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^j \mid j \in I_N, n \in \mathbb{N}\} \subset [0,1]$ such that $\sum_{j=1}^N \beta_n^j = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid n, k \in \mathbb{N}, k \leq n\}$ such that $\sum_{k=1}^n \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Define a sequence*

$\{x_n\}$ by $u, x_1 \in C$ and

$$y_n^j = \alpha_n x_n + (1-\alpha_n)T_j x_n \text{ for } j \in I_N;$$
$$C_n^j = \left\{z \in C \;\middle|\; \|z - y_n^j\| \le \|z - x_n\|\right\} \text{ for } j \in I_N;$$
$$x_{n+1} = \delta_n u + (1-\delta_n)\sum_{k=1}^{n}\gamma_{n,k}\sum_{j=1}^{N}\beta_k^j P_{C_k^j} x_n$$

for each $n \in \mathbb{N}$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H. Suppose the following conditions hold:

(i) $\liminf_{n\to\infty}\alpha_n < 1$;
(ii) $\beta_n^j > 0$ *for all* $j \in I_N$;
(iii) $\lim_{n\to\infty}\gamma_{n,k} > 0$ *for all* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty}\sum_{k=1}^{n}|\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
(iv) $\lim_{n\to\infty}\delta_n = 0$, $\sum_{n=1}^{\infty}\delta_n = \infty$ *and* $\sum_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ converges strongly to $P_{\bigcap_{j=1}^{N}\mathrm{Fix}\,T_j}u$.

In geodesic spaces, the convex combination of more than three points are order-dependent in general. In 2018, Hasegawa and Kimura [5] introduced another definition of convex combination which is order-independent for three points. Using this notion, Kimura and Ogihara [6] introduced *the combining projection method of balanced type* and proved a convergence to a fixed point of a nonexpansive mapping.

In this paper, we consider approximating a common fixed point of a countable or finite family of mappings in Hadamard spaces. In section 3, we introduce a Mann iterative scheme of a balanced mapping of a countable family of nonexpansive mappings and prove a delta-convergent to a common fixed point. In section 4, we introduce the combining projection method of balanced type of a finite family of mappings in a real Hilbert ball.

2 Preliminaries

Let X be a metric space. A set $\mathrm{Fix}\,T$ is all fixed points of a mapping T of X into itself. A mapping T is *nonexpansive* if the inequality $d(Tx,Ty) \le d(x,y)$ holds for all $x, y \in X$. A mapping T is *quasinonexpansive* if $\mathrm{Fix}\,T \ne \varnothing$ and the inequality $d(Tx,z) \le d(x,z)$ for $x \in X$ and $z \in \mathrm{Fix}\,T$. A nonexpansive mapping T with $\mathrm{Fix}\,T \ne \varnothing$ is quainonexpanisve. Indeed, for $x \in X$ and $z \in \mathrm{Fix}\,T$, by nonexpansiveness of T, it follows that

$$d(Tx,z) = d(Tx,Tz) \le d(x,z)$$

and hence T is quasinonexpansive.

Let f be a function of X into $\mathbb{R}$. A set $\mathrm{Argmin}_{y\in X} f(y)$ is defined by

$$\operatorname*{Argmin}_{y\in X} f(y) = \left\{z \in X \;\middle|\; f(z) = \inf_{y\in X} f(y)\right\}.$$

Let $\{x_n\}$ be a bounded sequence of X and $x_0 \in X$. Then, x_0 is *an asymptotic center of* $\{x_n\}$ if the equality

$$\limsup_{n\to\infty} d(x_n, x_0) = \inf_{y\in X} \limsup_{n\to\infty} d(x_n, y)$$

holds. The set of all asymptotic centers of $\{x_n\}$ is denoted by $\mathrm{AC}(\{x_n\})$. Further, $\{x_n\}$ is *delta-convergent to* x_0 if for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, $\mathrm{AC}(\{x_{n_i}\}) = \{x_0\}$, which is denoted by $x_n \stackrel{\Delta}{\rightharpoonup} x_0$.

A metric space X is *a uniquely geodesic space* if for all $x, y \in X$, there exists a unique mapping γ of $[0, d(x, y)]$ into X such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$. Let $x, y \in X$. Then, we can take a unique point $z = \gamma((1 - t)d(x, y))$ for each $t \in [0, 1]$, which is called *a convex combination between* x *and* y and is denoted by $z = tx \oplus (1 - t)y$.

Let X be a uniquely geodesic space and $x, y, z \in X$. Then, *a geodesic triangle of vertices* x, y, z is defined by $\operatorname{Im}\gamma_{xy} \cup \operatorname{Im}\gamma_{yz} \cup \operatorname{Im}\gamma_{zx}$, which is denoted by $\triangle(x, y, z)$. For $x, y, z \in X$, *a comparison triangle* to $\triangle(x, y, z) \subset X$ of vertices $\bar{x}, \bar{y}, \bar{z} \in \mathbb{E}^2$ is defined by $\operatorname{Im}\gamma_{\bar{x}\bar{y}} \cup \operatorname{Im}\gamma_{\bar{y}\bar{z}} \cup \operatorname{Im}\gamma_{\bar{z}\bar{x}}$ with $d(x, y) = d_{\mathbb{E}^2}(\bar{x}, \bar{y})$, $d(y, z) = d_{\mathbb{E}^2}(\bar{y}, \bar{z})$ and $d(z, x) = d_{\mathbb{E}^2}(\bar{z}, \bar{x})$, which is denoted by $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$. A point $\bar{p} \in \operatorname{Im}\gamma_{\bar{x}\bar{y}}$ is called *a comparison point of* $p \in \operatorname{Im}\gamma_{xy}$ if $d(x, p) = d_{\mathbb{E}^2}(\bar{x}, \bar{p})$. A uniquely geodesic space X is called a CAT(0) space if for all $x, y, z \in X$, $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, it follows that $d(p, q) \le d_{\mathbb{E}^2}(\bar{p}, \bar{q})$. A complete CAT(0) space is called *a Hadamard space.*

The following lemmas are important properties of a CAT(0) space.

Lemma 2.1. *Let* X *be a* CAT(0) *space. Then,*

$$d(tx \oplus (1 - t)y, z)^2 \le td(x, z)^2 + (1 - t)d(y, z)^2 - t(1 - t)d(x, y)^2$$

for each $x, y, z \in X$ *and* $t \in [0, 1]$.

Lemma 2.2 (Kirk and Panyanak [8])**.** *Let* X *be a Hadamard space. Then every bounded sequence has a subsequence which is delta-convergent to* $x_0 \in X$.

Lemma 2.3 (Dhompongsa, Kirk and Sims [2])**.** *Let* X *be a Hadamard space and* $\{x_n\}$ *a bounded sequence of* X*. Then the asymptotic center of* $\{x_n\}$ *consists of one point.*

In 2018, Hasegawa and Kimura introduced a notion of *a balanced mapping* and consider its properties in Hadamard spaces.

Theorem 2.1 (Hasegawa and Kimura [5])**.** *Let* X *be a Hadamard space,* T_i *a nonexpansive mapping of* X *into itself for* $i = 1, 2, \dots, N$ *such that* $\bigcap_{i=1}^N \operatorname{Fix} T_i \neq \varnothing$*, and* $\{\alpha^i : i = 1, 2, \dots, N\} \subset \,]0, 1[$ *such that* $\sum_{i=1}^N \alpha^i = 1$*. Let*

$$Ux = \operatorname*{Argmin}_{y\in X} \sum_{i=1}^N \alpha^i d(T_i x, y)^2$$

for each $x \in X$. Then, the following conditions hold:

(i) U *is nonexpansive;*
(ii) $\operatorname{Fix} U = \bigcap_{i=1}^{N} \operatorname{Fix} T_i$.

In the following theorem, Kimura and Ogihara [6] prove properties of a balanced mapping.

Theorem 2.2 (Kimura and Ogihara [6]). *Let X be a Hadamard space, C a nonempty bounded subset of X, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for all $n \in \mathbb{N}$. Let*

$$U_n x = \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \alpha_n^k d(T_k x, y)$$

for each $n \in \mathbb{N}$ and $x \in X$. Suppose the following conditions:

- $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$;
- $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Then the following conditions hold:

(i) $\sum_{n=1}^{\infty} \sup_{x \in C} d(U_{n+1}x, U_n x) < \infty$;
(ii) there exists a mapping $U \colon X \to X$ such that $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$;
(iii) $\lim_{n\to\infty} \sup_{x \in C} d(U_n x, Ux) = 0$;
(iv) U is nonexpansive and $\operatorname{Fix} U = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k$.

In the following, we introduced a Halpern iteration with a balanced mapping of countable family of nonexpansive mappings in Hadamard spaces.

Theorem 2.3 (Hasegawa [4], Kimura and Ogihara [6]). *Let X be a Hadamard space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for all $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Let*

$$U_n x = \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \alpha_n^k d(T_k x, y)^2$$

for all $x \in X$ and $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ by $u, x_1 \in X$ and

$$x_{n+1} = \delta_n u \oplus (1 - \delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
(ii) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ is convergent to $P_F u$, where P_F is the metric projection of X onto F.

The following lemma is important to prove a delta-convergence theorem of a countable family of nonexpansive mappings.

Lemma 2.4 (Tan and Xu [12]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \in \mathbb{N}$. If $\sum_{k=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

3 Delta-convergence theorem using a balanced mapping

In this section, we prove an approximation theorem of a common fixed point of a countable family of nonexpansive mappings in Hadamard spaces.

Theorem 3.1. *Let X be a Hadamard space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $F = \bigcap_{k=1}^{\infty} \operatorname{Fix} T_k \neq \varnothing$, $\{\alpha_n^k \mid k, n \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$, and $\{t_n \mid n \in \mathbb{N}\} \subset [0,1[$. Let*

$$U_n = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{n} \alpha_n^k d(T_k x, y)^2$$

for each $n \in \mathbb{N}$ and $x \in X$. Define a sequence $\{x_n\}$ of X by $x_1 \in X$ and

$$x_{n+1} = t_n x_n \oplus (1 - t_n) U_n x_n$$

for all $n \in \mathbb{N}$. Suppose the following conditions:

(i) $\lim_{n\to\infty} \alpha_n^k > 0$ for $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
(ii) $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$.

Then $\{x_n\}$ is delta-convergent to $x_0 \in F$.

Proof. Let $p \in F$. Then $p \in \bigcap_{k=1}^{n} \operatorname{Fix} T_k$ for $n \in \mathbb{N}$. Since U_n is quasinonexpansive for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, p) \leq t_n d(x_n, p) + (1 - t_n) d(U_n x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists $\lim_{n\to\infty} d(x_n, p)$. Further, we get

$$\begin{aligned} d(x_{n+1}, p)^2 &\leq t_n d(x_n, p)^2 + (1 - t_n) d(U_n x_n, p)^2 - t_n(1 - t_n) d(x_n, U_n x_n)^2 \\ &\leq d(x_n, p)^2 - t_n(1 - t_n) d(x_n, U_n x_n)^2 \end{aligned}$$

and hence

$$t_n(1 - t_n) d(x_n, U_n x_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.$$

Since $\sum_{k=1}^{\infty} t_n(1 - t_n) = \infty$, we get $\liminf_{n\to\infty} d(x_n, U_n x_n) = 0$. We next show $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. Let C be a nonempty bounded subset of X including $\{x_n\}$. By the nonexpansiveness of U_n for $n \in \mathbb{N}$, we get

$$d(x_{n+1}, U_{n+1} x_{n+1}) \leq d(x_{n+1}, U_n x_n) + d(U_n x_n, U_n x_{n+1}) + d(U_n x_{n+1}, U_{n+1} x_{n+1})$$

$$
\begin{aligned}
&\leq d(x_{n+1}, U_n x_n) + d(x_{n+1}, x_n) + d(U_n x_{n+1}, U_{n+1} x_{n+1}) \\
&= d(x_n, U_n x_n) + d(U_n x_{n+1}, U_{n+1} x_{n+1}) \\
&\leq d(x_n, U_n x_n) + \sup_{x \in C} d(U_n x, U_{n+1} x)
\end{aligned}
$$

for all $n \in \mathbb{N}$. By Theorem 2.2, $\sum_{n=1}^{\infty} \sup_{x\in C} d(U_n x, U_{n+1} x) < \infty$. By Lemma 2.4, there exists $\lim_{n\to\infty} d(x_n, U_n x_n)$ and hence $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. By Theorem 2.2, there exists a mapping $U\colon X \to X$ such that $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$, U is nonexpansive and $\operatorname{Fix} U = F$. Take a subsequence $\{x_{n_i}\}$ of X with $\mathrm{AC}(\{x_{n_i}\}) = \{y_0\}$ arbitrary. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \overset{\Delta}{\rightharpoonup} z_0 \in X$. Then, we get

$$
\begin{aligned}
d(x_{n_{i_j}}, Ux_{n_{i_j}}) &\leq d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + d(U_{n_{i_j}} x_{n_{i_j}}, Ux_{n_{i_j}}) \\
&\leq d(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}) + \sup_{x\in C} d(U_{n_{i_j}} x, Ux).
\end{aligned}
$$

From (iii) of Theorem 2.2, we get $\lim_{n\to\infty} d(x_{n_{i_j}}, Ux_{n_{i_j}}) = 0$. Then, we get

$$
\begin{aligned}
\limsup_{j\to\infty} d(x_{n_{i_j}}, Uz_0) &\leq \limsup_{j\to\infty} (d(x_{n_{i_j}}, Ux_{n_{i_j}}) + d(Ux_{n_{i_j}}, Uz_0)) \\
&= \limsup_{j\to\infty} d(Ux_{n_{i_j}}, Uz_0) \\
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0)
\end{aligned}
$$

and hence $z_0 \in \operatorname{Fix} U = F$. Put $\mathrm{AC}(\{x_n\}) = \{y_0'\}$. Then, we get

$$
\begin{aligned}
\limsup_{n\to\infty} d(x_n, z_0) &= \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0) \\
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, y_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, y_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, y_0') \\
&\leq \limsup_{n\to\infty} d(x_n, y_0') \leq \limsup_{n\to\infty} d(x_n, y_0)
\end{aligned}
$$

and hence $y_0' = y_0 = z_0 \in F$. Consequently, we complete the proof. □

4 The combining projection method of balanced type in a real Hilbert ball

In this section, we introduce the combining projection method of balanced type and prove its convergence to a common fixed point by applying Theorem 2.3 in a real Hilbert ball.

Let $(H, \|\cdot\|)$ be a Real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $B = \{x \in H \mid \|x\| < 1\}$. we define $\rho(\cdot,\cdot)\colon B \times B \to \mathbb{R}$ by

$$\rho(x,y) = \tanh^{-1}\sqrt{1 - \frac{(1-\|x\|^2)(1-\|y\|^2)}{1-|\langle x,y\rangle|^2}}$$

for $x, y \in B$. Then, (B, ρ) is a metric space, which is called *a real Hilbert ball.* Further, we know that a real Hilbert ball is an example of Hadamard spaces, and a half space $\{z \in B \mid \rho(x,z) \le \rho(y,z)\}$ is convex for $x, y \in B$; see [1, 3, 10].

In the following theorem, we introduce a sequence generated by *the combining projection method of balanced type* using a finite mappings, and prove its convergence in a real Hilbert ball.

Theorem 4.1. *Let B be a real Hilbert ball with the metric with ρ, T_i a family of a quasinonexpansive and continuous mappings of B into itself for $i = 1, 2, \dots, N$ such that $F = \bigcap_{i=1}^{N} \operatorname{Fix} T_i \neq \varnothing$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^i \mid i = 1, 2, \dots, N, n \in \mathbb{N}\} \subset]0,1[$ such that $\sum_{i=1}^{N} \beta_n^i = 1$ for $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid n, k \in \mathbb{N}, k \le n\} \subset]0,1[$ such that $\sum_{k=1}^{n} \gamma_{n,k} = 1$ for $n \in \mathbb{N}$, and $\{\gamma_n \mid n \in \mathbb{N}\} \subset [0,1]$. Let $u \in B$ and define sequences $\{x_n\}$ and $\{y_n^i\}$, sequences $\{V_k\}$ and $\{U_n\}$ of mappings of B into itself, and a sequence $\{C_n^i\}$ of a subset of B by $x_1 \in B$ and*

$$\begin{aligned}
&y_n^i = \alpha_n x_n \oplus (1-\alpha_n) T_i x_n \ \textit{for } i = 1, 2, \cdots, N;\\
&C_n^i = \big\{z \in B \bigm| \rho(y_n^i, z) \le \rho(x_n, z)\big\} \ \textit{for } i = 1, 2, \dots, N \textit{ and } n \in \mathbb{N};\\
&V_k x = \operatorname*{Argmin}_{y \in B} \sum_{i=1}^{N} \beta_i^k \rho(P_{C_k^i} x_n, y)^2 \ \textit{for } k \le n \textit{ and } x \in X;\\
&U_n x = \operatorname*{Argmin}_{y \in B} \sum_{k=1}^{n} \gamma_{n,k} \rho(V_k x, y)^2 \ \textit{for } x \in X;\\
&x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n
\end{aligned}$$

for each $n \in \mathbb{N}$, where a mapping P_K is the metric projection of B into a nonempty closed convex subset K of B. Suppose the following conditions:

(i) $\liminf_{n\to\infty} \alpha_n < 1$;
(ii) $\beta_n^i > 0$ *for* $i = 1, 2, \dots, N$ *and* $n \in \mathbb{N}$;
(iii) $\lim_{n\to\infty} \gamma_{n,k} > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
(iv) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, $\{x_n\}$ is convergent to $P_F u$.

Proof. Since $\{z \in B \mid \rho(z,u) \le \rho(z,v)\}$ is convex for $u, v \in B$ and a metric ρ is continuous, C_n^i is closed and convex for $i = 1, 2, \dots, N$ and $n \in \mathbb{N}$. Let $p \in F$. Since T_i is quasinonexpansive for $i = 1, 2, \dots, N$, we have F is closed convex and we get

$$\rho(y_k^i, p) = \rho(\alpha_k x_k \oplus (1-\alpha_k) T_i x_k, p)$$

$$\leq \alpha_k \rho(x_k, p) + (1-\alpha_k)\rho(T_i x_k, p)$$
$$\leq \rho(x_k, p)$$

and hence $p \in C_k^i$ for $i \in 1, 2 \ldots, N$ and $k \in \mathbb{N}$. This implies that

$$\varnothing \neq \bigcap_{i=1}^{N} \operatorname{Fix} T_i \subset \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{N} C_k^i.$$

Since a metric projection $P_{C_k^i}$ is nonexpansive for $i = 1, 2, \ldots, N$ and $k \in \mathbb{N}$, and Theorem 2.1, we get V_k is nonexpansive for $k \in \mathbb{N}$ and

$$\operatorname{Fix} V_k = \bigcap_{i=1}^{N} \operatorname{Fix} P_{C_k^i} = \bigcap_{i=1}^{N} C_k^i$$

for $k \in \mathbb{N}$ and hence

$$\bigcap_{k=1}^{\infty} \operatorname{Fix} V_k = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{N} C_k^i \supset \bigcap_{i=1}^{N} \operatorname{Fix} T_i \neq \varnothing.$$

Put $C_0 = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{N} C_k^i$. Since Theorem 2.3, $\{x_n\}$ is convergent to $x_0 = P_{C_0} u$. Since $x_0 \in C_0$, we get

$$\rho(y_n^i, x_0) \leq \rho(x_n, x_0)$$

and hence $y_n^i \to x_0$ for $i = 1, 2, \ldots, N$. By (i), there exists $\{\alpha_{n_j}\} \subset \{\alpha_n\}$ such that $\lim_{j\to\infty} \alpha_{n_j} \in [0, 1[$. Then, it follows that

$$\rho(x_{n_j}, T_i x_{n_j}) = \frac{1}{1-\alpha_{n_j}} \rho(x_{n_j}, y_{n_j}^i) \leq \frac{1}{1-\alpha_{n_j}} (\rho(x_{n_j}, x_0) + \rho(x_0, y_{n_j}^i))$$

for $i = 1, 2, \ldots, N$. Letting $j \to \infty$, we get $\lim_{j\to\infty} \rho(x_{n_j} T_i x_{n_j}) = 0$. Since T_i is continuous for $i = 1, 2, \ldots, N$, we obtain

$$\rho(x_0, T_i x_0) \leq \rho(x_0, x_{n_j}) + \rho(x_{n_j}, T_i x_{n_j}) + \rho(T_i x_{n_j}, T_i x_0)$$

for $i = 1, 2, \ldots, N$ and $j \in \mathbb{N}$. Letting $j \to \infty$, we get $x_0 = T_i x_0$ for $i = 1, 2, \ldots, N$. This implies that $x_0 \in \bigcap_{i=1}^{N} \operatorname{Fix} T_i$. Consequently, we complete the proof. □

Remark. In Theorem 4.1, we can replace B to a Hadamard space by adding the condition that a half space

$$\{z \in X \mid d(z, x) \leq d(z, y)\}$$

is convex for $x, y \in X$.

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References

[1] M. Bačák, *Convex analysis and optimization in Hadamard space*, Walter de Gruyter, Boston, 2014.

[2] S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed points of uniformly lipschitzian mappings*, Nonlinear Anal. **65** (2006), 762–772.

[3] K. Goebel and S. Reich,*Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, vol. 83 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 1984.

[4] T. Hasegawa, *Convergence theorems with a balanced mapping on Hadamard spaces*, Master thesis, Toho University, 2019.

[5] T. Hasegawa and Y. Kimura, *Convergence to a fixed point of a balanced mapping by the Mann algorithm in a Hadamard space*, Linear and Nonlinear Anal. **4** (2018), 445–452.

[6] Y. Kimura and T. Ogihara, *Strong convergence theorems and a projection method using a balanced mapping in Hadamard spaces*, Filomat **37** (2023), 9287–9297.

[7] Y. Kimura, W. Takahashi and J.C. Yao, *Strong convergence of an iterative scheme by a new type of projection method for a family of quasinonexpansive mapping*, J Optim Theory Appl. **149** (2011), 239–253.

[8] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic space*, Nonlinear Anal. **68** (12) (2008), 3689–3696.

[9] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive groups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[10] I. Shafrir, *Theorems of ergodic type for ρ-nonexpansive mappings in the Hilbert ball*, Ann. Mat. Pure. Appl. **163** (1993), 313–379.

[11] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[12] Tan, K-K, Xu, HK, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178**, 301–308 (1993).