An approximation sequence and its limits by a projection method on a complete geodesic spaces
完備測地距離空間における射影法を用いた解近似とその極限

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1 Introduction

Let K be a nonempty subset of a metric space X and $f\colon K \times K \to \mathbb{R}$ a bifunction. An equilibrium problem is to find a point $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$. The set of solution to the equilibrium problem $\operatorname{Equil} f$, that is,

$$\operatorname{Equil} f = \left\{ x \in K \;\middle|\; \inf_{y \in K} f(x, y) \geq 0 \right\}.$$

Equilibrium problems were first studied intensively by Blum and Oettli on topological vector spaces and Banach spaces. They proposed a mapping called a resolvent of a bifunction for an equilibrium problem and showed that its domain is the whole space. Further, Combettes and Hirstoaga studied resolvents of equilibrium problems in Hilbert spaces and they obtained several important properties of the resolvent. The following is one of the most important theorems.

Theorem 1.1 (Combettes–Hirstoaga [1]). *Let H be a Hilbert space, and K a nonempty, closed convex subset of H. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4).*

(E1) $f(x, x) = 0$ *for all* $x \in K$*;*
(E2) $f(x, y) + f(y, x) \leq 0$ *for all* $x, y \in K$*;*
(E3) $f(x, \cdot)\colon K \to \mathbb{R}$ *is lower semicontinuous and convex for all* $x \in K$*;*
(E4) $f(\cdot, y)\colon K \to \mathbb{R}$ *is upper hemicontinuous for all* $y \in K$*.*

Define the resolvent J_f by

$$J_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) + \langle z - x, y - z \rangle) \geq 0 \right\}$$

for $x \in H$. Then J_f has the following properties:

1. *The domain of J_f is H;*
2. *J_f is single-valued and firmly nonexpansive;*
3. *the set of all fixed points of J_f coincides with* Equil f *and it is closed and convex.*

Let X be a metric space and assume that $C \subset X$ is nonempty, closed, and convex. Then, for all $x \in X$, there exists a unique point $z \in C$ such that

$$d(x,z) = \inf_{y \in C} d(x,y).$$

Using this point, we define a metric projection $P_C \colon X \to C$ by $P_C x = z$ for $x \in X$.

In 2008, Takahashi, Takeuchi, and Kubota prove a strong convergence theorem by shrinking projection method as follows:

Theorem 1.2 (Takahashi, Takeuchi, and Kubota [5])**.** *Let H be a Hilbert space and $C \subset H$ a nonempty closed convex set. Let $T \colon C \to C$ be a nonexpansive mapping such that $F(T) \neq \varnothing$. Let $u \in H$ and $\{\alpha_n\} \subset [0,1[$ be a sequence. Let $\{x_n\}$ be a sequence and $\{C_n\}$ subsets of H defined by $C_1 = C$, $x_1 \in C$ and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} &= \{ z \in C \mid \|y_n - z\| \leq \|x_n - z\| \} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} u. \end{aligned}$$

Then, $x_n \to P_{F(T)} u \in C$.

Further in 2023, Kimura proved the Δ-convergence theorem by modified shrinking projection method for a nonexpansive mapping in CAT(0) space.

In this paper, we obtain a strong convergence theorem of an iterative sequence to a solution to an equilibrium problem on a CAT(1) space. We use projection method to generate the approximate sequence.

2 Preliminaries

Let X be a metric space. For $x, y \in X$ we define a geodesic between these points by a mapping $c \colon [0, d(x,y)] \to X$ such that $c(0) = x$, $c(d(x,y)) = y$, and $d(c(u), c(v)) = |u - v|$ for any $u, v \in [0, d(x,y)]$. We say X is uniquely π-geodesic if for any $x, y \in X$ satisfying $d(x,y) < \pi$, there exists a unique geodesic c between these points. In this case, we can define a convex combination between $x, y \in X$ if $d(x,y) < \pi$. That is, for such $x, y \in X$ and $t \in [0,1]$, we define $tx \oplus (1-t)y = c((1-t)d(x,y))$. A π-geodesic space X is called a CAT(1) space if

$$\cos d(tx \oplus (1-t)y, z) \sin d(x,y) \geq \cos d(x,z) \sin td(x,y) + \cos d(y,z) \sin(1-t)d(x,y)$$

for all $x, y \in X$ with $d(x,y) < \pi$ and $t \in [0,1]$. We say X is admissible if $d(u,v) < \pi/2$ for all $u, v \in X$.

Let X be a metric space and $T\colon X \to X$. If the point $x \in X$ satisfying $x = Tx$, then x is called a fixed point of T and we denote the set of all fixed points of T by $F(T)$. An admissible complete CAT(1) space X has the convex hull finite property if every continuous selfmapping on $\operatorname{cl} \operatorname{co} E$ has a fixed point for every finite subset E of X, where $\operatorname{cl} \operatorname{co} E$ is the closure of the convex hull of X.

Theorem 2.1 (Kimura [2]). *Let X be an admissible complete CAT(1) space having the convex hull finite property and $K \subset X$ a nonempty closed convex set. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies the conditions (E1)–(E4). Define the resolvent R_f of f by*

$$R_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z,y) - \log \cos d(x,y) + \log \cos d(x,z)) \geq 0 \right\}$$

for all $x \in X$. Then the following hold:

1. *$R_f\colon X \to K$ is defined as a single-valued mapping;*
2. *R_f satisfies the following inequality for any $x, y \in X$:*

$$\frac{\cos d(x, R_f y)}{\cos d(x, R_f x)} + \frac{\cos d(y, R_f x)}{\cos d(y, R_f y)} \leq 2 \cos d(R_f x, R_f y);$$

3. *$F(R_f) = \operatorname{Equil} f$ and it is closed and convex.*

Let X be CAT(1) space and $\{x_n\} \subset X$ a sequence. An asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{ z \in X \;\middle|\; \inf_{x \in X} \limsup_{n \to \infty} d(x, x_n) = \limsup_{n \to \infty} d(z, x_n) \right\}.$$

A sequence $\{x_n\} \subset X$ is said to be Δ-convergent to $x_0 \in X$ if $\mathrm{AC}(\{x_{n_i}\}) = \{x_0\}$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. It is denoted by $x_n \stackrel{\Delta}{\rightharpoonup} x_0$.

Lemma 2.1 (Kimura and Oguchi [4]). *Let X be an admissible complete CAT(1) space having the convex hull finite property. Let K is a nonempty closed convex subset of X, and suppose that $f\colon K \times K \to \mathbb{R}$ satisfies (E1)–(E4). Let $\{\lambda_n\} \subset \,]0, \infty[$ and $\{x_n\} \subset X$ be sequences satisfying $\inf_n \lambda_n > 0$, $x_n \stackrel{\Delta}{\rightharpoonup} x_0$ and $d(x_n, R_{\lambda_n f} x_n) \to 0$. Then, $x_0 \in \operatorname{Equil} f$.*

3 Main Result

We show a convergence theorem for an equilibrium problem defined on an admissible complete CAT(1) space.

Theorem 3.1. *Let X be an admissible complete* $\mathrm{CAT}(1)$ *space having the convex hull finite property and $C \subset X$ a nonempty closed convex set. Suppose that $f\colon C \times C \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$. *Let $\{\lambda_n\} \subset]0, \infty[$ be a sequence satisfying* $\inf_n \lambda_n > 0$. *Suppose that $\{z \in X \mid d(x, z) \leq d(y, z)\}$ is convex for all $x, y \in X$. Let $R_{\lambda_n f}\colon X \to C$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence and C_n a convex subset of X defined by $x_1 \in C$, $C_1 = C$, and*

$$
\begin{aligned}
&C_{n+1} = \{z \in C \mid d(R_{\lambda_n f} x_n, z) \leq d(x_n, z)\} \cap C_n,\\
&x_{n+1} = P_{C_{n+1}} x_n
\end{aligned}
$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ is well-defined and $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$.

Proof. We first show by induction that $F(R_{\lambda_n f} x) \subset C_n$ and x_n is well-defined for all $n \in \mathbb{N}$. $F(R_{\lambda_1 f}) \subset C_1$ is obvious. Suppose $F(R_{\lambda_k f}) \subset C_k$. Let $x \in C$ and $z \in F(R_{\lambda_k f})$. Then

$$
\frac{\cos d(x, R_{\lambda_k f} z)}{\cos d(x, R_{\lambda_k f} x)} + \frac{\cos d(z, R_{\lambda_k f} x)}{\cos d(z, R_{\lambda_k f} z)} \leq 2 \cos d(R_{\lambda_k f} x, R_{\lambda_k f} z).
$$

Since $z \in F(R_{\lambda_k f})$, we have $\cos d(R_{\lambda_k f} x, z) \cos d(x, R_{\lambda_k f} x) \geq \cos d(x, z)$ and thus $d(x, R_{\lambda_k f} x) \leq d(x, z)$. This implies that $z \in C_{k+1}$ and hence $F(R_{\lambda_{k+1} f}) \subset C_{k+1} \neq \varnothing$. Therefore $F(R_{\lambda_n f}) \subset C_n \neq \varnothing$ for all $n \in \mathbb{N}$. Further, we know that C_n is closed and convex for all $n \in \mathbb{N}$ and thus $\{x_n\}$ is well-defined.

Next, we show that $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$. Let $u \in F(R_{\lambda_n f})$. Since $\{P_{C_n}\}$ is quasi-nonexpansive, $d(x_{n+1}, u) = d(P_{C_n+1}, u) \leq d(x_n, u)$ and thus $\{d(x_n, u)\}$ converges to some $c \in [0, \pi/2[$. From the definition of CAT(1) space, we have

$$
\begin{aligned}
&\cos d(x_{n+1}, x_n) \sin d(u, x_{n+1})\\
&= \cos d(P_{C_{n+1}} x_n, x_n) \sin d(u, P_{C_{n+1}} x_n)\\
&\geq \cos d(tu \oplus (1-t) P_{C_{n+1}} x_n, x_n) \sin d(u, P_{C_{n+1}} x_n)\\
&\geq \cos d(u, x_n) \sin t d(u, P_{C_{n+1}} x_n) + \cos d(P_{C_{n+1}} x_n, x_n) \sin(1-t) d(u, P_{C_{n+1}} x_n)\\
&= \cos d(u, x_n) \sin t d(u, x_{n+1}) + \cos d(x_{n+1}, x_n) \sin(1-t) d(u, x_{n+1})
\end{aligned}
$$

and hence

$$
(\sin d(u, x_{n+1}) - \sin(1-t) d(u, x_{n+1})) \cos d(x_{n+1}, x_n) \geq \cos d(u, x_{n+1}) \sin t d(u, x_{n+1}).
$$

Using sum to product formulas, we have

$$
\begin{aligned}
&(\sin d(u, x_{n+1}) - \sin(1-t) d(u, x_{n+1})) \cos d(x_{n+1}, x_n)\\
&\qquad\qquad = 2 \cos \frac{(2-t) d(u, x_{n+1})}{2} \sin \frac{t d(u, x_{n+1})}{2} \cos \frac{t}{2} d(x_n, x_{n+1}).
\end{aligned}
$$

From half angle formulas, we get

$$\begin{aligned}&2\cos\frac{(2-t)d(u,x_{n+1})}{2}\sin\frac{td(u,x_{n+1})}{2}\cos d(x_n,x_{n+1})\\&\geq \cos d(u,x_n)\sin d(u,x_{n+1})\\&=\cos d(u,x_n)2\sin\frac{t}{2}d(u,x_{n+1})\cos\frac{t}{2}d(u,x_{n+1}).\end{aligned}$$

This implies that

$$\cos\frac{(2-t)d(u,x_{n+1})}{2}\cos d(x_n,x_{n+1})\geq \cos d(u,x_n)\cos\frac{t}{2}d(u,x_{n+1}),$$

and letting $t\to 0$, we get

$$\cos d(u,x_{n+1})\cos d(x_n,x_{n+1})\geq \cos d(u,x_n).$$

Since $d(x_n,u)\to c$,

$$1\geq \cos d(x_{n+1},x_n)\geq \frac{\cos d(u,x_n)}{\cos d(u,x_{n+1})}\to\frac{\cos c}{\cos c}=1$$

as $n\to\infty$, and thus $d(x_{n+1},x_n)\to 0$. Since $x_{n+1}\in C_{n+1}$ and $d(x_{n+1},x_n)\to 0$, we have

$$\begin{aligned}0\leq d(R_{\lambda_n f}x_n,x_n)&\leq d(R_{\lambda_n f}x_n,x_{n+1})+d(x_{n+1},x_n)\\&\leq d(x_n,x_{n+1})+d(x_{n+1}+x_n)\to 0\end{aligned}$$

and hence

$$d(R_{\lambda_n f}x_n,x_n)\to 0.$$

Since $\{x_n\}$ is spherically bounded, every subsequence $\{x_{n_i}\}$ of $\{x_n\}$ is spherically bounded. Let $\{x_0\}=\mathrm{AC}(\{x_n\})$ and $\{w_0\}=\mathrm{AC}(\{x_{n_i}\})$. We can take a subsequence $\{x_{n_{i_j}}\}\subset\{x_{n_i}\}$ such that Δ-convergence to some $z_0\in C$. Since $d(R_{\lambda_n f}x_n,x_n)\to 0$, by Lemma 2.1 we get $z_0\in\operatorname{Equil}f$. From the definition of asymptotic center, we get

$$\begin{aligned}\limsup_{n\to\infty}d(x_n,z_0)&=\limsup_{j\to\infty}d(x_{n_{i_j}},z_0)\\&\leq\limsup_{j\to\infty}d(x_{n_{i_j}},w_0)\\&\leq\limsup_{i\to\infty}d(x_{n_i},w_0)\\&\leq\limsup_{i\to\infty}d(x_{n_i},x_0)\\&\leq\limsup_{n\to\infty}d(x_n,x_0)\leq\limsup_{n\to\infty}d(x_n,z_0).\end{aligned}$$

Therefore $x_0=w_0=z_0$ and thus we get $\{x_0\}=\mathrm{AC}(\{x_{n_i}\})$ for all $\{x_{n_i}\}\subset\{x_n\}$. Consequently, $x_n\xrightarrow{\Delta}x_0\in\operatorname{Equil}f$. □

Theorem 3.2. *Let X be an admissible complete* $\mathrm{CAT}(1)$ *space having the convex hull finite property and $C \subset X$ a nonempty closed convex set. Suppose that $f\colon C \times C \to \mathbb{R}$ satisfies the conditions (E1)–(E4), and* $\operatorname{Equil} f \neq \varnothing$. *Let $\{\lambda_n\} \subset \,]0, \infty[$ be a sequence satisfying* $\inf_n \lambda_n > 0$. *Suppose that $\{z \in X \mid d(x, z) \leq d(y, z)\}$ is convex for all $x, y \in C$. Let $R_{\lambda_n f}\colon X \to C$ be the resolvent of $\lambda_n f$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence and C_n be a convex subset of X defined by $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned} C_{n+1} &= \{z \in C \mid d(R_{\lambda_n f} x_n, z) \leq d(x_n, z)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x_n \end{aligned}$$

for each $n \in \mathbb{N}$. Let $D \subset X$ be closed and convex such that $\operatorname{Equil} f \subset D \subset \bigcap_{n=1}^{\infty} C_n$. *Then, $P_D x_n \to x_0$, where x_0 is the Δ-limit of $\{x_n\}$.*

Proof. We suppose that $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$ and show $P_D x_n \to x_0$. Since

$$d(P_D x_{n+1}, x_{n+1}) \leq d(P_D x_n, x_{n+1}) = d(P_D x_n, P_{C_{n+1}} x_n) \leq d(P_D x_n, x_n),$$

$\{\cos d(P_D x_n, x_n)\}$ is a Cauchy sequence. Hence, there exists $\{\alpha_n\} \subset \mathbb{R}$ such that $\alpha_n \to 0$ and $\cos d(P_D x_m, x_m) - \cos d(P_D x_n, x_n) \leq \alpha_n$ if $m \geq n$. We have

$$\frac{\alpha_n}{\cos d(P_D x_m, x_m)} \geq \frac{\cos d(P_D x_m, x_m)}{\cos d(P_D x_m, x_m)} - \frac{\cos d(P_D x_n, x_n)}{\cos d(P_D x_m, x_m)},$$

and thus

$$\frac{\cos d(P_D x_n, x_n)}{\cos d(P_D x_m, x_m)} \geq 1 - \frac{\alpha_n}{\cos d(P_D x_m, x_m)} \geq 1 - \frac{\alpha_n}{\cos d(P_D x_1, x_1)}.$$

Further, since $\cos d(P_D x_m, x_m) \cos d(P_D x_m, P_D x_n) \geq \cos d(P_D x_n, x_m)$, we have

$$\begin{aligned} \cos d(P_D x_m, P_D x_n) &\geq \frac{\cos d(P_D x_n, x_m)}{\cos d(P_D x_m, x_m)} \\ &\geq \frac{\cos d(P_D x_n, x_n)}{\cos d(P_D x_m, x_m)} \geq 1 - \frac{\alpha_n}{\cos d(P_D x_1, x_1)}. \end{aligned}$$

Hence, $\{P_D x_n\}$ is a Cauchy sequence and we get $\{P_D x_n\}$ converges to some $y_0 \in C$. Since $x_n \xrightarrow{\Delta} x_0 \in \operatorname{Equil} f$, we have

$$\begin{aligned} \limsup_{n\to\infty} d(y_0, x_n) &\leq \limsup_{n\to\infty} (d(y_0, P_D x_n) + d(P_D x_n, x_n)) \\ &= \limsup_{n\to\infty} d(P_D x_n, x_n) \leq \limsup_{n\to\infty} d(x_0, x_n). \end{aligned}$$

Therefore, $y_0 = x_0$ and we get $P_D x_n \to x_0 \in \operatorname{Equil} f$. $\square$

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