Fibonacci optimization and its related field
— duality — (II)

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Abstract

We consider a $2n$-variable parametric minimization problem, where a parameter $\lambda > 0$. Then it holds that the parametric minimization problem derives two problems, which are λ-parametric minimization problem (primal) and λ-parametric maximization problem (dual). Both the optimal solutions are expressed in terms of *Gibonacci* sequence, which is a parametric generalization of the *Fibonacci* one. Either solution is characterized by the backward Gibonacci sequence and its complementary – *Hibonacci* sequence –. In particular, when a parameter $\lambda = 1$, we show that Gibonacci sequence and Hibonacci sequence are represented by *Fibonacci* number. Moreover, for $\lambda = 4$, both the sequences are represented by *Sibonacci* number.

1 Introduction

Recently, in [23, 24], Iwamoto and Kimura show that a *parametric linear system of equations* plays a fundamental part in establishing a mutual relation between minimization problem (primal) and maximization problem (dual). The system is of $2n$-equation on $2n$-variable, called *zero-minimum condition*. It yields a couple of second-order finite (n-) linear difference equations on n-variable, which constitute the respective *optimal conditions*. The respective equations have a minimum solution for primal and a maximum one for dual. Both the optimal solutions are expressed in terms of *Gibonacci* sequence, which is a parametric generalization of the *Fibonacci* one. Either solution is characterized by the backward Gibonacci sequence and its complementary – *Hibonacci* sequence –.

As a historical background, see (i) Bellman and others [1–7, 28], [9, 11, 30, 31] for dynamic optimization, (ii) Iwamoto, Kimura, Fujita and Kira [12–22, 25–27] for complementary duality, and (iii) [8, 10, 29, 32] for Fibonacci number.

In this paper, we consider a $2n$-variable parametric minimization problem (Q), where a parameter $\lambda > 0$. Then it holds that the problem derives two problems, which are λ-parametric minimization problem (primal) and λ-parametric maximization problem (dual). In the case of $\lambda = 1$, we show that both Gibonacci and Hibonacci sequences are represented by *Fibonacci* number. Thus it turns out that the optimal solutions of the primal and dual are represented by Fibonacci number. Moreover, in the case of $\lambda = 4$, both the sequences are represented by *Sibonacci* number. Similarly the optimal solutions are represented by Sibonacci number.

Section 2 gives a $2n$-variable parametric minimization problem, where a parameter λ ranges over $(0, \infty)$. The objective function turns out to be nonnegative. It attains zero iff a linear system of $2n$-equations on $2n$-variables has a solution. Section 3 presents a pair of λ-parametric minimization problem and λ-parametric maximization problem for $\lambda = 1$. In Section 4, we discuss λ-parametric optimization problems for $\lambda = 4$.

2 Complementary approach

Let $c \in R^1$ be a given constant. The first minimization problem has a fixed initial state $x_0 = c$:

$$
\text{Q} \quad
\begin{aligned}
\text{minimize} \quad & -2\lambda x_0\mu_1 + \sum_{k=1}^{n-1} \big[(x_{k-1} - x_k)^2 + x_k^2 + \lambda^2\mu_k^2 + (\mu_k - \mu_{k+1})^2 \\
& \qquad\qquad + 2(\lambda - 1)x_k(\mu_k - \mu_{k+1})\big] \\
& \qquad + (x_{n-1} - x_n)^2 + x_n^2 + (\lambda^2 + 1)\mu_n^2 + 2(\lambda - 1)x_n\mu_n \\
\text{subject to} \quad & \text{(i)} \ \ x \in R^n, \ x_0 = c, \quad \text{(ii)} \ \ \mu \in R^n.
\end{aligned}
$$

Let us define the objective function by $h : R^n \times R^n \to R^1$

$$
\begin{aligned}
h(x, \mu) \ = \ & -2\lambda c\mu_1 + \sum_{k=1}^{n-1} \big[(x_{k-1} - x_k)^2 + x_k^2 + \lambda^2\mu_k^2 + (\mu_k - \mu_{k+1})^2 \\
& \qquad + 2(\lambda - 1)x_k(\mu_k - \mu_{k+1})\big] \\
& \qquad + (x_{n-1} - x_n)^2 + x_n^2 + (\lambda^2 + 1)\mu_n^2 + 2(\lambda - 1)x_n\mu_n.
\end{aligned}
$$

We have an *evaluation* as follows.

Lemma 1 [23, 24] *Let $(x,\ \mu)$ be feasible. Then it holds that*

$$h(x, \mu) \ \geq \ 0. \tag{1}$$

The sign of equality holds iff

$$
\text{(Zm)} \qquad
\begin{array}{lll}
c - x_1 \ = \ \lambda\mu_1, & x_1 \ = \ \mu_1 - \mu_2 & \\
x_{k-1} - x_k \ = \ \lambda\mu_k, & x_k \ = \ \mu_k - \mu_{k+1} & 2 \leq k \leq n-1 \\
x_{n-1} - x_n \ = \ \lambda\mu_n, & x_n \ = \ \mu_n &
\end{array}
$$

holds.

This is a linear system of $2n$-equation on $2n$-variable $(x,\ \mu)$. We call (Zm) a *zero-minimum condition.*

Lemma 2 [23, 24] *Let*

$$\gamma := 2 + \lambda, \quad \xi := 1 + \lambda \quad (\lambda \neq 0).$$

Then the zero-minimum condition (Zm) *yields a pair of linear systems of n-equation on n-variable:*

Case $n = 1$

$$(\mathrm{EQ}) \qquad c = \xi x_1 \qquad c = \xi \mu_1.$$

Case $n = 2$

$$(\mathrm{EQ}) \qquad \begin{aligned} c &= \gamma x_1 - x_2 \\ x_1 &= \xi x_2 \end{aligned} \qquad \begin{aligned} c &= \xi \mu_1 - \mu_2 \\ \mu_1 &= \gamma \mu_2. \end{aligned}$$

Case $n \geq 3$

$$(\mathrm{EQ}) \qquad \begin{aligned} c &= \gamma x_1 - x_2 \\ x_{k-1} &= \gamma x_k - x_{k+1} \\ x_{n-1} &= \xi x_n \end{aligned} \qquad \begin{aligned} c &= \xi \mu_1 - \mu_2 \\ \mu_{k-1} &= \gamma \mu_k - \mu_{k+1} \\ \mu_{n-1} &= \gamma \mu_n. \end{aligned} \qquad 2 \leq k \leq n-1$$

Conversely the pair (EQ) *yields* (Zm) *under the condition that either system has a unique solution. This condition is assured by the nonsingularity of the relevant $n \times n$ martices A_n, B_n i.e.,* [1]

$$|A_n| \neq 0, \ |B_n| \neq 0.$$

□

The pair (EQ) is divided into two linear systems:

$$(\mathrm{EQ}_x) \qquad \begin{aligned} c &= x_0 \\ x_{k-1} &= \gamma x_k - x_{k+1} \\ x_{n-1} &= \xi x_n \end{aligned} \qquad 1 \leq k \leq n-1$$

and

$$(\mathrm{EQ}_\mu) \qquad \begin{aligned} c &= \xi \mu_1 - \mu_2 \\ \mu_{k-1} &= \gamma \mu_k - \mu_{k+1} \\ \mu_{n-1} &= \gamma \mu_n \end{aligned} \qquad 2 \leq k \leq n-1$$

[1]It holds that $|A_n| = |B_n|$.

3 Case $\lambda = 1$

Now we consider a series of individual minimization problems by letting the parameter λ vary on the positive half-line $(0, \infty)$.

First we consider the Case $\lambda = 1$. Then $\gamma := 2 + \lambda,\ \xi := 1 + \lambda$ yields

$$\gamma \ = \ 3, \quad \xi \ = \ 2.$$

We consider

$$\mathrm{Q}_1 \qquad \begin{array}{ll} \text{minimize} & -2x_0\mu_1 + \displaystyle\sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + x_k^2 + \mu_k^2 + (\mu_k - \mu_{k+1})^2\right] \\ & \qquad\qquad\qquad + (x_{n-1} - x_n)^2 + x_n^2 + 2\mu_n^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n,\ x_0 = c, \quad \text{(ii)}\ \ \mu \in R^n. \end{array}$$

Now we define the objective function $h : R^n \times R^n \to R^1$ by

$$\begin{aligned} h(x, \mu) \ = \ & -2c\mu_1 + \sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + x_k^2 + \mu_k^2 + (\mu_k - \mu_{k+1})^2\right] \\ & \qquad\qquad + (x_{n-1} - x_n)^2 + x_n^2 + 2\mu_n^2. \end{aligned}$$

Corollary 1 *Let $(x,\ \mu)$ be feasible. Then it holds that*

$$h(x, \mu) \ \geq \ 0. \tag{2}$$

The sign of equality holds iff a zero-minimum condition

$$(\mathrm{Zm}_1) \qquad \begin{array}{llll} c - x_1 \ = \ \mu_1, & x_1 \ = \ \mu_1 - \mu_2 & \\ x_{k-1} - x_k \ = \ \mu_k, & x_k \ = \ \mu_k - \mu_{k+1} & 2 \leq k \leq n-1 \\ x_{n-1} - x_n \ = \ \mu_n, & x_n \ = \ \mu_n & \end{array}$$

holds.

Corollary 2 *The zero-minimum condition* (Zm_1) *has a unique solution* $(x,\ \mu)$*;*

$$\begin{aligned} x &= (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{H_n}(H_{n-1},\ H_{n-2},\ \ldots, H_{n-k},\ \ldots,\ H_1,\ H_0), \end{aligned} \tag{3}$$

$$\begin{aligned} \mu &= (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{H_n}(G_n,\ G_{n-1},\ \ldots, G_{n+1-k},\ \ldots,\ G_2,\ G_1), \end{aligned} \tag{4}$$

where $\{G_n\}$ *is called two-step Gibonacci sequence and the sequence* $\{H_n\}$ *is called Hibonacci (see [23, 24]):*

$$G_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}, \quad H_n = 2G_n - G_{n-1},$$
$$\alpha = \frac{3-\sqrt{5}}{2}, \quad \beta = \frac{3+\sqrt{5}}{2}.$$

Hence Q_1 *attainas the zero minimum at* (x, μ).

We remark that the *Golden number*

$$\phi = \frac{1+\sqrt{5}}{2} \sim 1.618$$

and its *conjugate*

$$\overline{\phi} := 1 - \phi = -\phi^{-1} = \frac{1-\sqrt{5}}{2} \sim -0.382$$

are the solutions to the quadratic equation

$$t^2 - t - 1 = 0.$$

It holds that

$$\alpha = \frac{3-\sqrt{5}}{2} = \overline{\phi}^2 = \phi^{-2}, \quad \beta = \frac{3+\sqrt{5}}{2} = \phi^2$$
$$G_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} = \frac{\phi^{2n} - \overline{\phi}^{2n}}{\phi - \overline{\phi}} = F_{2n} \tag{5}$$
$$H_n = 2G_n - G_{n-1} = 2F_{2n} - F_{2n-2} = F_{2n+1}.$$

Thus both G_n and H_n are Fibonacci:

$$G_n = F_{2n}, \quad H_n = F_{2n+1}. \tag{6}$$

Corollary 3 *The zero-minimum condition* (Zm_1) *has a unique solution* (x, μ)*;*

$$x = (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n)$$
$$= \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n+1-2k},\ \ldots,\ F_3,\ F_1), \tag{7}$$

$$\mu = (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n)$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n+2-2k},\ \ldots,\ F_4,\ F_2). \tag{8}$$

Hence Q_1 *attainas the zero minimum at* (x, μ).

Corollary 4 *It holds that*

$$\begin{array}{rrcl} \text{(i)} & h(x,\mu) & \geq & 0 \quad \forall (x,\,\mu) \in R^n \times R^n \\ \text{(ii)} & h(x,\mu) & = & 0 \Longleftrightarrow (x,\mu) \ \textit{satisfies} \ (\mathrm{EQ}_1). \end{array}$$

The objective function

$$\begin{aligned} h(x,\mu) \ = \ -2c\mu_1 + \sum_{k=1}^{n-1} \left[(x_{k-1}-x_k)^2 + x_k^2 + \mu_k^2 + (\mu_k - \mu_{k+1})^2 \right] \\ + (x_{n-1}-x_n)^2 + x_n^2 + 2\mu_n^2 \end{aligned}$$

attains the zero-minimum. Further we have a *triple zero property* as follows.

Corollary 5 *Let a feasible* (x,μ) *satisfy* (Zm_1). *Then it holds that*

$$(\mathrm{tZ}_1) \qquad \begin{aligned} & h(x,\mu) \\ = \ & -c(c-x_1) + \sum_{k=1}^{n} \left[(x_{k-1}-x_k)^2 + x_k^2 \right] \\ = \ & -c\mu_1 + \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k-\mu_{k+1})^2 \right] + 2\mu_n^2 \\ = \ & 0. \end{aligned}$$

that is

$$(\mathrm{tZ}_1) \qquad \begin{aligned} & h(x,\mu) \\ = \ & -F_{2n+1}(F_{2n+1}-F_{2n-1}) + \sum_{k=1}^{n} \left[(F_{2n-2k+3}-F_{2n-2k+1})^2 + F_{2n-2k+1}^2 \right] \\ = \ & -F_{2n+1}F_{2n} + \sum_{k=1}^{n-1} \left[F_{2n-2k+2}^2 + (F_{2n-2k+2}-F_{2n-2k})^2 \right] + 2F_2^2 \\ = \ & 0. \end{aligned}$$

This yields a pair of quadratic equalities for the Fibonacci sequence $\{F_n\}$.

Corollary 6 *It holds that*

$$\begin{aligned} \sum_{k=0}^{n-1} \left[F_{2k+1}^2 + (F_{2k+3}-F_{2k+1})^2 \right] \ &= \ (F_{2n+1}-F_{2n-1})F_{2n+1}, \\ \sum_{k=0}^{n-1} \left[(F_{2k+2}-F_{2k})^2 + F_{2k+2}^2 \right] \ &= \ F_{2n}F_{2n+1}. \end{aligned} \tag{9}$$

Proof. It suffices to note that

$$F_{2k+3} - F_{2k+1} = F_{2k+2}, \qquad F_{2k+2} - F_{2k} = F_{2k+1}.$$

□

The pair is reduced to

$$\sum_{k=1}^{n-1} \left(F_{2k-1}^2 + F_{2k}^2\right) = F_{2n}F_{2n+1}$$

that is

$$\sum_{k=1}^{2n} F_k^2 = F_{2n}F_{2n+1}. \tag{10}$$

This is what we called *Lucas formula* [13, 29].

3.1 Fibonacci Duality

First we consider

$$\mathrm{P}_1 \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2\right] \\ \text{subject to} & \text{(i)} \;\; x \in R^n, \; x_0 = c \end{array}$$

and

$$\mathrm{D}_1 \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \displaystyle\sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - 2\mu_n^2 \\ \text{subject to} & \text{(i)} \;\; \mu \in R^n. \end{array}$$

Then both P_1 and D_1 are dual to each other. An equality condition is

$$(\mathrm{EC}_1) \quad \begin{array}{rcl@{\qquad}rcl} c - x_1 &=& \mu_1 & x_1 &=& \mu_1 - \mu_2 \\ x_{k-1} - x_k &=& \mu_k & x_k &=& \mu_k - \mu_{k+1} \quad k = 2, 3, \ldots, n-1 \\ x_{n-1} - x_n &=& \mu_n & x_n &=& \mu_n. \end{array}$$

The primal P_1 attains a minimum $m = \dfrac{F_{2n}}{F_{2n+1}} c^2$ at $x = (x_1, x_2, \ldots, x_n)$, while the dual D_1 does a maximum $M = \dfrac{F_{2n}}{F_{2n+1}} c^2$ at $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$:

$$\begin{aligned} x_k &= c\,\frac{F_{2n+1-2k}}{F_{2n+1}} \\ \mu_k &= c\,\frac{F_{2n+2-2k}}{F_{2n+1}} \end{aligned} \tag{11}$$

that is

$$\begin{aligned} x = (x_1,\, x_2,\, \ldots, x_k, \ldots, x_n) \;&=\; \frac{c}{F_{2n+1}}(F_{2n-1},\; F_{2n-3},\; \ldots, F_{2n+1-2k},\; \ldots,\; F_1) \\ \mu = (\mu_1,\, \mu_2,\, \ldots, \mu_k, \ldots, \mu_n) \;&=\; \frac{c}{F_{2n+1}}(F_{2n},\; F_{2n-2},\; \ldots, F_{2n+2-2k},\; \ldots,\; F_2). \end{aligned} \tag{12}$$

4 Case $\lambda = 4$

Let us consider the Case $\lambda = 4$. Then $\gamma := 2 + \lambda,\ \xi := 1 + \lambda$ yields

$$\gamma \;=\; 6, \quad \xi \;=\; 5.$$

We consider

$$\mathrm{Q}_4 \qquad \begin{aligned} &\text{minimize} \quad -8x_0\mu_1 + \sum_{k=1}^{n-1} \big[(x_{k-1} - x_k)^2 + x_k^2 + 16\mu_k^2 + (\mu_k - \mu_{k+1})^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad + 6x_k(\mu_k - \mu_{k+1})\big] \\ &\qquad\qquad\qquad\qquad + (x_{n-1} - x_n)^2 + x_n^2 + 17\mu_n^2 + 6x_n\mu_n \\ &\text{subject to} \quad \text{(i)}\ \ x \in R^n,\ x_0 = c, \quad \text{(ii)}\ \ \mu \in R^n. \end{aligned}$$

This objective function is denoted by $h(x, \mu)$.

Corollary 7 *Let $(x,\, \mu)$ be feasible. Then it holds that*

$$h(x, \mu) \;\geq\; 0. \tag{13}$$

The sign of equality holds iff a zero-minimum condition

$$(\mathrm{Zm}_4) \qquad \begin{aligned} c - x_1 \;&=\; 4\mu_1, \quad & x_1 \;&=\; \mu_1 - \mu_2 & \\ x_{k-1} - x_k \;&=\; 4\mu_k, \quad & x_k \;&=\; \mu_k - \mu_{k+1} \qquad & 2 \leq k \leq n-1 \\ x_{n-1} - x_n \;&=\; 4\mu_n, \quad & x_n \;&=\; \mu_n & \end{aligned}$$

holds.

Corollary 8 *The zero-minimum condition* (Zm_4) *has a unique solution $(x,\, \mu)$;*

$$\begin{aligned} &x \;=\; (x_1,\; x_2,\; \ldots,\; x_k,\; \ldots,\; x_{n-1},\; x_n) \\ &= \frac{c}{H_n}(H_{n-1},\; H_{n-2},\; \ldots, H_{n-k},\; \ldots,\; H_1,\; H_0), \end{aligned} \tag{14}$$

$$\begin{aligned} &\mu \;=\; (\mu_1,\; \mu_2,\; \ldots,\; \mu_k,\; \ldots,\; \mu_{n-1},\; \mu_n) \\ &= \frac{c}{H_n}(G_n,\; G_{n-1},\; \ldots, G_{n+1-k},\; \ldots,\; G_2,\; G_1) \end{aligned} \tag{15}$$

where

$$G_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}, \quad H_n = 5G_n - G_{n-1}.$$
$$\alpha = 3 - 2\sqrt{2}, \quad \beta = 3 + 2\sqrt{2}.$$

Hence Q_4 *attainas the zero minimum at* (x, μ).

We remark that the *Silver number*

$$\tau = 1 + \sqrt{2} \sim 2.414$$

and its *conjugate*

$$\overline{\tau} := 2 - \tau = -\tau^{-1} = 1 - \sqrt{2} \sim -0.414$$

are the solutions to the quadratic equation

$$t^2 - 2t - 1 = 0.$$

The *Sibonacci sequence* $\{S_n\}$ is defined as the solution to the second-order linear difference equation

$$x_{n+2} - 2x_{n+1} - x_n = 0 \qquad x_1 = 1, \ x_0 = 0.$$

Hence it satisfies

$$S_{n+1} = 2S_n + S_{n-1} \qquad S_1 = 1, \ S_0 = 0.$$

It holds that

$$\alpha = 3 - 2\sqrt{2} = \overline{\tau}^2 = \tau^{-2}, \quad \beta = 3 + 2\sqrt{2} = \tau^2$$
$$G_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} = \frac{\tau^{2n} - \overline{\tau}^{2n}}{2(\tau - \overline{\tau})} = \frac{S_{2n}}{2} \tag{16}$$
$$H_n = 5G_n - G_{n-1} = \frac{1}{2}(5S_{2n} - S_{2n-2}) = 2S_{2n} + S_{2n-1} = S_{2n+1}.$$

Thus both G_n and H_n are Sibonacci:

$$G_n = \frac{S_{2n}}{2}, \quad H_n = S_{2n+1}. \tag{17}$$

Corollary 9 *It holds that*

$$\text{(i)} \quad h(x,\mu) \geq 0 \quad \forall (x, \mu) \in R^n \times R^n$$
$$\text{(ii)} \quad h(x,\mu) = 0 \Longleftrightarrow (x,\mu) \ \textit{satisfies} \ (\mathrm{EQ}_4).$$

The objective function $h(x,\mu)$ attains the zero-minimum. Further we have a *triple zero property* as follows.

Corollary 10 *Let a feasible* (x,μ) *satisfy* (Zm$_4$). *Then it holds that*

$$
\text{(tZ}_4\text{)}\qquad
\begin{aligned}
&h(x,\mu)\\
&= -c(c-x_1)+\sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2+4x_k^2\right]\\
&= -c\mu_1+\sum_{k=1}^{n-1}\left[4\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+5\mu_n^2\\
&= 0.
\end{aligned}
$$

Hence we have

$$
\text{(tZ}_4\text{)}\qquad
\begin{aligned}
&h(x,\mu)\\
&= -H_n(H_n-H_{n-1})+\sum_{k=1}^{n}\left[(H_{n+1-k}-H_{n-k})^2+4H_{n-k}^2\right]\\
&= -H_nG_n+\sum_{k=1}^{n-1}\left[4G_{n+1-k}^2+(G_{n+1-k}-G_{n-k})^2\right]+5G_1^2\\
&= 0.
\end{aligned}
$$

This yields a pair of quadratic equalities for the Gibonacci and Hibonacci sequences $\{G_n\}$, $\{H_n\}$.

Corollary 11 *It holds that*

$$
\begin{aligned}
\sum_{k=0}^{n-1}\left[4H_k^2+(H_{k+1}-H_k)^2\right] &= H_n(H_n-H_{n-1}),\\
\sum_{k=0}^{n-1}\left[(G_{k+1}-G_k)^2+4G_{k+1}^2\right] &= H_nG_n.
\end{aligned}
\tag{18}
$$

The pair is reduced to

$$
\begin{aligned}
\sum_{k=0}^{n-1}\left[4S_{2k+1}^2+(S_{2k+3}-S_{2k+1})^2\right] &= S_{2n+1}(S_{2n+1}-S_{2n-1}),\\
\sum_{k=0}^{n-1}\left[(S_{2k+2}-S_{2k})^2+4S_{2k+2}^2\right] &= S_{2n}(S_{2n+2}-S_{2n}).
\end{aligned}
\tag{19}
$$

Thus we have an equality on quadratic sum for $\{S_n\}$

$$\sum_{k=1}^{2n} S_k^2 \;=\; \frac{1}{2} S_{2n} S_{2n+1}. \tag{20}$$

This is $\underline{\lambda = 4}$ (Sibonacci)–version of *Lucas formula* [13, 29], which is $\underline{\lambda = 1}$ (Fibonacci)–version.

4.1 Sibonacci Duality

Second we consier a pair The *dual-pair* (a pair which is dual to each other) is

$$\mathrm{P}_2 \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + 4x_k^2\right] \\ \text{subject to} & \text{(i)} \;\; x \in R^n, \; x_0 = c \end{array}$$

$$\mathrm{D}_2 \quad \begin{array}{ll} \text{Maximize} & 8c\mu_1 - \displaystyle\sum_{k=1}^{n-1} \left[16\mu_k^2 + 4(\mu_k - \mu_{k+1})^2\right] - 20\mu_n^2 \\ \text{subject to} & \text{(i)} \;\; \mu \in R^n. \end{array}$$

Then both P_2 and D_2 are dual to each other. An equality condition is

$$(\mathrm{EC}_4) \qquad \begin{array}{rclrcll} c - x_1 &=& 4\mu_1 & x_1 &=& \mu_1 - \mu_2 & \\ x_{k-1} - x_k &=& 4\mu_k & x_k &=& \mu_k - \mu_{k+1} & k = 2, 3, \ldots, n-1 \\ x_{n-1} - x_n &=& 4\mu_n & x_n &=& \mu_n. & \end{array}$$

The primal P_2 attains a minimum $m = \left(1 - \dfrac{H_{n-1}}{H_n}\right) c^2$ at $x = (x_1, x_2, \ldots, x_n)$, while the dual D_2 does a maximum $M = 4\dfrac{G_n}{H_n}\, c^2$ at $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$:

$$x_k \;=\; c\,\frac{H_{n-k}}{H_n}, \qquad \mu_k \;=\; c\,\frac{G_{n+1-k}}{H_n} \tag{21}$$

that is

$$\begin{aligned} x = (x_1,\, x_2,\, \ldots, x_k, \ldots, x_n) \;&=\; \frac{c}{H_n}(H_{n-1},\; H_{n-2},\; \ldots, H_{n-k},\; \ldots,\; H_0) \\ \mu = (\mu_1,\, \mu_2,\, \ldots, \mu_k, \ldots, \mu_n) \;&=\; \frac{c}{H_n}(G_n,\; G_{n-1},\; \ldots, G_{n+1-k},\; \ldots,\; G_1) \end{aligned} \tag{22}$$

where

$$G_n \;=\; \frac{\beta^n - \alpha^n}{\beta - \alpha}, \quad H_n \;=\; 5G_n - G_{n-1}.$$

$$\alpha \;=\; 3 - 2\sqrt{2}\,, \quad \beta \;=\; 3 + 2\sqrt{2}\,.$$

Then

$$4G_n = H_n - H_{n-1}, \quad H_0 = G_1 = 1. \tag{23}$$

Hence the the optimum point (x, μ) satisfies (EC_1) and the optimum values are same $m = M$.

We note that both G_n and H_n are Sibonacci:

$$G_n = \frac{S_{2n}}{2}, \quad H_n = S_{2n+1}.$$

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