TOWARDS EQUIVARIANT SCHUBERT CALCULUS FOR p-COMPACT GROUPS

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Abstract. This report is based on our talk delivered in Geometry and Topology of Transformation Groups held at RIMS on June 14th, 2023. The main aim of this report is to explain our attempt to establish *equivariant Schubert calculus* for certain complex reflection groups as well as p-compact groups. Details will appear in our forthcoming paper [18]. This is joint work with Hiroshi Naruse.

1. Motivations of our work

First of all, let us briefly introduce three papers that motivate our current work.

1.1. **Totaro's work (2003).** We begin with Totaro's work [23]: Let $W := G(r,1,n) = \mathbb{Z}/r\mathbb{Z} \wr S_n = (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$ be the complex reflection group acting naturally on $V = \mathbb{C}^n$. Firstly, Totaro introduces the ring:

$$F(r,n) := \mathbb{Z}[x_1, \dots, x_n]/(e_i(x_1^r, \dots, x_n^r)(1 \leq i \leq n)),$$

where $e_i(x_1^r, \dots, x_n^r)$ denotes the ith elementary symmetric polynomials in $x_1^r, \dots, x_n^r$. Then it is known that $F(r,n) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the coinvariant ring of W, that is, $F(r,n) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[V]_W := \mathbb{C}[V] \otimes_{\mathbb{C}[V]^W} \mathbb{C} = \mathbb{C}[V]/(\mathbb{C}[V]^W_+)$, where $(\mathbb{C}[V]^W_+)$ is the ideal in $\mathbb{C}[V]$ generated by W-invariant polynomials of strictly positive degrees. Moreover, for $r = 2$, $F(2,n)$ is the integral cohomology ring of the isotropic flag manifold $Sp(n)/T^n$, and hence he suggested thinking of $F(r,n)$ as the cohomology of a certain "generalized flag manifold". Then he considers the following subring of $F(r,n)$:

$$C(r,n) := \mathbb{Z}[e_1, \dots, e_n]/(e_i(x_1^r, \dots, x_n^r)(1 \leq i \leq n)) \subset F(r,n),$$

where $e_i = e_i(x_1, \dots, x_n)$ denotes the ith elementary symmetric polynomial in the variables $x_1, \dots, x_n$. One has $C(r,n) \otimes_{\mathbb{Z}} \mathbb{C} \cong (\mathbb{C}[V]_W)^{S_n} = \mathbb{C}[V]^{S_n} \otimes_{\mathbb{C}[V]^W} \mathbb{C} = \mathbb{C}[V]^{S_n}/(\mathbb{C}[V]^W_+)$. Moreover, for $r = 2$, $C(2,n)$ is isomorphic to the integral cohomology ring of the Lagrangian Grassmannian $Sp(n)/U(n)$, and therefore $C(r,n)$ can be thought of as the cohomology of a certain "generalized Lagrangian Grassmannian". Totaro's primary purpose is to construct a basis for the ring $C(r,n)$, thus generalizing the classical result in Schubert calculus for Lagrangian Grassmannians (Józefiak [12], Pragacz [20]). More specifically, he uses the Hall–Littlewood functions $Q_\lambda(\boldsymbol{z};t)$ (λ: partition), specialized t to be ζ, to give a basis, which he calls *Hall–Littlewood basis*, for the ring $C(r,n) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]$. Here $\zeta = \zeta_r$ is a primitive rth root of unity. Then his result is stated as follows: A partition λ is said to be *r-regular* if no part of λ occurs r or more times, in other words, if $m_i(\lambda) < r$ for all i, where $m_i(\lambda)$ is the multiplicity

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of i. Denote by $\mathcal{P}_{(r)}$ the set of r-regular partitions. He introduces two rings:

$$\begin{aligned} C(r) &:= C(r,\infty) = \mathbb{Z}[e_1, e_2, \ldots]/(e_i(x_1^r, x_2^r, \ldots)(i \geq 1)), \\ C'(r) &:= \bigoplus_{\lambda \in \mathcal{P}_{(r)}} \mathbb{Z}[\zeta] Q_\lambda(\boldsymbol{z}; \zeta). \end{aligned}$$

Then he showed the following result:

Lemma 1.1 (Totaro [23], Lemma 3.2)**.** *There is an isomorphism*

$$C(r) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta] \xrightarrow{\sim} C'(r),\ e_i \longmapsto Q_{(i)}(\boldsymbol{z}; \zeta)\ (i \geq 1).$$

It follows from Lemma 1.1 that we have

$$\begin{aligned} C(r,n) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta] &\xleftarrow{\sim} C(r) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]/(e_i\ (i > n)) \\ &\xleftarrow{\sim} C'(r)/(Q_{(i)}(\boldsymbol{z}; \zeta)\ (i > n)) \\ &\cong \bigoplus_{\lambda \in \mathcal{P}_{(r)},\ \lambda_1 \leq n} \mathbb{Z}[\zeta] Q_\lambda(\boldsymbol{z}; \zeta)\ \text{(Hall–Littlewood basis)}. \end{aligned}$$

1.2. **Ortiz's work (2015).** Next we briefly review Ortiz's work [19]: Let V be a complex vector space of dimension n, and $W \subset GL(V)$ be a finite *complex reflection group* (or *pseudoreflection group*, *unitary reflection group*). We fix a basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ for V, and denote by $\{X_1, \ldots, X_n\}$ its dual basis. Let $S_X := \mathrm{Sym}(V^*) \cong \mathbb{C}[X_1, \ldots, X_n]$ be the symmetric algebra of the dual space V^* of V. As usual, W acts on S_X by the contragredient action: $(w \cdot f)(x) = f(w^{-1} \cdot x)$ $(w \in W,\ f \in S_X,\ x \in V)$. Let $S_Y := \mathbb{C}[Y_1, \ldots, Y_n]$ be another polynomial ring with trivial W-action. Denote by $s(W)$ the set of (pseudo)reflections of W, and $|s|$ the order of a (pseudo)reflection $s \in s(W)$. For a (pseudo)reflection $s \in s(W)$, let α_s be the corresponding "root", and λ_s be a non-trivial eigenvalue of $s \in s(W)$, which is a primitive $|s|$th root of unity. Under the above setting, he introduces the following:

Definition 1.2 (GKM ring (algebra) of W (structure algebra of W))**.**

$$\Psi_W^O := \left\{ \psi = (\psi_v)_{v \in W} \in \bigoplus_{v \in W} S_Y \;\middle|\; \sum_{j=0}^{|s|-1} \frac{\psi_{s^j v}}{(\lambda_s^i)^j} \in (\alpha_s)^i S_Y \quad \left(\begin{array}{l} \forall v \in W,\ \forall s \in s(W) \\ 1 \leq \forall i \leq |s| - 1 \end{array} \right) \right\}.$$

Note that when $|s| = 2$ (therefore $\lambda_s = -1$) for $\forall s \in s(W)$, the above condition reduces to the usual GKM condition:

$$\psi_v - \psi_{sv} \in \alpha_s S_Y = (\alpha_s)\ (\forall v \in W,\ \forall s \in s(W)).$$

Since $s^j \cdot \alpha_s = \lambda_s^j \alpha_s$, the above condition is equivalent to the following:

$$\sum_{j=0}^{|s|-1} \frac{\psi_{s^j v}}{(s^j \cdot \alpha_s)^i} \in S_Y\ \ (\forall v \in W,\ \forall s \in s(W),\ 1 \leq \forall i \leq |s| - 1)\,.$$

This condition is referred to as *Ortiz's GKM condition* in the sequel.

Define a map $(-)_Y : S_X \longrightarrow S_Y$ by $f(X_1, \ldots, X_n) \longmapsto f(Y_1, \ldots, Y_n)$. Then the (algebraic) *localization map* is defined as follows:

Definition 1.3 ((Algebraic) Localization map)**.**

$$\phi_W^O : S_X \otimes S_Y \longrightarrow \bigoplus_{v \in W} S_Y,\ f \otimes g \longmapsto ((v \cdot f)_Y g)_{v \in W}.$$

It follows easily from the definition that the image of ϕ_W^O is contained in the GKM ring: $\mathrm{Im}\ \phi_W^O \subset \Psi_W^O \subset \bigoplus_{v\in W} S_Y$. In the case when $W = G(r,1,n)$, Ortiz showed that $\mathrm{Im}\ \phi_W^O = \Psi_W^O$ by building an S_Y-basis $\{\varphi^v\}_{v\in W}$ of Ψ_W^O consisting of elements of $\mathrm{Im}\ \phi_W^O$.

Theorem 1.4 (Ortiz [19], Theorem 6.1)**.** *Let W be $G(r,1,n)$ and let S_X^W denote the ring of W-invariant polynomials in S_X. Then, the localization map $\phi_W^O : S_X \otimes S_Y \longrightarrow \bigoplus_{v\in W} S_Y$ induces the following isomorphism:*

$$S_X \otimes S_Y/(f - f_Y \mid f \in (S_X^W)_+) \xrightarrow{\sim} \Psi_W^O \subset \bigoplus_{v\in W} S_Y, \tag{1.1}$$

where $(S_X^W)_+ = \{f \in S_X^W \mid \deg(f) > 0\}$.

Note that the left-hand side of (1.1) is the *double coinvariant ring* of $W = G(r,1,n)$. As for the *root system* for $G(r,1,n)$, see e.g., Bremke–Malle [6], Rampetas–Shoji [21].

1.3. **McDaniel's work (2016).** Lastly, we review McDaniel's work [16]: In order to state his result, we repeat again the notation and terminologies introduced in the previous subsection (but slightly changed): Let V be a complex vector space of dimension n, and $S := \mathbb{C}[V]$ be the ring of polynomial functions on V. A finite pseudo-reflection group $W \subset GL(V)$ acts on S by $(w \cdot f)(x) = f(w^{-1} \cdot x)$. Denote by S^W the ring of W-invariant polynomials, and $S_W := S/(S_+^W)$ the coinvariant ring of W, where (S_+^W) is the ideal in S generated by W-invariant polynomials of strictly positive degrees. Then he considers the *double coinvariant ring* (*equivariant coinvariant ring*) $S \otimes_{S^W} S$.

Definition 1.5 ((Algebraic) Localization map)**.**

$$\phi_W^M : S \otimes_{S^W} S \longrightarrow \bigoplus_{v\in W} S,\ f \otimes g \longmapsto (f(v \cdot g))_{v\in W}.$$

Definition 1.6 (McDaniel's GKM ring)**.**

$$\Psi_W^M := \left\{ \psi = (\psi_v)_{v\in W} \in \bigoplus_{v\in W} S \;\middle|\; \sum_{j=0}^{|s|-1} \frac{\psi_{vs^j}}{(vs^j \cdot \alpha_s)^i} \in S \ \left(\begin{array}{c} \forall v \in W,\ \forall s \in s(W) \\ 1 \le \forall i \le |s|-1 \end{array} \right) \right\}.$$

In order to prove $\mathrm{Im}\,\phi_W^M = \Psi_W^M$, he make use of the (generalized) *right divided difference operators*:

Definition 1.7 (Right divided difference operators)**.**

$$ {}_i\partial_s : \bigoplus_{v\in W} S \longrightarrow \bigoplus_{v\in W},\ \psi \longmapsto {}_i\partial_s(\psi),$$

$$({}_i\partial_s(\psi))_v := \sum_{j=0}^{|s|-1} \frac{\psi_{vs^j}}{(vs^j \cdot \alpha_s)^i}.$$

When $|s| = 2$ (therefore $\lambda_s = -1$ and $i = 1$), we have

$$({}_1\partial_s(\psi))_v = \frac{\psi_v - \psi_{vs}}{v \cdot \alpha_s},$$

which is the usual right divided difference operators (up to sign). Notice that using the right divided difference operators ${}_i\partial_s$, McDaniel's GKM ring is rewritten as

$$\Psi_W^M = \left\{ \psi = (\psi_v)_{v\in W} \in \bigoplus_{v\in W} S \;\middle|\; ({}_i\partial_s(\psi))_v \in S \ \left(\begin{array}{c} \forall v \in W,\ \forall s \in s(W) \\ 1 \le \forall i \le |s|-1 \end{array} \right) \right\}.$$

Then his main result is stated as follows:

Theorem 1.8 (McDaniel [16], Theorem 1.1). *The localization map* $\phi_W^M : S \otimes_{S^W} S \longrightarrow \bigoplus_{v\in W} S$ *induces the following isomorphism:*

$$S \otimes_{S^W} S \xrightarrow{\sim} \Psi_W^M \subset \bigoplus_{v\in W} S.$$

Note that McDaniel did not give an S-basis for Ψ_W^M.

2. Factorial Hall–Littlewood P- and Q-polynomials

In order to establish *equivariant Schubert calculus* for the complex reflection groups $G(r,1,n)$ and $G(r,r,n)$, a generalization of the factorial Schur Q- and P-functions will be needed. Such a generalization has been introduced by our recent paper [17].

2.1. Definition of the factorial H–L P- and Q-polynomials.

Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ be a countably infinite sequence of independent variables, and $\boldsymbol{b} = (b_1, b_2, \ldots)$ another set of independent variables. For a positive integer $k \geq 1$, we set

$$[x|\boldsymbol{b}]^k := \prod_{i=1}^{k}(x + b_i) \quad \text{and} \quad [[x;t|\boldsymbol{b}]]^k := (x - tx)[x|\boldsymbol{b}]^{k-1}.$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of length $\ell(\lambda) = \ell$, we set

$$[\boldsymbol{x}|\boldsymbol{b}]^\lambda := \prod_{i=1}^{\ell}[x_i|\boldsymbol{b}]^{\lambda_i} \quad \text{and} \quad [[\boldsymbol{x};t|\boldsymbol{b}]]^\lambda := \prod_{i=1}^{\ell}[[x_i;t|\boldsymbol{b}]]^{\lambda_i} = \prod_{i=1}^{\ell}(x_i - tx_i)[x_i|\boldsymbol{b}]^{\lambda_i - 1}.$$

Definition 2.1 (Factorial Hall–Littlewood P- and Q-polynomials). *For a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \leq m$, we define*

$$\begin{aligned}
HP_\lambda(\boldsymbol{x}_m;t|\boldsymbol{b}) \;&:= \sum_{\overline{w}\in S_m/(S_1)^\ell\times S_{m-\ell}} w \cdot \left[[\boldsymbol{x}|\boldsymbol{b}]^\lambda \times \prod_{\substack{1\leq i\leq \ell \\ i<j\leq m}} \frac{x_i - tx_j}{x_i - x_j} \right] \\
&= \sum_{\overline{w}\in S_m/(S_1)^\ell\times S_{m-\ell}} w \cdot \left[\prod_{i=1}^{\ell}\prod_{j=1}^{\lambda_i}(x_i + b_j) \times \prod_{\substack{1\leq i\leq \ell \\ i<j\leq m}} \frac{x_i - tx_j}{x_i - x_j} \right], \\
HQ_\lambda(\boldsymbol{x}_m;t|\boldsymbol{b}) \;&:= \sum_{\overline{w}\in S_m/(S_1)^\ell\times S_{m-\ell}} w \cdot \left[[[\boldsymbol{x};t|\boldsymbol{b}]]^\lambda \times \prod_{\substack{1\leq i\leq \ell \\ i<j\leq m}} \frac{x_i - tx_j}{x_i - x_j} \right] \\
&= (1-t)^\ell \times \\
&\quad \sum_{\overline{w}\in S_m/(S_1)^\ell\times S_{m-\ell}} w \cdot \left[\prod_{i=1}^{\ell} x_i \cdot \prod_{j=1}^{\lambda_i - 1}(x_i + b_j) \times \prod_{\substack{1\leq i\leq \ell \\ i<j\leq m}} \frac{x_i - tx_j}{x_i - x_j} \right].
\end{aligned}$$

We also define

$$HP_\lambda(\boldsymbol{x}_m;t) := HP_\lambda(\boldsymbol{x}_m;t|\boldsymbol{0}) \;\; \textit{and} \;\; HQ_\lambda(\boldsymbol{x}_m;t) := HQ_\lambda(\boldsymbol{x}_m;t|\boldsymbol{0}).$$

Example 2.2. (1) $HQ_\lambda(\boldsymbol{x}_m;t|\boldsymbol{b}) = (1-t)^{\ell(\lambda)} HP_\lambda(\boldsymbol{x}_m;t|0,\boldsymbol{b})$.

(2)

$$\begin{aligned}
&HP_{(1)}(\boldsymbol{x}_m;t|\boldsymbol{b}) = x_1+x_2+\cdots+x_m+\tfrac{1-t^m}{1-t}b_1,\\
&HQ_{(1)}(\boldsymbol{x}_m;t|\boldsymbol{b}) = (1-t)(x_1+x_2+\cdots+x_m),\\
&HP_{(1^2)}(\boldsymbol{x}_m;t|\boldsymbol{b})\\
&= (1+t)\left[m_{(1^2)}(\boldsymbol{x}_m)+\tfrac{1-t^{m-1}}{1-t}b_1 m_{(1)}(\boldsymbol{x}_m)+\tfrac{(1-t^{m-1})(1-t^m)}{(1-t)(1-t^2)}b_1^2\right],
\end{aligned}$$

where $m_\lambda(\boldsymbol{x}_m)$ denotes the monomial symmetric polynomial corresponding to the partition λ.

(3) *For each integer $k \geq 0$, let $v_k(t) := \prod_{i=1}^{k}\frac{1-t^k}{1-t}$. For a partition λ, we set $v_{\lambda>0}(t) := \prod_{i\geq 1} v_{m_i(\lambda)}(t)$. Then, for a partition λ of length $\leq m$, we see that*

$$HP_\lambda(\boldsymbol{x}_m;t) = v_{\lambda>0}(t)P_\lambda(\boldsymbol{x}_m;t) \quad \textit{and} \quad HQ_\lambda(\boldsymbol{x}_m;t) = Q_\lambda(\boldsymbol{x}_m;t),$$

where $P_\lambda(\boldsymbol{x}_m;t)$ and $Q_\lambda(\boldsymbol{x}_m;t)$ are the usual Hall–Littlewood polynomials in Macdonald's book [15, Chapter III].

2.2. **Basic properties of factorial H–L P- and Q-polynomials.** Factorial Hall-Littlewood P- and Q-polynomials have some nice properties that the ordinary factorial S-, P-, and Q-polynomials possess. First we show the *vanishing property* of factorial Hall–Littlewood P- and Q-polynomials: Let $\boldsymbol{b} = (b_1, b_2, \ldots)$ be a sequence of indeterminates, and t an indeterminate. We define

$$-\boldsymbol{b}_i^k(t) := (-b_i, -tb_i, \ldots, -t^{k-1}b_i) \quad \text{and} \quad -\boldsymbol{b}_i^0(t) := (\ ) \text{ (empty sequence)}.$$

For a partition $\mu = (\mu_1, \mu_2, \ldots)$, we define

$$-\boldsymbol{b}_\mu(t) := (-\boldsymbol{b}_{\mu_1}^{m_{\mu_1}}(t), \ldots, -\boldsymbol{b}_2^{m_2}(t), -\boldsymbol{b}_1^{m_1}(t)),$$

where $m_i = m_i(\mu)$ is the multiplicity of i $(1 \leq i \leq \mu_1)$.

Example 2.3. *Let $\mu = (5,5,5,4,1,1)$, then $m_1(\mu) = 2$, $m_2(\mu) = 0$, $m_3(\mu) = 0$, $m_4(\mu) = 1$, $m_5(\mu) = 3$, and*

$$-\boldsymbol{b}_\mu(t) = (-b_5, -tb_5, -t^2b_5, -b_4, -b_1, -tb_1).$$

For a partition μ of length $\ell(\mu) \leq m$, consider the substitution $\boldsymbol{x}_m = (x_1, x_2, \ldots, x_m)$ with $-\boldsymbol{b}_\mu(t)$. Then, using Definition 2.1, one can show the following:

Proposition 2.4 (Vanishing property)**.** *Let λ, μ be partitions of length at most m and set $\hat{\mu} := \mu + (1^m) = (\mu_1+1, \mu_2+1, \ldots, \mu_m+1)$. Then the factorial Hall–Littlewood P- and Q-polynomials satisfy the following vanishing property:*

(1) *If $\mu \not\supset \lambda$, we have*

$$HQ_\lambda(-\boldsymbol{b}_\mu(t), \underbrace{0,\ldots,0}_{m-\ell(\mu)}; t|\boldsymbol{b}) = 0 \quad \textit{and} \quad HP_\lambda(-\boldsymbol{b}_{\hat{\mu}}(t); t|\boldsymbol{b}) = 0.$$

(2) *If $\mu = \lambda$, we have*

$$HQ_\lambda(-\boldsymbol{b}_\lambda(t), \underbrace{0,\ldots,0}_{m-\ell(\lambda)}; t|\boldsymbol{b}) = \prod_{q=1}^{\lambda_1}\prod_{k=1}^{m_q(\lambda)}\left(\prod_{p=1}^{q}(-t^{k-1}b_q + t^{m_p(\lambda)}b_p)\right),$$

$$HP_\lambda(-\boldsymbol{b}_{\hat{\lambda}}(t); t|\boldsymbol{b}) = v_{\lambda>0}(t)\prod_{q=2}^{\hat{\lambda}_1}\prod_{k=1}^{m_q(\hat{\lambda})}\left(\prod_{p=1}^{q-1}(-t^{k-1}b_q + t^{m_p(\hat{\lambda})}b_p)\right).$$

Example 2.5.

$$
\begin{aligned}
&HQ_{(2,2,1,1)}(-\boldsymbol{b}_{(2,2,1,1)}(t), 0, \dots, 0\,|\boldsymbol{b})\\
&= (t-1)b_1 \times t(t^2-1)b_1 \times (t-1)b_2 \times t^2(t^2-1)b_2 \times (tb_1 - b_2) \times (t^2 b_1 - b_2).
\end{aligned}
$$

Next we show the *r-cancellation property* of the factorial Hall–Littlewood P- and Q-polynomials: Let $r \geq 2$ be a fixed positive integer and $\zeta = \zeta_r$ be a primitive rth root of unity. For an indeterminate a, define $\boldsymbol{a}^r(\zeta) := (a, a\zeta, a\zeta^2, \dots, a\zeta^{r-1})$.

Definition 2.6 (r-Cancellability)**.** *Let $r \leq m$. A symmetric polynomial $f(x_1, \dots, x_m)$ with coefficients in a certain ring $\supset \mathbb{Z}[\zeta]$ is said to have the r-cancellation property if $f(\boldsymbol{a}^r(\zeta), x_{r+1}, \dots, x_m)$ does not depend on a.*

By specializing t to be ζ, the factorial Hall–Littlewood polynomials $HP_\lambda(\boldsymbol{x}_m;\zeta|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_m;\zeta|\boldsymbol{b})$ are symmetric polynomials with coefficients in $\mathbb{Z}[\zeta]\otimes\mathbb{Z}[\boldsymbol{b}] = \mathbb{Z}[\zeta]\otimes\mathbb{Z}[b_1, b_2, \dots]$. Thus one can ask if these polynomials have the r-cancellation property. In a recent paper [17, §4], we have given *generating functions* for the factorial Hall–Littlewood P- and Q-polynomials. For instance, by Corollary 4.4 in [17], $HP_\lambda(\boldsymbol{x}_m;\zeta|\boldsymbol{b})$ is given as the coefficient of $\boldsymbol{u}^{-\lambda} = u_1^{-\lambda_1}\cdots u_\ell^{-\lambda_\ell}$ in

$$
\frac{1}{(1-\zeta)^\ell}\prod_{i=1}^{\ell}\left(\prod_{j=1}^{m}\frac{u_i - \zeta x_j}{u_i - x_j}\prod_{j=1}^{i-1}\frac{u_i - u_j}{u_i - \zeta u_j} - \zeta^{m-i+1}\right)\times\prod_{j=1}^{\lambda_i}\frac{u_i + b_j}{u_i}. \tag{2.1}
$$

Substituting $(x_1, \dots, x_r)$ with $\boldsymbol{a}^r(\zeta)$, we have

$$
\prod_{j=1}^{r}\frac{u_i - \zeta^j a}{u_i - \zeta^{j-1}a}\times\prod_{j=r+1}^{m}\frac{u_i - \zeta x_j}{u_i - x_j} = \prod_{j=r+1}^{m}\frac{u_i - \zeta x_j}{u_i - x_j}
$$

since $\zeta^r = 1$. Therefore, (2.1) does not depend on a nor $x_1, \dots, x_r$ after the substitution. From this, we have

Proposition 2.7 (r-Cancellation property)**.** *Assume that $r \leq m$. The factorial Hall–Littlewood polyonomials $HP_\lambda(\boldsymbol{x}_m;\zeta|\boldsymbol{b})$ and $HQ_\lambda(\boldsymbol{x}_m;\zeta|\boldsymbol{b})$ have the r-cancellation property.*

3. Finite loop spaces, Generalized homogeneous spaces, p-Compact groups

In this section, we briefly explain the topological background of our study.

3.1. **Finite loop spaces, generalized homogeneous spaces.**

First recall some classical results which go back to 1950s: For a compact Lie group G, denote by BG its classifying space. Then the following results due to Borel and Bott are classical:

Theorem 3.1 (Borel [3])**.** *Let G be a compact connected Lie group, H a closed connected subgroup of maximal rank. Then there is an isomorphism*

$$
H^*(G/H;\mathbb{Q}) \cong H^*(BH;\mathbb{Q}) \otimes_{H^*(BG;\mathbb{Q})} \mathbb{Q}.
$$

Theorem 3.2 (Bott [4], [5])**.** *Let G be a compact connected Lie group, and T a maximal torus. Then the integral cohomology $H^*(G/T)$ is concentrated in even degrees and torsion free.*

The original proof of these results relies on the differentiable structure of the compact Lie groups, and therefore is not purely homotopy theoretic. Thus it is natural to ask if there exist purely homotopy theoretic proofs of the theorems of Borel and Bott. One of first attempts to this direction was given by Rector [22]:

Definition 3.3. (1) *A loop space $\mathcal{X} = (\mathcal{X}, B\mathcal{X}, e)$ is a triple consisting of a topological space $\mathcal{X}$, a pointed (connected) topological space $B\mathcal{X}$, and a homotopy equivalence $e : \Omega B\mathcal{X} \xrightarrow{\sim} \mathcal{X}$. The space $B\mathcal{X}$ is called the classifying space of $\mathcal{X}$.*

(2) *A morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ of loop spaces is a pointed continuous map $Bf : B\mathcal{X} \longrightarrow B\mathcal{Y}$. The homotopy fiber of Bf over the basepoint of $B\mathcal{Y}$ is called the generalized homogenous space of Bf, and denoted by $\mathcal{Y}/f(\mathcal{X})$, or just $\mathcal{Y}/\mathcal{X}$ if f is understood.*

(3) *We will call a topological space X finite (more precisely, $\mathbb{Z}$-finite) if the integral cohomology $H^*(X)$ is finitely generated $\mathbb{Z}$-module.*

Finite loop spaces seem to be nice homotopy theoretic analogues of Lie groups. However, as Rector proved, for example, there are uncountable number of homotopically distinct finite loop spaces $B\mathcal{X}$ with $\Omega B\mathcal{X} \simeq SU(2)$.

3.2. p-**Compact groups.** The above mentioned defect of finite loop spaces was overcome by the concept of *p-compact groups* due to Dwyer–Wilkerson [9]. Without entering the details, we briefly summarize this concept: Let p be a fixed prime number. A space $\mathcal{X}$ is *$\mathbb{F}_p$-finite* if $H^*(\mathcal{X}; \mathbb{F}_p)$ is a finite dimensional graded vector space over $\mathbb{F}_p$. The *p-completion* construction of Sullivan, Bousfield–Kan produces for each space $\mathcal{X}$, a map $\mathcal{X} \longrightarrow \widehat{\mathcal{X}_p}$. A space $\mathcal{X}$ is called *$\mathbb{F}_p$-complete* if this map is a homotopy equivalence. Denote by $\widehat{\mathbb{Z}_p}$ the ring of p-adic integers, $\widehat{\mathbb{Q}_p}$ the field of p-adic numbers.

Definition 3.4 (Dwyer–Wilkerson [9])**.** *A p-compact group is a triple $(\mathcal{X}, B\mathcal{X}, e)$, where $B\mathcal{X}$ is a pointed, connected p-complete space, $\mathcal{X}$ is $\mathbb{F}_p$-finite, and $e : \mathcal{X} \simeq \Omega B\mathcal{X}$ is a homotopy equivalence.*

p-Compact groups have the following nice properties:

(1) Any p-compact group $\mathcal{X}$ has a maximal torus $i : \mathcal{T} \longrightarrow \mathcal{X}$.
(2) The Weyl group $W_{\mathcal{X}} = W_{\mathcal{X}}(\mathcal{T})$ of $\mathcal{X}$ is a finite p-adic reflection group, that is, a pair $(W_{\mathcal{X}}, L_{\mathcal{X}})$, where $L_{\mathcal{X}}$ is a $\widehat{\mathbb{Z}_p}$-lattice and $W_{\mathcal{X}}$ is a finite subgroup $GL(L_{\mathcal{X}})$ generated by reflections.

Furthermore, the classification of p-compact groups has been completed by Andersen–Grodal–Møller–Viruel [1], Andersen–Grodal [2].

3.3. $\mathcal{T}$**-equivariant cohomology of homogeneous spaces of** p**-compact groups.** Let $\mathcal{X}$ be a p-compact group with a maximal torus $\mathcal{T}$ of rank ℓ. The *generalized* or *p-compact flag manifold* $\mathcal{X}/\mathcal{T}$ is the homotopy fiber of the map $Bi : B\mathcal{T} \longrightarrow B\mathcal{X}$. As before, $W_{\mathcal{X}}$ denotes the Weyl group of $\mathcal{X}$. Then, by using the Eilenberg–Moore spactral sequence associated with the fibration $\mathcal{X}/\mathcal{T} \hookrightarrow B\mathcal{T} \longrightarrow B\mathcal{X}$, we obtain the following *Borel presentation* of the $\mathcal{T}$-equivariant cohomology of $\mathcal{X}/\mathcal{T}$:

Proposition 3.5 (Borel presentation)**.**

$$H^*_{\mathcal{T}}(\mathcal{X}/\mathcal{T}; \widehat{\mathbb{Q}_p}) \cong H^*(B\mathcal{T}; \widehat{\mathbb{Q}_p}) \otimes_{H^*(B\mathcal{X}; \widehat{\mathbb{Q}_p})} H^*(B\mathcal{T}; \widehat{\mathbb{Q}_p}).$$

Moreover, it is known that

$$H^*(B\mathcal{T}; \widehat{\mathbb{Q}_p}) \cong \widehat{\mathbb{Q}_p}[X_1, \dots, X_\ell] \quad \text{and} \quad H^*(B\mathcal{X}; \widehat{\mathbb{Q}_p}) \cong \widehat{\mathbb{Q}_p}[X_1, \dots, X_\ell]^{W_{\mathcal{X}}}.$$

Therefore, we have

$$\begin{aligned} H^*_{\mathcal{T}}(\mathcal{X}/\mathcal{T};\widehat{\mathbb{Q}_p}) &\cong H^*(B\mathcal{T};\widehat{\mathbb{Q}_p}) \otimes_{H^*(B\mathcal{X};\widehat{\mathbb{Q}_p})} H^*(B\mathcal{T};\widehat{\mathbb{Q}_p}) \\ &\cong \frac{\widehat{\mathbb{Q}_p}[X_1,\ldots,X_\ell,Y_1,\ldots,Y_\ell]}{(f-f_Y \mid f\in (\widehat{\mathbb{Q}_p}[X_1,\ldots,X_\ell]^{W_{\mathcal{X}}})_+)} \\ &= \text{double coinvarint algebra of } W_{\mathcal{X}}. \end{aligned}$$

Example 3.6. *Let $\mathcal{X}(r,1,n)$ be the p-compact group corresponding to $G(r,1,n) = (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$. Then we have*

$$H^*_{\mathcal{T}}(\mathcal{X}(r,1,n)/\mathcal{T};\widehat{\mathbb{Q}_p}) \cong \frac{\widehat{\mathbb{Q}_p}[X_1,\ldots,X_n,Y_1,\ldots,Y_n]}{(e_i(X_1^r,\ldots,X_n^r)-e_i(Y_1^r,\ldots,Y_n^r)\ (1\le i\le n))}.$$

Thus the double coinvariant ring of $G(r,1,n)$ can be realized as the $\mathcal{T}$-equivariant cohomology ring of a "generalized flag manifold" $\mathcal{X}(r,1,n)/\mathcal{T}$ (cf. Totaro's work introduced in §1.1).

Theorem 3.7 (Andersen–Grodal–Møller–Viruel [1], Theorem 1.5)**.** *Let $\mathcal{X}$ be a connected p-compact group, p odd, with a maximal torus $\mathcal{T}$ and the Weyl group $W_{\mathcal{X}}$. Then $H^*(\mathcal{X}/\mathcal{T};\widehat{\mathbb{Z}_p})$ is a free $\widehat{\mathbb{Z}_p}$-module of rank $|W_{\mathcal{X}}|$, concentrated in even degrees.*

Using this theorem, one can prove the following *Schubert presentation* of the $\mathcal{T}$-equivariant cohomology of $\mathcal{X}/\mathcal{T}$:

Theorem 3.8 (Schubert presentation)**.** *Let $\mathcal{X}$, p, $\mathcal{T}$, $W_{\mathcal{X}}$ be as above. Then $H^*_{\mathcal{T}}(\mathcal{X}/\mathcal{T};\widehat{\mathbb{Z}_p})$ is a free $H^*(B\mathcal{T};\widehat{\mathbb{Z}_p})$-module of rank $|W_{\mathcal{X}}|$, concentrated in even degrees.*

Thus, there exists a free $H^*(B\mathcal{T};\widehat{\mathbb{Z}_p})$-basis, say $\{\sigma_w\}_{w\in W_{\mathcal{X}}}$, so that, *at the cohomological level*, we have the following presentation:

$$H^*_{\mathcal{T}}(\mathcal{X}/\mathcal{T};\widehat{\mathbb{Z}_p}) \cong \bigoplus_{w\in W_{\mathcal{X}}} H^*(B\mathcal{T};\widehat{\mathbb{Z}_p})\sigma_w.$$

More generally, one can consider more general homogeneous spaces of p-compact groups. In fact, Castellana constructed various morphisms between p-compact groups that are analogues of the classical Whitney sum map ([8, Theorems A, B]. For instance, from Theorem A, one has a morphism $j : \mathcal{U}(n) := \mathcal{X}(1,1,n) \longrightarrow \mathcal{X}(r,1,n)$. The homotopy fiber $\mathcal{X}(r,1,n)/\mathcal{U}(n)$ of the map $Bj : B\mathcal{U}(n) \longrightarrow B\mathcal{X}(r,1,n)$ can be considered as a generalization of the classical Lagrangian Grassmannian. We call the homogeneous space $\mathcal{X}(r,1,n)/\mathcal{U}(n)$ the *generalized Lagrangian Grassmannian*. By using the Eilenberg–Moore spectral sequence again, one can compute the $\mathcal{T}$-equivariant cohomology of $\mathcal{X}(r,1,n)/\mathcal{U}(n)$ in a similar way to Example 3.6.

4. Towards equivariant Schubert calculus for the p-compact group $\mathcal{X}(r,1,n)$

As mentioned in §1.1, Totaro suggested to think the ring $F(r,n)$ or the coinvariant ring of $G(r,1,n)$ as the cohomology ring of a certain "generalized flag manifold". Indeed, as we saw in Example 3.6, the double coinvariant ring of $G(r,1,n)$ can be realized as the $\mathcal{T}$-equivariant cohomology ring of the p-compact flag manifold $\mathcal{X}(r,1,n)/\mathcal{T}$.

In this last section, we briefly explain our attempt to establish equivariant Schubert calculus on generalized homogeneous spaces associated with the p-compact group $\mathcal{X}(r,1,n)$. We proceed the argument along the same lines as those developed by Ikeda–Naruse [11], Ikeda–Mihalcea–Naruse [10]. Thus, let $\mathcal{X}(r,1,n)$ be the p-compact group corresponding to $G(r,1,n) = (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$ with a maximal torus $\mathcal{T}$, and $\mathcal{U}(n)$ ($= \mathcal{X}(1,1,n)$) be the "subgroup" corresponding to $S_n \subset G(r,1,n)$. Our primary goal is to describe the $\mathcal{T}$-equivariant cohomology $H^*_{\mathcal{T}}(\mathcal{X}(r,1,n)/\mathcal{U};R)$ and $H^*_{\mathcal{T}}(\mathcal{X}(r,1,n)/\mathcal{T};R)$ ($R = \mathbb{C}, \widehat{\mathbb{Q}_p}$, or $\widehat{\mathbb{Z}_p}$).

4.1. **Complex reflection group** $G(r,1,n)$**.** We summarize the basic data of the complex reflection group $G(r,1,n)$ needed to describe our idea (For details, see e.g., Lehrer–Taylor [14, Chapter 2]). The group $G(r,1,n)$ is defined as the semi-direct product of $(\mathbb{Z}/r\mathbb{Z})^n$ with S_n, namely $G(r,1,n) := (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$. Thus an element of $G(r,1,n)$ is a pair $(\boldsymbol{r},\sigma)$, where $\boldsymbol{r} = (r_1,\ldots,r_n) \in (\mathbb{Z}/r\mathbb{Z})^n$ and $\sigma = [\sigma(1)\ \sigma(2)\ \cdots\ \sigma(n)] \in S_n$. The product of $(\boldsymbol{r},\sigma)$ and $(\boldsymbol{s},\tau)$ is given by

$$(\boldsymbol{r},\sigma)\cdot(\boldsymbol{s},\tau) := (\tau\cdot\boldsymbol{r}+\boldsymbol{s},\sigma\tau),$$

where $\tau\cdot\boldsymbol{r} := (r_{\tau(1)}, r_{\tau(2)},\ldots,r_{\tau(n)})$ and the addition is taken mod r. It is often very useful to view an element $(\boldsymbol{r},\sigma)$ as the *r-colored permutation of n letters*. In this manner, we write an element $(\boldsymbol{r},\sigma)$ as $[\sigma(1)^{(r_1)}\ \sigma(2)^{(r_2)}\ \cdots\ \sigma(n)^{(r_n)}]$. Occasionally, we write *j bars* over i instead of $i^{(j)}$. For example, an element $((1,0,2),[2\ 1\ 3]) \in G(3,1,3)$ is written as $[2^{(1)}\ 1^{(0)}\ 3^{(2)}] = [\bar{2}\ 1\ \bar{\bar{3}}]$. It is known that the group $G(r,1,n)$ has the following presentation by generators: $G(r,1,n) = \langle s_0,s_1,s_2,\ldots,s_{n-1}\rangle$, where $s_0 := [\bar{1}\ 2\ \cdots\ n]$, and $s_i := [1\ \cdots\ i+1\ i\ \cdots n] = (i\ i+1)$ for $1 \le i \le n-1$. Thus the subgroup generated by s_i ($1 \le i \le n-1$) can be identified with the symmetric group S_n. The generator s_0 is subject to the following relations:

$$s_0^r = 1,\ s_0s_1s_0s_1 = s_1s_0s_1s_0,\ s_0s_i = s_is_0\ (i \ge 2).$$

Let us introduce the following elements: $t_i := [1\ \cdots i-1\ \bar{i}\ i+1\ \cdots\ n]$ for $1 \le i \le n$. With respect to the generators, these elements are given by $t_1 = s_0$ and $t_i = s_{i-1}t_{i-1}s_{i-1}$ ($i = 2,\ldots,n$). Then, the set $s(G(r,1,n))$ of (pseudo)reflections consists of the following elements of two types:

(i) $t_i^a t_j^{-a} s_{i,j} = t_i^a t_j^{-a}(i\,j)$ ($1 \le i \le n$, $0 \le a < r$), where $s_{i,j} = (i\,j)$ is the transposition of i and j.

(ii) t_i^a ($1 \le i \le n$, $1 \le a < r$).

With these notations, we make the following definition:

Definition 4.1 (Grassmannian elements of $G(r,1,n)$)**.** *An element $w = [m_1^{(r_1)}\ \cdots\ m_n^{(r_n)}] \in G(r,1,n)$ is said to be Grassmannian if there exists k with $0 \le k \le n$ such that $r_1,\ldots,r_k > 0$, $r_{k+1} = \cdots = r_n = 0$, and $m_1 > m_2 > \cdots > m_k$, $m_{k+1} < m_{k+2} < \cdots < m_n$. The set of all Grassmannian elements in $G(r,1,n)$ will be denoted by $G(r,1,n)^0$.*

Example 4.2. *Grassmannian elements of $G(3,1,2)$ are given as follows*:

$$[1\ 1] = 1,\quad [\bar{1}\ 2] = t_1,\quad [\bar{\bar{1}}\ 2] = t_1^2,\quad [\bar{2}\ 1] = t_2s_1,\quad [\bar{2}\ \bar{1}] = t_1t_2s_1,$$
$$[\bar{2}\ \bar{\bar{1}}] = t_1^2t_2s_1,\quad [\bar{\bar{2}}\ 1] = t_2^2s_1,\quad [\bar{\bar{2}}\ \bar{1}] = t_1t_2^2s_1,\quad [\bar{\bar{2}}\ \bar{\bar{1}}] = t_1^2t_2^2s_1.$$

Let us relate the set of Grassmannian elements $G(r,1,n)^0$ with a certain combinatorial object: In the above definition, putting $\sigma := [m_1\ m_2\ \cdots\ m_n] \in S_n$, we see that $w = t_{m_1}^{r_1}t_{m_2}^{r_2}\cdots t_{m_k}^{r_k}\sigma$. We associate the elements w with an r-regular partition $\lambda_{(w)} := (m_1^{r_1}, m_2^{r_2},\ldots,m_k^{r_k})$. Note that the largest part of $\lambda_{(w)}$ is $\le n$ since $n \ge m_1 > \cdots > m_k \ge 1$. Let $\mathcal{P}_{(r)}^{\le n}$ be the set of r-regular partitions λ such that $\lambda_1 \le n$. Then we have the follwing: There exists a one-to-one correspondence between $G(r,1,n)^0$ and $\mathcal{P}_{(r)}^{\le n}$. With the above notation, the correspondence is given by

$$w = t_{m_1}^{r_1}t_{m_2}^{r_2}\cdots t_{m_k}^{r_k}\sigma \in G(r,1,n)^0 \longleftrightarrow \lambda_{(w)} = (m_1^{r_1}, m_2^{r_2},\cdots,m_k^{r_k}) \in \mathcal{P}_{(r)}^{\le n}.$$

We also use the following description: Since $G(r,1,n)$ is defined as the semi-direct product of $(\mathbb{Z}/r\mathbb{Z})^n$ with S_n, an element $v \in G(r,1,n)$ is written uniquely as $v = t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n} \sigma$, where $0 \le r_i < r$ $(1 \le i \le n)$ and $\sigma \in S_n$. Thus one can take the set $\{t_1^{r_1} \cdots t_n^{r_n} \mid 0 \le r_i < r \ (1 \le i \le n)\}$ as coset representatives of $G(r,1,n)/S_n$. Then we have the following: There exists a one-to-one correspondence between $\mathcal{P}_{(r)}^{\le n}$ and $G(r,1,n)/S_n$. The correspondence is given by

$$v = t_1^{r_1} \cdots t_n^{r_n} \in G(r,1,n)/S_n \ \longleftrightarrow\ \lambda_{(v)} = (n^{r_n} \ \cdots 1^{r_1}) \in \mathcal{P}_{(r)}^{\le n}.$$

4.2. **Algebraic localization map.** It would be convenient to work with an *algebraic model* of the $\mathcal{T}$-equivariant cohomology of $\mathcal{X}(r,1,n)/\mathcal{U}(n)$. Moreover, it would be desirable to construct the theory independent of n. Thus we will take $n = \infty$. Let $r \ge 2$ be a fixed positive integer, and $\zeta = \zeta_r$ be a primitive rth root of unity. Let $G(r,1,\infty) = \cup_n G(r,1,n)$ and $S_\infty = \cup_n S_n$. Let $\Lambda(\boldsymbol{x}) = \Lambda_{\mathbb{C}}(\boldsymbol{x})$ denote the ring of symmetric functions over $\mathbb{C}$ in countaby many variables $\boldsymbol{x} = (x_1, x_2, \ldots)$, and consider the following subring:

$$\Gamma_{(r)}(\boldsymbol{x}) := \mathbb{C}[p_k \ (k \not\equiv 0 \mod r)] = \mathbb{C}[p_1, p_2, \ldots, p_{r-1}, p_{r+1}, \ldots, p_{2r-1}, p_{2r+1}, \ldots] \subset \Lambda(\boldsymbol{x}),$$

where $p_k = p_k(\boldsymbol{x})$ is the kth power sum symmetric function. By Definition 2.1, one can check directly that the polynomials $HQ_\lambda(\boldsymbol{x}_m; t|\boldsymbol{b})$, $HP_\lambda(\boldsymbol{x}_m; t)$ and $HQ_\lambda(\boldsymbol{x}_m; t)$ have the usual *stability property*: Let λ be a partition of length $\le m$. Then

$$\begin{aligned} HQ_\lambda(x_1, \ldots, x_m, 0; t|\boldsymbol{b}) &= HQ_\lambda(x_1, \ldots, x_m; t|\boldsymbol{b}), \\ HP_\lambda(x_1, \ldots, x_m, 0; t) &= HP_\lambda(x_1, \ldots, x_m; t), \\ HQ_\lambda(x_1, \ldots, x_m, 0; t) &= HQ_\lambda(x_1, \ldots, x_m; t). \end{aligned}$$

From this, we can let the number m of variables go to infinity, to define symmetric functions $HQ_\lambda(\boldsymbol{x}; t|\boldsymbol{b})$, $HP_\lambda(\boldsymbol{x}; t)$, and $HQ_\lambda(\boldsymbol{x}; t)$ respectively. Then the following fact holds:

Proposition 4.3 (cf. Macdonald [15], III, §7, Examples 7)**.** *Let $\mathcal{P}_{(r)} = \mathcal{P}_{(r)}^{\le\infty}$ denote the set of r-regular partitions. Then the set $\{HP_\lambda(\boldsymbol{x}; \zeta)\}_{\lambda \in \mathcal{P}_{(r)}}$ (resp. $\{HQ_\lambda(\boldsymbol{x}; \zeta)\}_{\lambda \in \mathcal{P}_{(r)}}$) is a $\mathbb{C}$-basis for $\Gamma_{(r)}(\boldsymbol{x})$.*

Set $\mathbb{C}[\boldsymbol{b}] = \mathbb{C}[b_1, b_2, \ldots]$, and consider the following ring:

$$\Gamma_{(r)}(\boldsymbol{x}|\boldsymbol{b}) := \Gamma_{(r)}(\boldsymbol{x}) \otimes \mathbb{C}[\boldsymbol{b}].$$

By Definition 2.1, the highest homogeneous component of $HQ_\lambda(\boldsymbol{x}; \zeta|\boldsymbol{b})$ in $\boldsymbol{x}$ coincides with $HQ_\lambda(\boldsymbol{x}; \zeta)$. Therefore, taking Proposition 4.3 into account, the functions $HQ_\lambda(\boldsymbol{x}; \zeta|\boldsymbol{b})$ $(\lambda \in \mathcal{P}_{(r)})$ form a basis for $\Gamma_{(r)}(\boldsymbol{x}|\boldsymbol{b})$ over $\mathbb{C}[\boldsymbol{b}]$.

Definition 4.4 (Algebraic localization map)**.** *The algebraic localization map*

$$\begin{aligned} \Phi :\ \Gamma_{(r)}(\boldsymbol{x}|\boldsymbol{b}) &\longrightarrow \bigoplus_{v \in G(r,1,\infty)/S_\infty} \mathbb{C}[\boldsymbol{b}], \\ f(\boldsymbol{x}|\boldsymbol{b}) &\longmapsto \Phi(f(\boldsymbol{x}|\boldsymbol{b})) = (\Phi_v(f(\boldsymbol{x}|\boldsymbol{b})))_{v \in G(r,1,\infty)/S_\infty} \end{aligned}$$

is defined by

$$\Phi_v(f(\boldsymbol{x}|\boldsymbol{b})) := f(-\boldsymbol{b}_v(\zeta), 0, 0, \ldots).$$

For $v = t_1^{r_1} \cdots t_n^{r_n} \in G(r,1,\infty)/S_\infty$, the sequence $-\boldsymbol{b}_v(\zeta)$ is given by

$$-\boldsymbol{b}_v(\zeta) = (-\boldsymbol{b}_1^{r_1}(\zeta), \ldots, -\boldsymbol{b}_n^{r_n}(\zeta)),$$

where $-\boldsymbol{b}_i^j(\zeta) = (-b_i, -\zeta b_i, \ldots, -\zeta^{j-1} b_i)$ and $-\boldsymbol{b}_i^0(\zeta) := (\ \ )$.

Since we have bijections $G(r,1,\infty)/S_\infty \cong \mathcal{P}_{(r)} \cong G(r,1,\infty)^0$ as mentioned in the previous subsection, we do not distinguish these three sets. Note that the map Φ is the algebraic counterpart of the geometric *localization map* at "torus fixed points". Therefore we often use the notation $f(\boldsymbol{x}|\boldsymbol{b})|_v$ instead of $\Phi_v(f(\boldsymbol{x}|\boldsymbol{b}))$, emphasizing the restriction to each torus fixed point. For example, by Example 2.2, we know that

$$HQ_{(1)}(\boldsymbol{x};t|\boldsymbol{b}) = (1-t)(x_1+x_2+\cdots).$$

Therefore, for $r=3$ and $t=\omega := e^{\frac{2\pi i}{3}}$, the images of the algebraic localization map are given by

$$\begin{aligned}
HQ_{(1)}|_{\emptyset} &= 0,\\
HQ_{(1)}|_{(1)} &= (\omega-1)b_1,\\
HQ_{(1)}|_{(1,1)} &= (\omega^2-1)b_1,\\
HQ_{(1)}|_{(2)} &= (\omega-1)b_2,\\
HQ_{(1)}|_{(2,1)} &= (\omega-1)b_1+(\omega-1)b_2,\\
HQ_{(1)}|_{(2,1,1)} &= (\omega^2-1)b_1+(\omega-1)b_2,\\
HQ_{(1)}|_{(2,2)} &= (\omega^2-1)b_2,\\
HQ_{(1)}|_{(2,2,1)} &= (\omega-1)b_1+(\omega^2-1)b_2,\\
HQ_{(1)}|_{(2,2,1,1)} &= (\omega^2-1)b_1+(\omega^2-1)b_2.
\end{aligned}$$

The GKM ring $\Psi^O_{G(r,1,\infty)}$ can be defined in a similar manner to Definition 1.2. Since we are interested in the generalized Lagrangian Grassmannian $\mathcal{X}(r,1,n)/\mathcal{U}(n)$, we also introduce a "parabolic" analogue of the algebra $\Psi^O_{G(r,1,\infty)}$ (see e.g., Kumar [13, Definition 11.1.16]). Denote this ring by $\Psi^O_{G(r,1,\infty)/S_\infty}$. This is an algebraic counterpart of the $\mathcal{T}$-equivariant cohomology ring $H^*_{\mathcal{T}}(\mathcal{X}(r,1,n)/\mathcal{U}(n);\mathbb{C})$. Then, using Proposition 2.4 and Ortiz's GKM condition, we can prove the following result:

Theorem 4.5. *The algebraic localization map Φ is injective, and induces the isomorphism $\Phi : \Gamma_{(r)}(\boldsymbol{x}|\boldsymbol{b}) \xrightarrow{\sim} \Psi^O_{G(r,1\infty)/S_\infty}$.*

We know that the set $\{HQ_\lambda(\boldsymbol{x};\zeta|\boldsymbol{b})\}_{\lambda\in\mathcal{P}_{(r)}}$ is a $\mathbb{C}[\boldsymbol{b}]$-basis for $\Gamma_{(r)}(\boldsymbol{x}|\boldsymbol{b})$. Thus, putting $\psi_\lambda := \Phi(HQ_\lambda(\boldsymbol{x};\zeta|\boldsymbol{b}))$, then we obtain the following corollary:

Corollary 4.6. *The set $\{\psi_\lambda\}_{\lambda\in\mathcal{P}_{(r)}}$ defined as above is a $\mathbb{C}[\boldsymbol{b}]$-basis for the GKM ring $\Psi^O_{G(r,1,\infty)/S_\infty}$.*

Further consideration shows that the basis $\{HQ_\lambda(\boldsymbol{x};\zeta|\boldsymbol{b})\}_{\lambda\in\mathcal{P}_{(r)}}$ for $\Gamma_{(r)}(\boldsymbol{x}|\boldsymbol{b})$, or equivalently, $\{\psi_\lambda\}_{\lambda\in\mathcal{P}_{(r)}}$ for $\Psi^O_{G(r,1,\infty)/S_\infty}$ has some plausible properties that the usual equivariant Schubert classes of G/P have, where G is a complex semisimple Lie group and P a parabolic subgroup (cf. Ikeda–Naruse [11, Proposition 2.1], Kumar [13, Theorem 11.1.7]). In this sense, our basis is what should be called the *Schubert basis* for the generalized Lagrangian Grassmannian $\mathcal{X}(r,1,n)/\mathcal{U}(n)$.

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