STRONG CONVERGENCE THEOREMS FOR A SOLUTION OF SPLIT FEASIBILITY PROBLEM IN BANACH SPACES

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Abstract. We treat with Split Feasibility Problem in Banach spaces. We show diverse characterizations of a solution of SFP with a metric projection, a generalized projection and a sunny generalized nonexpansive retraction. Then we show strong convergence theorems for a solution of SFP by hybrid methods with these projections. These theorems imply that the convergence to a point of solutions of SFP also mean the convergence to a common fixed point of some nonlinear mappings.

1. Introduction

Let E and F be Banach spaces, let $A : E \to F$ be a bounded linear operator, and let C and D be convex and closed subsets of E and F, respectively. Then we called the following problem Split Feasibility Problem (SFP):
Find a point

$$z \in C \cap A^{-1}(D).$$

In the case of finite dimensional linear spaces, C. Byrne treated the following iterative algorithm with metric projections ([6]);

$$x_{n+1} = P_C(x_n + rA^T(P_D - I)Ax_n) \to z \in C \cap A^{-1}(D),$$

where $n \in \mathbb{N}$, A is represented as a matrix which can be selected to impose consistency with measured data and, P_C and P_D are metric projections to C and D, respectively.

Characterization of a solution of SFP and a strong convergence theorem for a solution of SFP with metric projections are given by W. Takahashi ([19], 2014), as follows:

Theorem 1.1. *For closed convex subsets C and D of uniformly convex and smooth Banach spaces E and F, resp. Let $A : E \to F$ be a bounded linear operator. Then the following are equivalent:*

$$(i) \quad z \in C \cap A^{-1}(D)$$
$$(ii) \quad z = P_C(I_E - rJ_E^{-1}A^*J_F(I_F - P_D)A)z,$$

where P_C and P_D are metric projections, I_E and I_D are identity mappings, J_E and J_F are duality mappings on E and F, respectively.

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Theorem 1.2. *Let E and F be a uniformly convex and smooth Banach space, let J_E and J_F be duality mappings on E and F, respectively, let C and D be nonempty, closed and convex subsets of E and F, respectively, let $A : E \to F$ be a bounded and linear operator with $A \neq 0$, let A^* be the adjoint operator of A and let $r \in (0, \infty)$. Suppose that $C \cap A^{-1}(D) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n & = & (I_E - rJ^{-1}A^*J_F(I_F - P_D)A)x_n, \\ C_n & = & \{z \in C : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n & = & \{z \in Q_{n-1} : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} & = & P_{C_n \cap Q_n} x_1 \end{cases}$$

for any $n \in \mathbf{N}$, where $D_0 = C$. Then $\{x_n\}$ is strongly convergent to a point $z_0 \in C \cap A^{-1}(D)$ for any $r \in (0, r_{\|A\|^2})$, where $z_0 = P_{C \cap A^{-1}(D)} x_1$.

In this paper, we show some diverse extensions of Takahashi's result with not only a metric projection, but also a generalized projection and a sunny generalized nonexpansive retraction. At first we show diverse characterizations of a solution of SFP by these projections. Then we show strong convergence theorems for a solution of SFP by using hyblid methods with these projections.

2. Preliminaries

For a closed and convex subset C of a Banach space, we introduce definitions and properties of a metric projection P_C, a generalized projection Π_C and a sunny generalized nonexpansive retraction R_C.

A metric projection P_C is defined by the following.

Definition 2.1. Let E be a strictly convex and smooth Banach space, and C a nonempty, closed and convex subset of E. For any $x \in E$, there exists a unique element $z \in C$ such that

$$\|x - z\| = \min_{y \in C} \|x - y\|,$$

and so we denote

$$z = P_C(x)$$

and we call P_C a metric projection of E onto C.

A generalized projection Π_C is defined by a following function $V(\cdot, \cdot)$.

Definition 2.2. Let E be a smooth Banach space, and for all x, $y \in E$ we denote

$$V(x, y) = \|x\|^2 + \|y\|^2 - 2\langle x, J(y)\rangle,$$

where J is a normalized duality mapping, that is,

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2, \quad \|x^*\| = \|x\|\}$$

and $\langle , \rangle$ is a duality pair.

Remark 1. $J(x)$ is a singleton for any $x \in E$ when E is a smooth Banach space. It is trivial that $0 \leq (\|x\| - \|y\|)^2 \leq V(x, y)$ for all $x, y \in E$. It is well-known that if E is a smooth and strictly convex Banach space, $V(x, y) = 0 \iff x = y$. If E is a Hilbert space, $V(x, y) = \|x - y\|^2$. In a general Banach space, this function V

is a special version of Bregman distance $D_f(x,y)$ with respect to $f = \|\cdot\|^2$, where $D_f(x,y) = f(x) - f(y) - \langle x - y, \bigtriangledown f(y)\rangle$ and $\bigtriangledown$ is a gradient.

There are many mappings which are characterized by this function V. We give two examples, as follows.

Example 1. A mapping $T : E \to E$ is called a generalized nonexpansive if for any fixed point p of T

$$V(Tx,p) \le V(x,p)$$

for any $x \in E$. There are a lot of results with respect to convergence theorems for fixed points of T. (cf. [9])

Example 2. A mapping $T : E \to E$ is called a V-strongly nonexpansive mapping if there exists a $\lambda > 0$ such that the following inequality holds for all $x, y \in E$

$$V(Tx,Ty) \le V(x,y) - V((I-T)x,(I-T)y),$$

where I is an identity function on E.

Remark 2. We also obtained some convergence theorems for a common fixed point with respect to this V-strongly nonexpansive mapping and maximal monotone operators in Banach spaces (see [11]), and it is showed that this mapping is a firmly nonexpansive in a Hilbert space, however it is not a nonexpansive mapping in some Banach space (see [13]).

Now we give a definition of a generalized projection Π_C.

Definition 2.3. Let E be a smooth and strictly convex Banach space, and C a nonempty, closed and convex subset of E. For any $x \in E$, since there exists a unique element $z \in C$ such that

$$V(z,x) = \min_{y \in C} V(y,x),$$

we denote this z by $\Pi_C(x)$, *i.e.* $z = \Pi_C(x)$, and we call the $\Pi_C : E \to C$ a generalized projection of E onto C.

At the last, we give a definition and properties of a sunny generalized nonexpansive retraction onto C.

Definition 2.4. Let E be a Banach space and C a nonempty subset of E. A mapping R from E onto C is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for any $x \in E$ and any $t \in [0,\infty)$. If $Rx = x$ for any $x \in C$,
the mapping R is called a retraction

The following lemmas of the properties with respect to the three projections are well-known (cf. [9]).

Lemma 2.1. *Let E be a strictly convex, reflexive and smooth Banach space, and $C \subset E$ a nonempty closed subset. Then the followings are equivalent:*
(i) *There exists a generalized nonexpansive retraction $R : E \to C$,*

(ii) There exists a sunny generalized nonexpansive retraction $R : E \to C$*,*
(iii) $J(C)$ *is closed and convex.*

We denote a sunny generalized nonexpansive retraction $E \to C$ *by* R_C*.*
The followings are equivalent:
(iv) $\quad z = R_C(x) \iff V(x,z) = \min_{y \in C} V(x,y)$

Lemma 2.2. *Let* E *be a strictly convex and smooth Banach space, let* $C \subset E$ *be nonempty and closed. Let* J *be a duality mapping on* E*.*
(A) For any $(x,z) \in E \times C$*,*

$$z = P_C(x) \Longleftrightarrow \langle z-y, J(x-z) \rangle \geq 0 \quad (y \in C)$$

(B) For any $(x,z) \in E \times C$*,*

$$z = \Pi_C(x) \Longleftrightarrow \langle z-y, Jx - Jz \rangle \geq 0 \quad (y \in C)$$

(C) For any $(x,z) \in E \times C$*,*

$$z = R_C(x) \Longleftrightarrow \langle z-x, Jy - Jx \rangle \geq 0 \quad (y \in C)$$

In the next section, we shall show diverse extensions of Takahashi's result in 2014, by these three projections.

3. Our Results

In this section we show our results with respect to the characterizations of a solution of SFP and the strong convergence theorems by the previous three projections. For the sake of proofs we treat the problem under the following situation ♣ .

♣ : Let E and F be strictly convex, reflexive and smooth Banach spaces,
I_E and I_F identity mappings on E and F, respectively.
J_E and J_F duality mappings on E and F, respectively.
$A : E \to F$ a bounded linear operator, A^* the adjoint operator of A,
$C \subset E$ and $D \subset F$ nonempty, closed and convex subsets,
P_C, Π_C and R_C be a metric projection, a generalized projection and a generalized retraction onto C, respectively.

Then we obtain diverse characterizations of a solution of SFP by the metric projection, the generalized projection and the sunny generalized nonexpansive retraction ([14]), as follows.

Theorem 3.1. ([14]) *Suppose that* $C \cap A^{-1}(D) \neq \emptyset$ *and* $r \in (0,\infty)$ *under* ♣*. The followings are equivalent:*

$$\begin{aligned}
&(1) \quad z \in C \cap A^{-1}(D)\\
&(2) \quad z = P_C(I_E - rJ_E^{-1}A^*J_F(I_F - P_D)A)z\\
&(3) \quad z = P_C(I_E - rJ_E^{-1}A^*(J_F - J_F\Pi_D)A)z\\
&(4) \quad z = \Pi_C J_E^{-1}(J_E - rA^*J_F(I_F - P_D)A)z\\
&(5) \quad z = \Pi_C J_E^{-1}(J_E - rA^*(J_F - J_F\Pi_D)A)z
\end{aligned}$$

Theorem 3.2. ([14]) *Suppose that $C \cap A^{-1}(D) \neq \emptyset$ and $r \in (0, \infty)$ under ♣. If $J_F(D)$ is closed, the followings are equivalent:*

$$\begin{aligned} &(1) \quad z \in C \cap A^{-1}(D) \\ &(6) \quad z = P_C(I_E - rJ_E^{-1}A^*(I_{F^*} - R_{J_F(D)}J_F A)z \\ &(7) \quad z = \Pi_C J_E^{-1}(J_E - rA^*(I_{F^*} - R_{J_F(D)})J_F A)z \end{aligned}$$

If $J_E(C)$ is closed, the following are equivalent to (1) :

$$\begin{aligned} &(8) \quad z = J_E^{-1}R_{J_E(C)}(J_E - rA^*J_F(I_F - P_D)A)z \\ &(9) \quad z = J_E^{-1}R_{J_E(C)}(J_E - rA^*(J_E - J_F\Pi_D)A)z \\ &(10) \quad z = J_E^{-1}R_{J_E(C)}(J_E - rA^*(I_{F^*} - R_{J_F(D)})J_F A)z, \end{aligned}$$

where (10) *also needs the condition that $J_F(D)$ is closed.*

Next we show the strong convergence theorems for a solution of SFP by hyblid methods with the generalized projection and the sunny generalized nonexpansive retraction, which extend Takahashi's Theorem for diverse projections methods. For the sake of the proofs we need Xu's results ([21]).

Lemma 3.1. ([21]) *Let E be a Banach space, then the following are equivalent:*
(i) *E is uniformly convex*
(ii) *For all $p \in (1, \infty)$ and $\rho \in (0, \infty)$, there exists a continuous, strictly increasing and convex function $g_{p,\rho} : [0, \infty) \to [0, \infty)$ such that $g_{p,\rho}(0) = 0$ and*

$$\langle x - y, x^* - y^* \rangle \geq g_{p,\rho}(\|x - y\|)$$

for all $x, y \in B_\rho(E)$, $x^* \in J_p(x)$ and $y^* \in J_p(y)$.
Where

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \quad \|x^*\| = \|x\|^{p-1}\}.$$

Lemma 3.2. ([21]) *Let E be a smooth Banach space. Then the following are equivalent:*
(iii) *E is uniformly smooth*
(iv) *For all $q \in (1, \infty)$ and $\rho \in (0, \infty)$, there exists a continuous, strictly increasing and convex function $g^*_{q,\rho} : [0, \infty) \to [0, \infty)$ such that $g_{q,\rho}(0) = 0$ and*

$$\langle x - y, J_q(x) - J_q(y) \rangle \leq g^*_{q,\rho}(\|x - y\|)$$

for all $x, y \in B_\rho(E)$. The $J_q(x)$ is a singleton;

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \quad \|x^*\| = \|x\|^{q-1}\}.$$

By these lemmas, we establish the following conditions ♠1 and ♠2 for our purpose, with respect to a Banach space we treat on theorems.

♠1 : It is a uniformly convex and smooth Banach space, and it satisfies

$$\inf_{\rho \in (0,\infty)} g_{2,\rho}(1) > 0.$$

$\spadesuit 2$: It is a uniformly smooth Banach space, and there exists $r_0 \in (0,\infty)$ and $\alpha \in (1,\infty)$ such that

$$\sup_{r\in(0,r_0),\rho\in(0,\infty)} \frac{g^*_{2,\rho}(r)}{r^\alpha} < \infty.$$

Examples satisfying these conditions are given in [14].

Now we show the extensions of Takahashi's convergence theorem by hybrid methods with the generalized projection and the sunny generalized nonexpansive retraction. First we show a theorem with the generalized projection.

Theorem 3.3. ([14]) *Let E be a uniformly convex and smooth Banach space, let F be a strictly convex, reflexive and uniformly smooth Banach space, J_E and J_F duality mappings on E and F, resp., $C \subset E$ and $D \subset F$ nonempty, closed and convex subsets, $A : E \to F$ a bounded and linear operator with $A \neq 0$, A^* the adjoint operator of A, and let $r \in (0,\infty)$. Suppose E^* and F^* satisfy the condition $\spadesuit 2$ and $\spadesuit 1$, respectively, and $C \cap A^{-1}(D) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n &= J_E^{-1}(J_E - rA^*(J_F - J_F\Pi_D)A)x_n, \\ C_n &= \{z \in C : \langle z_n - z, J_Ex_n - J_Ez_n\rangle \geq 0\}, \\ D_n &= \{z \in D_{n-1} : \langle x_n - z, J_Ex_1 - J_Ex_n\rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n\cap D_n}x_1 \end{cases}$$

for any $n \in \mathbf{N}$, where $D_0 = C$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}(D)$ for any $r \in (0, r_{\|A\|^2})$, where $z_0 = \Pi_{C\cap A^{-1}(D)}x_1$.

Next we show a theorem with the sunny generalized nonexpansion retraction.

Theorem 3.4. ([14]) *Assume the condition of Theorem 3.3 holds, with respect to $E, F, J_E, J_F, C \subset E, D \subset F$. And suppose $J_F(D)$ is closed , let I_{F^*} be the identity mapping on F^*, $A : E \to F$ a bounded and linear operator with $A \neq 0$, A^* the adjoint operator of A, and let $r \in (0,\infty)$. Suppose E^* and F^* satisfy the condition $\spadesuit 2$ and $\spadesuit 1$, resp., and $C \cap A^{-1}(D) \neq \emptyset$. Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated for any $n \in \mathbf{N}$ as following; let $D_0 = C$ and*

$$\begin{cases} z_n &= J_E^{-1}(J_E - rA^*(I_{F^*} - R_{J_F(D)})J_FA)x_n, \\ C_n &= \{z \in C : \langle z_n - z, J_Ex_n - J_Ez_n\rangle \geq 0\}, \\ D_n &= \{z \in D_{n-1} : \langle x_n - z, J_Ex_1 - J_Ex_n\rangle \geq 0\}, \\ x_{n+1} &= J_E^{-1}R_{J_E(C_n\cap D_n)}J_Ex_1 \end{cases}$$

Then, there exists $r_{\|A\|^2} \in (0,\infty)$ such that $x_n \to z_0 \in C \cap A^{-1}(D)$ for any $r \in (0, r_{\|A\|^2})$, where $z_0 = J_E^{-1}R_{J_E(C\cap A^{-1}(D))}J_Ex_1$.

Remark 3. Theorem 3.3 and 3.4 also imply that their $\{x_n\}$s converge strongly to a point of common fixed points of (2), (3), (4), (5) and (6) of Theorems 3.1 and 3.2.

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