On some quasilinear parabolic equations with non-monotone multivalued terms*

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Abstract

We survey some recent results obtained in the joint paper [33] with M. Ôtani concerning the existence of solutions to a class of semilinear evolution equations with nonmonotone mutivalued terms.

1 Introduction

The aim of this note is to survey some recent results concerning the existence of solutions to the parabolic differential inclusion in $Q_T := [0,T] \times \Omega$:

$$\frac{\partial}{\partial t} u(t,x) - \triangle_p u(t,x) \in -\partial\phi(u(t,x)) + G(t,x,u(t,x)),\ (t,x) \in Q_T \tag{1}$$

coupled with the initial-boundary conditions

$$\begin{cases} u(t,x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{2}$$

Here Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $T > 0$, $\triangle_p$ is the p-Laplace differential operator defined by

$$\triangle_p u = div\left(\|\nabla u\|_{\mathbb{R}^N}^{p-2} \nabla u\right) \quad \text{with } p > \max\left(1, \tfrac{2N}{N+2}\right),$$

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$\partial\phi$ denotes the subdifferential of a proper lower semicontinuous convex function $\phi : \mathbb{R} \to [0,\infty]$ with $\phi(0) = \min_{u\in\mathbb{R}} \phi(u) = 0$, and $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is a nonmonotone multivalued mapping.

Our prototype of (1)-(2) is the case where $\phi \equiv 0$ and $G(t,x,u) = |u|^{q-2}u$, denoted by $(\mathrm{E})_p$, which is studied in [37, 26, 27, 28].

If $p > q$, then for every $u_0 \in W_0^{1,p}(\Omega)$, the existence of a global weak (resp. strong) solution is shown in [37] (resp. [27]). As for the case where $p < q$, Tsutsumi [37] showed the existence of a time-global weak solution, for the Sobolev-subcritical range of $q \in (p, p^*)$, provided that u_0 is sufficiently small in $W_0^{1,p}(\Omega)$. Here $p^* = \infty$ for $p \geq N$ and $p^* = \frac{Np}{N-p}$ for $p < N$.

However, concerning the existence of strong solutions, q is assumed to be more restrictive than the Sobolev-subcritical in [27, 28], namely, $q \in (p, p_*]$ with $p_* < \infty$ for $N \leq p$, and $p_* = 1 + \frac{NP}{2(N-p)}$ for $p < N$. Under this condition, the existence of local solutions is shown in [26, 28] and the existence of small global solution is shown in [27, 26, 28].

One of the main purposes of our work is to give a new device which enablea us to discuss the existence of strong solutions of (1)-(2) for the Sobolev-subcritical range of $q \in (2, p^*)$. In fact, as a corollary of Theorem 6.1 to be given in §6.1, we have:

Theorem *Let* $\max(1, \frac{2N}{N+2}) < p$ *and* $q \in (2, p^*)$, *then for any* $u_0 \in W_0^{1,p}(\Omega)$, *there exists* $T_0 \in (0,T]$ *such that* $(\mathrm{E})_p$ *admits a solution* u *in* $(0,T_0)$ *satisfying*

$$\begin{cases} u \in C([0,T_0]; W_0^{1,p}(\Omega)), \\ \frac{\partial u}{\partial t},\ \Delta_p u,\ g(u) = |u|^{q-2}u \in L^2(0,T_0; L^2(\Omega)). \end{cases}$$

Furthermore, this result is generalized for the case where g is replaced by upper semi-continuous or lower semi-continuous multi-valued functions in the subsequent subsections.

Differential inclusions appear naturally in the study of parabolic problems with discontinuous nonlinearities which arise from simplified models in the description of porous medium combustion (see [17], [18]), chemical reactor theory (see [19]), and game theory (see [15] and [24] for details and their references). To guarantee the existence of solutions, we need to extend the discontinuous nonlinearity to a multivalued mapping by filling the jumps at the discontinuity points of the nonlinearity.

In the context of elliptic systems, the problem has been studied extensively by many authors using different methods. More specifically, Rauch [35] used mollifications and truncation techniques, while the approach in Chang [14] is based on the nonsmooth critical point theory for locally Lipschitz functions, dealing with partial differential equations involving a discontinuous reaction term. The variational framework introduced in Chang [14] leads to several results, mentioned, for example, in the monographs [21] and [10].

In the context of parabolic systems, Carl [8] studied nonlinear dynamic problems with nonmonotone discontinuities by adapting Rauch's method to the dynamic situation, Papageorgiou [34] extended Carl's work and proposed a different approach based on evolution inclusions, a more suitable one to the multivalued character of the problem.

Several other methods have been developed and involved to study the existence of solutions to the initial-boundary value problems for parabolic equations with discontinuous or multivalued nonlinearities: the upper and lower solutions method together with a generalized iteration in [9], [7] and also in [11], [12] and [13] to prove the existence of extremal solutions. We also refer to [3] where existence results were obtained for a class of parabolic equations with either Carathéodory or discontinuous nonlinear terms; and [22] where the existence of solutions to parabolic problems with discontinuous and nonmonotone nonlinearity was obtained by passing to a multivalued version by filling in the gaps at the discontinuity points.

The goal of our paper is twofold: it has firstly to set up a framework which enables us to treat wider nonlinearity of $G(\cdot,\cdot,u)$, more precisely, to cover the growth condition on $G(\cdot,\cdot,u)$ up to the Sobolev-subcritical range, and secondly to adapt and improve the techniques and arguments developed in [31] and [32] in order to obtain existence results for the parabolic inclusion (1) with the initial and boundary conditions (2), generalizing corresponding results given by many authors, especially given in [28], [29], and [32] where the semi-linear case $p = 2$ is considered. Our approach uses tools from the multivalued analysis, together with the theory of nonlinear operators of monotone type and methods from the theory of nonlinear evolution equations.

We prove two types of local existence results: one for the case where the multivalued mapping $u \mapsto G(\cdot,\cdot,u)$ is upper semicontinuous (u.s.c.) with closed convex values and the second one deals with the case where $u \mapsto G(\cdot,\cdot,u)$ is lower semicontinuous (l.s.c.) with closed (not necessarily convex) values. We also discuss the extension of large or small local solutions along the lines of arguments developed in [28].

The existence of local solutions is obtained by following the strategy in [28], i.e., we apply Schauder-Tikhonov-type fixed point theorems for the mapping $\mathcal{G} : h \mapsto G(t,x,u_h)$, where u_h is the unique solution of the problem (1)-(2) with $G(t,x,u)$ replaced by h. With the aid of results in [32], it is shown that $\mathcal{G}$ becomes u.s.c. or l.s.c. from $\mathcal{X}_T^{\alpha,\beta} := L^\alpha(0,T;L^\beta(\Omega))$ with the weak topology for suitable $\alpha,\beta \in (1,\infty)$ into itself or $L^1(0,T;L^1(\Omega))$ according to the case where $u \mapsto G(\cdot,\cdot,u)$ is u.s.c. or l.s.c., respectively.

Another crucial step is to show that there exist $R > 0$ and (a sufficiently small) $T_0 > 0$ such that $\mathcal{G}$ maps $\{\, h \in \mathcal{X}_{T_0}^{\alpha,\beta}\ ;\ \|h\|_{\mathcal{X}_{T_0}^{\alpha,\beta}} \le R \,\}$ into itself. For this purpose, we rely on arguments similar to those developed in [28] based on some interpolation inequalities. In this procedure, we formulate the two different kinds of settings: Hilbert-space setting with $\alpha = \beta = 2$ (in §3.1) and Non-Hilbert-space setting (in §3.2).

The advantage of our treatment lies in the fact that for the existence of time-local strong solutions of (1)-(2), it allows the Sobolev-subcritical growth order of $G(t,x,u)$

with respect to u, which has been left as an open problem even for the case where $G(t,x,u)$ is a single-valued function.

The structure of our paper is the following: In Section 2, we prepare some notations and basic definitions from the nonlinear operator theory and the multivalued analysis used in the following sections. In Section 3, we prepare some auxiliary results concerning the property of the mapping $\mathcal{G}$ in a Hilbert-space setting (§3.1) and in a Non-Hilbert-space setting (§3.2). Section 4 is devoted to the study of the local existence of solutions to problem (1)-(2). We obtain two kinds of existence results: one for the case where G is closed convex valued, upper semicontinuous with respect to the third variable (§4.1), and the other for the case where the multivalued mapping G is lower semicontinuous with closed (not necessarily convex) values (§4.2). Both cases are discussed in the Hilbert-space setting and the Non-Hilbert-space setting. In Section 5, we study the global existence of solutions, namely, the existence of *large global solutions* without assuming the smallness of the given data (§5.1) and the existence of *small global solutions* when the given data are taken sufficiently small (§5.2). In Section 6, we exemplify the applicability of our results. In particular, it is shown that our framework can give a new result concerning an open problem for the classical equation (1), i.e., the case where $\phi \equiv 0$ and $G(t,x,u)$ is a single-valued function.

2 Notations and preliminaries

For easy reference, in this section, we recall some notations and basic definitions from the multivalued analysis and the nonlinear operator theory, which we shall use in the sequel. For further details, we refer to [1], [2], [4], [25], [31] and [32].

Let X and Y be Hausdorff topological spaces and let 2^Y be the family of all subsets of Y. A multivalued map $F : X \to 2^Y \backslash \{\emptyset\}$ is said to be *upper semicontinuous* (u.s.c. for short) on X, if for every closed subset C of Y, the set

$$F^-(C) := \{x \in X\,;\, F(x) \cap C \neq \emptyset\}$$

is closed in X. $F : X \to 2^Y \backslash \{\emptyset\}$ is said to be *lower semicontinuous* (l.s.c for short) on X, if

$$F^+(C) := \{x \in X\,;\, F(x) \subset C\}$$

is closed in X for each closed subset C of Y.

It is well known that $F : X \to 2^Y \backslash \{\emptyset\}$ is upper semicontinuous on X with compact values, then its *graph*

$$Gr(F) := \{(x,y) \in X \times Y\,;\, y \in F(x)\}$$

is closed in $X \times Y$. Conversely, if $F : X \to 2^Y \backslash \{\emptyset\}$ has a closed graph and if for each $x \in X$, there exists a neighborhood U of x such that $F(U) := \bigcup_{x \in U} F(x)$ is precompact, then F is u.s.c. on X (see Propositions 2.22 and 2.23 of [25]).

By a *section* of a multivalued map $F : X \to 2^Y \backslash \{\emptyset\}$ we mean any function $f : X \to Y$ such that

$$f(x) \in F(x) \text{ for all } x \in X.$$

Let (I, Σ, μ) be a σ-finite complete measurable space and $(Y, \|\cdot\|)$ be a separable Banach space. A closed valued multifunction $\Psi : I \to 2^Y \backslash \{\emptyset\}$ is said to be $\Sigma-$measurable (or simply, measurable) if for every open set $U \subset Y$, we have

$$\Psi^-(U) := \{\omega \in I\,;\, \Psi(\omega) \cap U \neq \emptyset\} \in \Sigma.$$

It is known that Ψ is measurable if and only if for every $y \in Y$, the map

$$\omega \mapsto d(y, \Psi(\omega)) := \inf\{\|z - y\|\,;\, z \in \Psi(\omega)\}$$

is a measurable $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{\infty\}$-valued function (see [25], Corollary 19, p.143). A multifunction $\Psi : I \to 2^Y$ with nonempty values is said to be graph measurable if

$$Gr(\Psi) := \{(\omega, z) \in I \times Y\,;\, z \in \Psi(\omega)\} \in \Sigma \otimes \mathcal{B}(Y),$$

where $\mathcal{B}(Y)$ denotes the Borel σ-algebra on Y. For multifunctions with closed values, the measurability implies the graph measurability, while the converse is true if Σ is complete.

For $1 \leq p \leq \infty$, we denote by S^p_Ψ the set of all sections of Ψ which belong to the Lebesgue-Bochner space $L^p(I\,;\,Y)$ that is

$$S^p_\Psi = \{v \in L^p(I\,;\,Y)\,;\, v(\omega) \in \Psi(\omega) \;\; \mu - \text{a.e.}\}.$$

It is easy to check that for a graph measurable multifunction $\Psi : \Omega \to 2^Y \backslash \{\emptyset\}$, the set S^p_Ψ is nonempty if and only if $\omega \mapsto \inf\{\|x\|\,;\, x \in \Psi(\omega)\}$ is majorized by a $L^p(I)$ -function (see [25], Lemma 3.2, p.175).

A set $K \subseteq L^p(I\,;\,Y)$ is said to be *decomposable* if for all u, $v \in K$ and all $A \in \Sigma$ we have

$$u\chi_A + v\chi_{I\backslash A} \in K,$$

where χ_A denotes the characteristic function of A. It is clear that the set $\mathcal{S}^p_\Psi$ is decomposable.

In the remaining part of this section, we collect some definitions and properties concerning maximal monotone mappings. Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$ and let $A : H \to 2^H$ be a maximal monotone operator with domain $D(A) := \{\, x \in H\,;\, A\,x \neq \emptyset\,\}$. The *minimal section* of A is the function $A^0 : H \to H$ satisfying the following conditions:

$$A^0(x) \in A(x) \text{ and } \left\|A^0(x)\right\|_H = \inf\{\|\xi\|_H\,;\, \xi \in A(x)\} \;\; \forall x \in D(A).$$

Recall that, the graph of any maximal monotone operator is *demiclosed*, i.e., closed in $H \times H_w$, where H_w denotes the space H furnished with the weak topology.

Let $\varphi : H \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. We say that φ is *proper* if its *effective domain*

$$D(\varphi) := \{x \in H \, ; \, \varphi(x) < +\infty\}$$

is non-empty. The multivalued map $\partial\varphi : H \to 2^H$ defined by

$$\partial\varphi(x) = \{g \in H \, ; \, \varphi(y) - \varphi(x) \geq (g, y - x)_H \text{ for all } y \in H\} \tag{3}$$

is called the *subdifferential* of φ (in the sense of convex analysis). It is known that the *subdifferential* $\partial\varphi$ of a proper lower semicontinuous convex function φ is a maximal monotone operator with domain

$$D(\partial\varphi) := \{x \in H \, ; \, \partial\varphi(x) \neq \emptyset\} \subset D(\varphi).$$

We shall use $\partial^0\varphi$ instead of $(\partial\varphi)^0$ to denote the minimal section of the maximal monotone operator $\partial\varphi$.

3 Auxiliary results

In what follows, we always assume that Ω is a bounded open subset of $\mathbb{R}^N$ with finite Lebesgue measure denoted by $|\Omega|$, $N \geq 1$, $T > 0$, $Q_T := [0, T] \times \Omega$ and put $\mathcal{X}_T^{\alpha,\beta} := L^\alpha\left(0, T; L^\beta(\Omega)\right)$ with $1 < \alpha, \beta < \infty$. We often denote $\mathcal{X}_T^{\alpha,\beta}$ simply by $\mathcal{X}$, if no confusion arises.

Definition 3.1. *Let $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ be a multivalued mapping. The multivalued map $\widetilde{G} : \mathcal{X} \to 2^{\mathcal{X}} \backslash \{\emptyset\}$ defined by*

$$\widetilde{G}(u) = \{g \in \mathcal{X} \, ; \, g(t,x) \in G(t, x, u(t,x)) \quad \text{a.e. } (t,x) \in Q_T\} \tag{4}$$

is called the realization of $G(\cdot, \cdot, u)$ in $\mathcal{X}$.

Definition 3.2. *We say that the realization $\widetilde{G}$ of G in $\mathcal{X}$ is a.e.-demiclosed if for any sequence $(u_n)_{n\in\mathbb{N}}$ of functions from Q_T into $\mathbb{R}$ which converges almost everywhere in Q_T to a function $u : Q_T \to \mathbb{R}$ and for any sequence $(g_n)_{n\in\mathbb{N}}$ of functions from Q_T into $\mathbb{R}$ such that*

$$g_n(t,x) \in G(t, x, u_n(t,x)) \text{ for each } n \in \mathbb{N} \text{ and almost all } (t,x) \in Q_T,$$

which converges weakly in $\mathcal{X}$ to a function $g \in \mathcal{X}$, then one has $g \in \widetilde{G}(u)$, that is,

$$g(t,x) \in G(t, x, u(t,x)) \text{ for almost all } (t,x) \in Q_T.$$

The following result plays an essential role in the later arguments.

Proposition 3.3. *Let $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ be a nonempty closed convex valued multifunction such that:*

For almost all $(t, x) \in Q_T$, $G(t, x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is upper semicontinuous.

Then the realization $\widetilde{G}$ of G in $\mathcal{X}$ is a.e.-demiclosed.

Proof. We can repeat the same argument as that in the proof of Proposition 3 in [32] with obvious modifications, namely by replacing $L^2(0, T; L^2(\Omega))$ with $\mathcal{X}_T^{\alpha,\beta}$. □

Let $\Psi(\mathbb{R}, \mathbb{R}_+)$ be the family of all lower semicontinuous convex functions ϕ : $\mathbb{R} \to [0, \infty]$ such that

$$\phi(0) = \min_{u \in \mathbb{R}} \phi(u) = 0.$$

Let $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ and define $\varphi : L^2(\Omega) \to \mathbb{R}_+$ by

$$\varphi(u) = \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega \phi(u(x)) \, dx & \text{if } u \in D(\varphi), \\ +\infty & \text{otherwise,} \end{cases}$$

with $\max(1, \frac{2N}{N+2}) < p$ and

$$D(\varphi) = \{ u \in W_0^{1,p}(\Omega) \; ; \; \widetilde{\phi}(u) := \int_\Omega \phi(u(x)) \, dx < +\infty \}.$$

Then φ becomes a proper lower semicontinuous functional defined on $L^2(\Omega)$ and we have

$$\partial\varphi(u) = -\triangle_p u + \partial\phi(u)$$

with domain

$$D(\partial\varphi) = \{ u \in D(\varphi) \; ; \; \triangle_p u \in L^2(\Omega), \, \exists b \in L^2(\Omega) \text{ such that } b(x) \in \partial\phi(u(x)) \text{ a.e. } x \in \Omega \}.$$

Moreover, for any $z = -\triangle_p u + b \in \partial\varphi(u)$ with $b \in \partial\phi(u)$ one has

$$\|z\|_{L^2}^2 \geq \|\triangle_p u\|_{L^2}^2 + \|b\|_{L^2}^2 \text{ and } (-\triangle_p u, b)_{L^2} \geq 0 \tag{5}$$

(see Lemma 1 of Ôtani-Staicu [31]).

3.1 A direct treatment in the L^2-framework

In this subsection, following the strategy in [28], we develop a direct treatment for our problem in $\mathcal{H}_T := L^2(0,T;L^2(\Omega))$. We first recall the following standard result from Kōmura-Brézis theory (see Theorem 3.6 of H. Brézis [4]).

Proposition 3.4. *Let $\phi \in \Psi(\mathbb{R},\mathbb{R}_+)$. Then for any $h \in \mathcal{H}_T$ and $u_0 \in D(\varphi)$, the problem*

$$(\mathrm{E})^h \begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \triangle_p u(t,x) \in -\partial\phi(u(t,x)) + h(t,x), & (t,x) \in Q_T, \\ u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

admits a unique solution $u_h \in C([0,T];L^2(\Omega))$ satisfying

$$\frac{\partial u_h}{\partial t},\ \triangle_p u_h, b_h \in \mathcal{H}_T = L^2\left(0,T;L^2(\Omega)\right),$$

where b_h is the section of $\partial\phi(u_h)$ satisfying $(\mathrm{E})^h$, i.e., $\frac{\partial u_h}{\partial t} - \Delta_p u_h + b_h = h$.

Then we can define a multivalued mapping $\mathcal{G}_{\mathcal{H}_S} : h \mapsto \widetilde{G}(u_h)$, the realization of $G(\cdot,\cdot,u_h)$ in $\mathcal{H}_S$, for all $S \in (0,T]$, i.e.,

$$\mathcal{G}_{\mathcal{H}_S}(h) := \{ g \in \mathcal{H}_S\,;\, g(t,x) \in G(t,x,u_h(t,x)) \text{ a.e. } (t,x) \in Q_S \} \tag{6}$$

with

$$|||\mathcal{G}_{\mathcal{H}_S}(h)|||_{\mathcal{H}_S} := \sup\{ \|g(t,x)\|_{\mathcal{H}_S}\,;\, g(t,x) \in G(t,x,u_h(t,x)) \}, \tag{7}$$

where u_h is the solution of $(\mathrm{E})^h$.

Here we introduce the following growth condition on G.

(GC)$_*$ We say that a multivalued map $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies **(GC)**$_*$, if there exist nonnegative numbers $k \in [0,1)$, q, C_q and a function $a(\cdot) \in L^1(Q_T)$ such that:

$$|||G(t,x,u)|||^2 \le |a(t,x)| + k\,|\partial^0\phi(u)|^2 + C_q\,|u|^{2(q-1)} \quad \text{for a.e. } (t,x) \in Q_T,\ \forall u \in D(\partial\phi), \tag{8}$$

where $|||G(t,x,u)||| := \sup\{ |\xi|\,;\, \xi \in G(t,x,u) \}$ and $q \in [2,p_*]$.

Here p_* denotes any finite number if $N \le p$; and $p_* = 1 + \frac{Np}{2(N-p)}$ if $p < N$.

Note that by virtue of the Sobolev-Poincaré embedding theorem, there exists a constant $K_q > 0$ such that

$$K_q \|u\|_{L^{2(q-1)}} \le \|\nabla u\|_{L^p} \quad \forall q \in [2,p_*] \quad \forall u \in W_0^{1,p}(\Omega). \tag{9}$$

Here for $R>0$ and $S\in(0,T]$, we put

$$\mathbf{K}_R^{\mathcal{H}_S} := \{\, h\in\mathcal{H}_S\,;\ \|h\|_{\mathcal{H}_S}^2 \le R^2\,\}. \tag{10}$$

Then we can show that $\mathcal{G}_{\mathcal{H}_S}$ maps $\mathbf{K}_R^{\mathcal{H}_S}$ into itself for suitably chosen $R>0$ and $S\in(0,T]$.

Proposition 3.5. *Let $\phi\in\Psi(\mathbb{R},\mathbb{R}_+)$ and $G: Q_T\times\mathbb{R}\to 2^{\mathbb{R}}$ satisfy $(\mathbf{GC})_*$ and let $u_0\in D(\varphi)$. Then there exist $R>0$ and $T_0>0$ depending on k, $\|a\|_{L^1(Q_T)}$, q and $\varphi(u_0)$ such that $\mathcal{G}_{\mathcal{H}_{T_0}}$ maps $\mathbf{K}_R^{\mathcal{H}_{T_0}}$ into itself.*

3.2 Non-Hilbert-space setting

In this subsection we develop another framework to treat our problem in a (non-Hilbert) Banach space. Throughout this subsection, we always assume that

$$\max\,(1,\frac{2N}{N+2})<p<N$$

and denote $W_0^{1,p}(\Omega)$ by V and its dual by V^*. Then V is compactly embedded in $L^2(\Omega)$, since $2<p^*=\frac{Np}{N-p}$.

To define the Banach spaces where we work, we need to introduce a couple of exponents given by

$$s=s(N,p):=\tfrac{p^*(p^*+p-2)}{p}=\tfrac{N(p^*+p-2)}{N-p},\quad r=r(N,p):=\tfrac{s}{s-p^*+1}.$$

Then we note the following relations.

Proposition 3.6. *We have:*

(i) $\max\,(2,p)<p^*<s$.

(ii) $1<s'=\frac{s}{s-1}<r<2$.

(iii) $2<r'=\frac{r}{r-1}<p^*$.

(iv) $(p^*)'=\frac{p^*}{p^*-1}<r$.

Now we define the Banach space $X_S^{\tilde{p},r}$ in which we work by

$$X_S^{\tilde{p},r}:=L^{\tilde{p}}(0,S;L^r(\Omega))\quad\text{with}\quad \tilde{p}:=\tfrac{p^*+p-2}{p-1}>p'>1.$$

We also introduce the following growth condition on $\partial\phi(\cdot)$.

$(\mathbf{GC})_\phi$ There exists a constant C_ϕ such that

$$|||\partial\phi(u)|||:=\sup\{\,|\xi|\,;\ \xi\in\partial\phi(u)\,\}\le C_\phi\left(|u|^{p*-1}+1\right)\quad\forall u\in\mathbb{R}. \tag{11}$$

Then in parallel with Proposition 3.4, we have the following proposition.

Proposition 3.7. *Assume that* $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ *satisfy* $(\mathbf{GC})_\phi$. *Let* $h \in X_S^{\tilde{p},r}$ *with* $S \in (0,T]$ *and* $u_0 \in L^{p^*}(\Omega)$. *Then* $(\mathrm{E})^h$ *with* $T = S$ *admits a unique solution* $u_h \in C([0,S]; L^2(\Omega))$ *satisfying*

$$\sup_{0\le t\le S} ||u_h(t)||_{p^*}^{p^*} + p^*\varepsilon_p \int_0^S ||u_h(t)||_{L^s}^{p^*+p-2} dt \le ||u_0||_{p^*}^{p^*} + p^* C(\varepsilon_p) ||h||_{X_S^{\tilde{p},r}}^{\tilde{p}}, \tag{12}$$

where $\varepsilon_p = \frac{p^*-1}{2}\left(\frac{p}{p^*+p-2}\right)^p K_{SP}$, $K_{SP} := \inf_{w\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{||\nabla w||_{L^p}^p}{||w||_{L^{p^*}}^p}$,

$C(\varepsilon_p) = \frac{p-1}{p^*-1}\left(\frac{p^*+p-2}{p^*-1}\right)^{-\frac{p^*+p-2}{p-1}} \varepsilon_p^{-\frac{p^*-1}{p-1}}$,

$$\begin{cases} u_h \in C([0,S]; L^{\bar{r}}(\Omega)) \quad \forall \bar{r} \in [1, p^*), \\ u_h \in L^\infty(0,S; L^{p^*}(\Omega)) \cap L^p(0,S;V), \\ \Delta_p u_h \in L^{p'}(0,S;V^*), \\ \partial u_h/\partial t \in L^{p'}(0,S;V^*) + X_S^{\tilde{p},r} + L^2(0,S;L^2(\Omega)), \\ b_h \in L^2(0,S;L^2(\Omega)), \end{cases}$$

where b_h *is the section of* $\partial\phi(u_h)$ *satisfying* $(\mathrm{E})^h$, *i.e.,* $\frac{\partial u_h}{\partial t} - \Delta_p u_h + b_h = h$.

Now we define a multivalued mapping $\mathcal{G}_{X_S^{\tilde{p},r}} : h \mapsto \widetilde{G}(u_h)$, the realization of $G(\cdot,\cdot,u_h)$ in $X_S^{\tilde{p},r}$, for $S \in (0,T]$, i.e.,

$$\mathcal{G}_{X_S^{\tilde{p},r}}(h) := \left\{ g \in X_S^{\tilde{p},r} \, ; \, g(t,x) \in G(t,x,u_h(t,x)) \text{ a.e. } (t,x) \in Q_S \right\}$$

with

$$|||\mathcal{G}_{X_S^{\tilde{p},r}}(h)|||_{X_S^{\tilde{p},r}} := \sup\left\{ ||g||_{X_S^{\tilde{p},r}} \, ; \, g(t,x) \in G(t,x,u_h(t,x)) \right\},$$

where u_h is the solution of $(\mathrm{E})^h$.

Here we introduce the following growth condition on G.

$(\mathbf{GC})^*$: We say that a multivalued map $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies $(\mathbf{GC})^*$, if there exist nonnegative numbers $q \in (p_*, p^*)$, C_q and a function $a(\cdot) \in X_T^{\tilde{p},r}$ such that:

$$|||G(t,x,u)||| \le |a(t,x)| + C_q\,|u|^{q-1} \quad \text{for a.e. } (t,x) \in Q_T\ ,\ \forall u \in D(\partial\phi)\ , \tag{13}$$

where $|||G(t,x,u)||| := \sup\{\, |\xi| \, ; \, \xi \in G(t,x,u) \,\}$.

For $R > 0$ and $S \in (0, T]$, we put

$$\mathbf{K}_R^{X_S^{\tilde{p},r}} := \{ h \in X_S^{\tilde{p},r} ; ||h||_{X_S^{\tilde{p},r}} \leq R \}. \tag{14}$$

Then, in parallel with Proposition 3.5, we can show that $\mathcal{G}_{X_S^{\tilde{p},r}}$ maps $\mathbf{K}_R^{X_S^{\tilde{p},r}}$ into itself for suitably chosen $R > 0$ and $S \in (0, T]$.

Proposition 3.8. *Assume that $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ satisfies $(\mathbf{GC})_\phi$ and $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ satisfies $(\mathbf{GC})^*$. Then for any $u_0 \in L^{p^*}(\Omega)$, there exist $R > 0$ and $T_0 \in (0, T]$ depending on $||a||_{X_T^{\tilde{p},r}}$, q and $||u_0||_{L^{p^*}}$ such that $\mathcal{G}_{X_{T_0}^{\tilde{p},r}}$ maps $\mathbf{K}_R^{X_{T_0}^{\tilde{p},r}}$ into itself.*

4 Local existence of solutions

4.1 The upper semicontinuous case

In this subsection, we give a couple of existence results for the problem (1) – (2) when the multivalued map G is upper semicontinuous with closed convex values. Namely, we assume the following:

$(\mathbf{H}_G^1)$: $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued map with closed convex values satisfying the following conditions:

(i) For almost all $(t, x) \in Q_T$, $G(t, x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is upper semicontinuous.

(ii) For each $u \in \mathbb{R}$, $G(\cdot, \cdot, u) : Q_T \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is $\mathcal{L}(Q_T)$ –measurable.

Our result in the L^2-framework is stated as follows:

Theorem 4.1. *Assume that $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ and let $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ be a multivalued mapping with closed convex values satisfying $(\mathbf{H}_G^1)$ and $(\mathbf{GC})_*$. Then for each $u_0 \in D(\varphi) = W_0^{1,p}(\Omega) \cap D(\widetilde{\phi})$, there exists $T_0 = T_0(\varphi(u_0)) \in (0, T]$ such that the initial boundary value problem (1)-(2) admits a solution u on $[0, T_0]$ satisfying*

$$\begin{cases} u \in C([0, T_0]; W_0^{1,p}(\Omega)), \\ \widetilde{\phi}(u(t)) \text{ is absolutely continuous on } [0, T_0], \\ \frac{\partial u}{\partial t}, \ \Delta_p u, \ b, \ g \in L^2(0, T_0; L^2(\Omega)), \end{cases} \tag{15}$$

where b and g are the sections of $\partial\phi(u)$ and $G(t, x, u(t, x))$, respectively, satisfying (1), i.e., $\frac{\partial u}{\partial t} - \Delta_p u + b = g$.

We now give a couple of results in the Non-Hilbert-space setting.

Theorem 4.2. *Assume that* $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ *satisfies* $(\mathbf{GC})_\phi$ *and let* $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ *be a multivalued mapping with closed convex values satisfying* $(\mathbf{H}_G^1)$ *and* $(\mathbf{GC})^*$*. Then for each* $u_0 \in L^{p^*}(\Omega)$*, there exists* $T_0 = T_0(||u_0||_{L^{p^*}}) \in (0, T]$ *such that the initial boundary value problem* (1)-(2) *admits a solution* u *on* $[0, T_0]$ *satisfying*

$$\begin{cases} u \in C([0,T_0]; L^{\bar{r}}(\Omega)) \quad \forall \bar{r} \in [1, p^*), \\ u \in L^\infty(0,T_0; L^{p^*}(\Omega)) \cap L^p(0,S;V), \\ \Delta_p u \in L^{p'}(0,T_0;V^*), \\ \partial u/\partial t \in L^{p'}(0,T_0;V^*) + X_{T_0}^{\tilde{p},r} + L^2(0,T_0;L^2(\Omega)), \\ b \in L^2(0,T_0;L^2(\Omega)), \;\; g \in X_{T_0}^{\tilde{p},r}, \end{cases} \tag{16}$$

where b *and* g *are the sections of* $\partial\phi(u)$ *and* $G(t,x,u(t,x))$*, respectively satisfying* (1)*, i.e.,* $\frac{\partial u}{\partial t} - \Delta_p u + b = g$.

Corollary 4.3. *Assume that* $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ *satisfies* $(\mathbf{GC})_\phi$ *and let* $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ *be a multivalued mapping with closed convex values satisfying* $(\mathbf{H}_G^1)$ *and* $(\mathbf{GC})^*$ *with* $a \in X_T^{\tilde{p},r} \cap \mathcal{H}_T$*. Then for each* $u_0 \in D(\varphi) = W_0^{1,p}(\Omega)$*, there exists* $T_0 = T_0(||u_0||_{L^{p^*}}) \in (0, T]$ *such that the initial boundary value problem* (1)-(2) *admits a solution* u *on* $[0, T_0]$ *satisfying* (15)*.*

4.2 The lower semicontinuous case

In this subsection, we are concerned with problem $(1) - (2)$ for the case where the multivalued map G is lower semicontinuous with closed (not necessarily convex) values. Namely, the multivalued map is assumed to satisfy the following condition:

$(\mathbf{H}_G^2)$: $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is a multivalued map with closed values such that:

(i) For almost all $(t,x) \in Q_T$, $G(t,x,\cdot) : \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is lower semicontinuous,

(ii) $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is $\mathcal{L}(Q_T) \otimes \mathcal{B}(\mathbb{R})$-measurable.

Then our result in the L^2 setting is stated as follows:

Theorem 4.4. *Let* $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ *and let* $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ *be a multivalued mapping with closed values satisfying* $(\mathbf{H}_G^2)$ *and* $(\mathbf{GC})_*$*. Then for each* $u_0 \in D(\varphi) = W_0^{1,p}(\Omega) \cap D(\widetilde{\phi})$*, there exists* $T_0 = T_0(\varphi(u_0)) \in (0, T]$ *such that the initial boundary value problem* (1)-(2) *admits a solution* u *on* $[0, T_0]$ *satisfying*

$$\begin{cases} u \in C([0,T_0]; W_0^{1,p}(\Omega)), \\ \widetilde{\phi}(u(t)) \textit{ is absolutely continuous on } [0,T_0], \\ \frac{\partial u}{\partial t}, \;\; \Delta_p u, \;\; b, \;\; g \in L^2(0,T_0;L^2(\Omega)), \end{cases} \tag{17}$$

where b *and* g *are the sections of* $\partial\phi(u)$ *and* $G(t,x,u(t,x))$*, respectively satisfying* (1)*, i.e.,* $\frac{\partial u}{\partial t} - \Delta_p u + b = g$*.*

As in Theorem 4.2 and Corollary 4.3, we obtain the following results in the Non-Hilbert-space setting.

Theorem 4.5. *Assume that* $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ *satisfies* $(\mathbf{GC})_\phi$ *and let* $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ *be a multivalued mapping with closed values satisfying* $(\mathbf{H}_G^2)$ *and* $(\mathbf{GC})^*$*. Then for each* $u_0 \in L^{p^*}(\Omega)$*, there exists* $T_0 = T_0(||u_0||_{L^{p^*}}) \in (0,T]$ *such that the initial boundary value problem* (1)-(2) *admits a solution* u *on* $[0,T_0]$ *satisfying*

$$\begin{cases} u \in C([0,T_0]; L^{\bar{r}}(\Omega)) \quad \forall \bar{r} \in [1,p^*), \\ u \in L^\infty(0,T_0; L^{p^*}(\Omega)) \cap L^p(0,T_0;V), \\ \Delta_p u \in L^{p'}(0,T_0;V^*), \\ \partial u/\partial t \in L^{p'}(0,T_0;V^*) + X_{T_0}^{\tilde{p},r} + L^2(0,T_0;L^2(\Omega)), \\ b \in L^2(0,T_0;L^2(\Omega)),\ g \in X_{T_0}^{\tilde{p},r}, \end{cases} \tag{18}$$

where b *and* g *are the sections of* $\partial\phi(u)$ *and* $G(t,x,u(t,x))$*, respectively satisfying* (1)*, i.e.,* $\frac{\partial u}{\partial t} - \Delta_p u + b = g$*.*

Corollary 4.6. *Assume that* $\phi \in \Psi(\mathbb{R}, \mathbb{R}_+)$ *satisfies* $(\mathbf{GC})_\phi$ *and let* $G : Q_T \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ *be a multivalued mapping with closed values satisfying* $(\mathbf{H}_G^2)$ *and* $(\mathbf{GC})^*$ *with* $a \in X_T^{\tilde{p},r} \cap \mathcal{H}_T$*. Then for each* $u_0 \in D(\varphi) = W_0^{1,p}(\Omega)$*, there exists* $T_0 = T_0(||u_0||_{L^{p^*}}) \in (0,T]$ *such that the initial boundary value problem* (1)-(2) *admits a solution* u *on* $[0,T_0]$ *satisfying* (17)*.*

To prove these results, the following fact plays a crucial role.

Lemma 4.7. *Let all assumptions in Theorem 4.4 or Theorem 4.5 be satisfied. Then for any* $S \in (0,T]$*, the mapping* $\mathcal{G}_{\mathcal{X}} : \mathbf{K}_R^{\mathcal{X}} \to \mathcal{X}$ *with* $\mathcal{X} = \mathcal{H}_S$ *or* $\mathcal{X} = X_S^{\tilde{p},r}$ *becomes lower semicontinuous from* $\mathcal{X}^w$ *into* $\mathcal{X}^w$ *and* $L^1(Q_S)$*. Here* $\mathcal{X}^w$ *denotes* $\mathcal{X}$ *endowed with the weak topology.*

Remark 4.8. (1) *Under the assumptions assumed in Lemma 4.7,* $\mathcal{G}_{\mathcal{X}}$ *is also lower semicontinuous from* $\mathcal{X}^w$ *into* $\mathcal{X}^s$*,* $\mathcal{X}$ *with the strong topology, for the following cases:*

(i) *The case where* $\mathcal{X} = \mathcal{H}_S$ *and* (8) *holds with* $q \in [2,p_*)$ *and* $k = 0$*.*

(ii) *The case where* $\mathcal{X} = X_S^{\tilde{p},r}$ *and* (13) *holds with* $q \in [p_*,p^*)$*.*

(2) *For the semi-linear case* $p = 2$*, under assumptions assumed in Theorem 4.4 with the condition on* q *in* $(\mathbf{GC})_*$ *replaced by the Sobolev subcritical condition, i.e.,* $q < 2^*$*,* $\mathcal{G}_{\mathcal{H}_S}$ *becomes lower semicontinuous from* $\mathcal{H}_S$ *into* $L^1(Q_S)$*. Moreover if* (8) *holds with* $k = 0$*, then* $\mathcal{G}_{\mathcal{H}_S}$ *becomes lower semicontinuous from* $\mathcal{H}_S^w$ *into* $\mathcal{H}_S$ *with the strong topology.*

5 Global existence of solutions

In this section, we discuss the global extension of local solutions given in the previous section. So in what follows, we always assume that G satisfies required assumptions with Q_T and $a \in L^1(Q_T)$ (or $a \in L^{\tilde{p}}(0,T;L^r(\Omega))$) for any $T > 0$.

Let $u(t)$ be a time-local solution of problem (1)-(2) on $[0, T_0]$ given in Theorems 4.1 and 4.4 (or Theorems 4.2, 4.5, Corollaries 4.3, 4.6). Then $u_e(t)$ defined on $[0, T_e)$ with $T_e > T_0$ is called an extension of $u(t)$ if

(i) $u_e(t)$ is a solution of (1)-(2) satisfying (15) (or (16)) with T_0 replaced by T_1 for all $T_1 \in (T_0, T_e)$.

(ii) $u_e(t) = u(t)$ for all $t \in [0, T_0]$,

and $u_m(t)$ defined on $[0, T_m)$ is called a maximal extension of $u(t)$, if $u_m(t)$ is an extension of $u(t)$ and there is no extension of $u_m(t)$, i.e., $u_m(t)$ can not be continued to the right of T_m as a solution of (1)-(2) with same regularity as (15) (or (16)).

We first prepare the following alternative lemma.

Lemma 5.1. *Let $u(t)$ be a time-local solution of problem (1)-(2) on $[0, T_0]$ given in Theorems 4.1 and 4.4 (or Theorems 4.2, 4.5, Corollaries 4.3, 4.6). Then (under the same assumptions of the above theorems) we have*

(1) There exists a least one maximal extension $u_m(t)$ of $u(t)$ defined on $[0, T_m)$ with $T_m \in (T_0, +\infty]$.

(2) Let $u_m(t)$ be any maximal extension of $u(t)$ defined on $[0, T_m)$, then the following alternative (i) or (ii) holds.

(i) $T_m = +\infty$,

(ii) $T_m < +\infty$ and

$$\lim_{t \nearrow T_m} \varphi\left(u_m\left(t\right)\right) = +\infty \quad (or \quad \lim_{t \nearrow T_m} ||u_m(t)||_{L^{p^*}} = +\infty). \tag{19}$$

By virtue of Lemma 5.1, to prove the existence of global solutions, it suffices to establish a priori bounds for $\varphi(u(t))$ or $||u(t)||_{L^{p^*}}$.

5.1 Large global solutions

Theorem 5.2. *Let $q = 2$ or $q < p$ be satisfied. Then any local solution of $(1) - (2)$ given in Theorems 4.1, 4.2, 4.4, 4.5 and Corollaries 4.3, 4.6 can be continued globally to $[0, +\infty)$.*

5.2 Small global solutions

In this subsection, we show the existence of small global solutions for sufficiently small data $a(t)$ and $u_0(x)$. In what follows we use the following notation:

$$\{f(t)\}_\ell := \begin{cases} \sup\limits_{0\le t<\infty} \displaystyle\int_t^{t+1} f(s)\,ds & \text{for } f \in L^1_{loc}(0,\infty), \\ \sup\limits_{0\le t<\infty} \displaystyle\int_t^{t+1} \widetilde{f}(s)\,ds & \text{for } f \in L^1(0,T) \;\text{ with } 0<T<\infty, \end{cases}$$

where $\widetilde{f}(s)$ is the zero extension of $f(s)$ to $[0,\infty)$.

Theorem 5.3. *Let* $\max\,(p,2) < q$, *then there exists a (sufficiently small) number* $r_0 > 0$ *such that if* $\{\,||a(\cdot,t)||_{L^1(\Omega)}\,\}_\ell \le r_0$ *and* $\varphi(u_0) \le r_0$ *(resp.* $\{\,||a(\cdot,t)||^{\tilde{p}}_{L^r(\Omega)}\,\}_\ell \le r_0$ *and* $||u_0||_{L^{p*}} \le r_0$*), then any local solution given in Theorems 4.1 and 4.4 (resp. Corollaries 4.3 and 4.6) can be continued globally to* $[0,+\infty)$.

To prove this result we prepare the following lemma which is essentially proved in the proof of Lemma 4.3 given in [28].

Lemma 5.4. *Let* $f \in L^1\,(0,T)$ *and* $j\,(\cdot)$ *be an absolutely continuous positive function on* $[0,S]$ *with* $S \in (0,T]$ *such that*

$$\frac{d}{dt}\,j\,(t) + \alpha j\,(t)^{1+\delta} \le |f\,(t)| \quad a.e. \quad t \in [0,S]\,, \tag{20}$$

where α *and* δ *are given positive parameters. Then we have*

$$\sup_{0\le t\le S} j(t) \le \max\,(j\,(0)\,, (\alpha\,\delta\,\omega)^{-\frac{1}{\delta}}) + 2\,(\omega+1)\,\{|f(t)|\}_\ell, \tag{21}$$

where ω *is an arbitrary positive constant.*

6 Examples

In this section, we exemplify the applicability of our results.

6.1 Classical problem

To demonstrate that our framework covers a broad range of the application even for classical problems, we consider some open problem for the following equation in $Q_\infty := [0,+\infty) \times \Omega$:

$$(\mathrm{P}) \begin{cases} \frac{\partial}{\partial t}u(t,x) - \Delta_p u(t,x) - g(t,x,u(t,x)) = f(t,x) & (t,x) \in Q_\infty, \\ u(t,x) = 0 & (t,x) \in [0,+\infty) \times \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

which corresponds to (1)-(2) with $Q_T = Q_\infty$, $\phi(\cdot) \equiv 0$ and $G(t,x,u) = g(t,x,u) + f(t,x)$ is a single-valued function. This problem with $u_0 \in W_0^{1,p}(\Omega)$, especially for the case $p = 2$, is extensively investigated by many authors. As for the local well-posedness in the L^2-framework for the semi-linear case, $p = 2$, is shown under the Sobolev-subcritical growth condition on g, i.e.,

$$|g(t,x,u)| \le C_q \left(|u|^{q-1} + 1\right) \quad \text{with } q < 2^* \quad \text{for a.e. } (t,x) \in Q_\infty, \tag{22}$$

where $2^* = \infty$ if $N \le 2$; and $2^* = \frac{2N}{N-2}$ if $2 < N$.

On the other hand, the study for the quasi-linear case, $p \ne 2$, is not amply pursued. Tsutsumi [37] and Ishii [26] studied the case where $p, q \in (2, \infty)$; $g(t,x,u) = |u|^{q-2}u$; $f \equiv 0$. In [37], it is shown by using Galerkin's method that there exists a global weak solution u of (P) satisfying $u \in L^\infty(0,T; W_0^{1,p}(\Omega))$, $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$ for all $T > 0$, for the following two cases:

(i) $q < p$ and $u_0 \in W_0^{1,p}(\Omega)$.

(ii) $p < q < p^*$ and u_0 belongs to the so-called "Stable Set" W, which is assured by the smallness of u_0 in $W_0^{1,p}(\Omega)$. Here p^* is the Sobolev critical exponent associated with the embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ given by $p^* = \infty$ for $p > N$ and $p^* = \frac{Np}{N-p}$ for $p < N$.

The existence of strong solutions u satisfying (15) is also discussed in [27, 26, 28]. The existence of a global strong solution for case (i) above is shown in [27]. For the case where $p < q$, the existence of a strong solution is discussed for more restrictive range of q, more precisely, under the growth condition: $q \le p_*$ (p_* is the exponent given in $(\mathbf{GC})_*$).

The existence of a local strong solution is shown in [26, 28] for any $u_0 \in W_0^{1,p}(\Omega)$, and the existence of a global strong solution is shown in [27, 26, 28] for *small* u_0 in $W_0^{1,p}(\Omega)$.

On the analogy of the semi-linear case $p = 2$, it is reasonable to support the conjecture that the existence of local solutions for (P) is assured under the Sobolev-subcritical growth condition $q < p^*$. In the former studies, however, this conjecture was not confirmed because of the lack of the elliptic estimate for $-\Delta_p$, which is the essential tool for the proof of this conjecture for the semi-linear case $p = 2$.

Our framework here provides another approach giving a positive answer to this conjecture, which does not rely directly on the elliptic estimate for $-\Delta_p$.

Theorem 6.1. *Let* $\max\,(1, \frac{2N}{N+2}) < p$ *and assume*

$(\mathbf{H}_g)$ $g : Q_T \times \mathbb{R} \to \mathbb{R}$ *satisfies the following conditions:*

(i) *For almost all* $(t,x) \in Q_\infty$, $g\,(t,x,\cdot) : \mathbb{R} \to \mathbb{R}$ *is continuous,*

(ii) *For each* $u \in \mathbb{R}$, $g(\cdot,\cdot,u) : Q_\infty \to \mathbb{R}$ *is* $\mathcal{L}(Q_\infty)-$*measurable.*

(iii) *There exists non-negative numbers* $q \in (1, p^*)$, $C_{q,1}$, $C_{q,2}$ *such that:*

$$|g(t,x,u)| \leq C_{q,1}\,|u|^{q-1} + C_{q,2} \quad a.e.\ (t,x,u) \in Q_\infty \times \mathbb{R}. \tag{23}$$

Then for any $u_0 \in W_0^{1,p}(\Omega)$ *and* $f \in X_T^{\tilde{p},r} \cap L^2(0,T;L^2(\Omega))$, *there exists* $T_0 \in (0,T]$ *such that* (P) admits a solution u satisfying

$$\begin{cases} u \in C([0,T_0];W_0^{1,p}(\Omega)), \\ \frac{\partial u}{\partial t},\ \Delta_p u,\ g(\cdot,\cdot,u) \in L^2(0,T_0;L^2(\Omega)). \end{cases} \tag{24}$$

Moreover, we have:

(i) Let $1 < q \leq 2$ or $2 < q < p$ be satisfied, then the local solution given above can be continued to $[0,+\infty)$.

(ii) Let max $(p,2) < q$ and $C_{q,2} = 0$, then there exists a (sufficiently small) number $r_0 > 0$ such that if $\{\|f(t)\|_{L^2}\}_\ell \leq r_0$ and $||u_0||_{W_0^{1,p}} \leq r_0$ (resp. $\{\|f(t)\|_{L^r}^{\tilde{p}}\}_\ell \leq r_0$ and $\|u_0\|_{p^*} \leq r_0$), then the local solution given above can be continued to $[0,+\infty)$, provided that $q \leq p_*$ (resp. $p_* < q < p^*$).

Proof. It is easy to see that $(\mathbf{H}_G^1)$, $(\mathbf{GC}_*)$ and $(\mathbf{GC}^*)$ are derived from $(\mathbf{H}_g)$ with $G = g + f$ and $a = f + C_{q,2}$. We note that the case $1 < q < 2$ can be reduced to the case $q = 2$, since

$$|u|^{q-1} \leq |u| + 1 \quad \forall u \in \mathbb{R}, \quad \forall q \in (1,2). \tag{25}$$

Then we can apply Theorem 4.1 and Corollary 4.3 for the existence of local solutions. To derive the continuation of local solutions, it suffices to apply Theorems 5.2 and 5.3. □

6.2 The case where $D(\phi) = \mathbb{R}^1$

Let $\beta(\cdot) = \partial\phi(\cdot)$ be a maximal monotone graph in $\mathbb{R}^1 \times \mathbb{R}^1$ such that

$$\phi(0) = 0 = \min_{u \in \mathbb{R}} \phi(u)$$

and there exists $C_1 > 0$ such that

$$C_1\left(|u|^{2(s_0-1)} - 1\right) \leq |\partial^0\phi(u)|^2 \ \text{ for all } u \in D(\phi) = \mathbb{R}^1,\ 1 < s_0 < \infty. \tag{26}$$

We also introduce a class of continuous functions $\mathcal{C}_q$ by the following: $f \in \mathcal{C}_q$ if and only if $f : \mathbb{R}^1 \to \mathbb{R}^1$ is continuous and satisfies

$$|f(u)| \leq C_q\,|u|^{q-1} + C_0 \quad \forall u \in \mathbb{R}^1, \tag{27}$$

where $1 < q <$ max (p^*, s_0), $C_q \geq 0$, $C_0 \geq 0$. In the following we consider the case where

$$G(t,x,u) = G_0(u) + f_e(t,x), \tag{28}$$

where $f_e(t,x)$ is a given forcing term defined on Q_∞.

6.2.1 The upper semicontinuous case

Take $f_1^+,\ f_1^-,\ f_2^+,\ f_2^- \in \mathcal{C}_q$, such that

$$f_1^-(u) < f_2^-(u)\ \ \forall u \in (-\infty, 0]\,,$$
$$f_1^+(u) < f_2^+(u)\ \ \forall u \in [0, +\infty)\,,$$
$$f_2^-(0) < f_1^+(0)\,,$$

and define

$$G_0(u) = \begin{cases} \left[f_1^-(u)\,, f_2^-(u)\right] & \text{if} \quad u < 0, \\ \left[f_1^+(u)\,, f_2^+(u)\right] & \text{if} \quad u > 0, \\ \left[f_1^-(0)\,, f_2^+(0)\right] & \text{if} \quad u = 0. \end{cases} \tag{29}$$

Then it is easy to see that $G(t,x,u) = G_0(u) + f_e(t,x)$ is a closed convex and upper semicontinuous multivalued function satisfying $(\mathbf{H}_G^1)$. We here note that there is no continuous section of $G_0(\cdot)$.

Since $f_1^+,\ f_1^-,\ f_2^+,\ f_2^- \in \mathcal{C}_q$, by virtue of (27), there exist $\widetilde{C}_0 \geq 0$, and $\widetilde{C}_q \geq 0$ such that

$$|||G(t,x,u)|||^2 \leq \widetilde{C}_0 + |f_e(t,x)|^2 + \widetilde{C}_q\,|u|^{2(q-1)} \quad (t,x,u) \in Q_\infty \times \mathbb{R}^1. \tag{30}$$

(I) Local solutions

(1) Let $q \in (1, \max\ (p_*, s_0))$, $u_0 \in D(\varphi) = W_0^{1,p}(\Omega) \cap D(\widetilde{\phi})$ and $f_e \in L^2_{loc}(0,\infty; L^2(\Omega)))$. Then we can apply Theorem 4.1 to assure the existence of local solutions satisfying (15).

In fact if $q \in (1, p_*)$, then in view of (30) and (25), we can easily check (8) of $(\mathbf{GC})_*$ with $k = 0$, $C_q = \widetilde{C}_q$ and $a = \widetilde{C}_0 + |f_e|^2 \in L^1(Q_T)$ for any $T > 0$. As for the case where $1 < q < s_0$, since for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|u|^{2(q-1)} \leq \varepsilon\,|u|^{2(s_0-1)} + C_\varepsilon \quad \text{for all} \quad u \in \mathbb{R}^1, \tag{31}$$

in view of (26), for sufficiently small $\varepsilon > 0$, we can show that (8) is satisfied with $k = \varepsilon\frac{\widetilde{C}_q}{C_1} \in (0,1)$, $a(t,x) = |f_e(t,x)|^2 + \widetilde{C}_0 + \widetilde{C}_q(C_\varepsilon + \varepsilon) \in L^1(Q_T)$ for any $T > 0$ and $C_q = 0$, $q = 2$.

(2) Let $q \in (1, p^*)$, $u_0 \in D(\varphi) = W_0^{1,p}(\Omega) \cap D(\widetilde{\phi})$ and $f_e \in L^2(Q_T) \cap X_T^{\tilde{p},r}$ for any $T > 0$. Furthermore we assume that ϕ satisfies $(\mathbf{GC})_\phi$. Then in view of (30) and (25), we can check (13) of $(\mathbf{GC})^*$ with $C_q = (\widetilde{C}_q)^{1/2}$ and $|a| = (|f_e|^2 + \widetilde{C}_0)^{1/2} \in L^2(Q_T) \cap X_T^{\tilde{p},r}$ for any $T > 0$. Hence Corollary 4.3 assures the existence of local solutions satisfying (15).

Furthermore we have:

(II) Large global solutions

(**3**) The case where $1 < q \leq 2$ or $2 < q < p$:
Theorem 5.2 assures that every local solution can be continued globally to $[0, +\infty)$. (Note that the case $1 < q < 2$ can be reduced to the case $q = 2$ by (25).)

(**4**) The case where $q < s_0 < +\infty$: As already mentioned above, (8) is satisfied with $C_q = 0,\ q = 2$. Then we can apply Theorem 5.2 with $q = 2$ to assure the existence of global solutions.

(III) **Small global solutions**

(**5**) The case where $\max\,(p, 2) < q < p^*$: We here assume that $f_1^{\pm}$ and $f_2^{\pm}$ satisfy (27) with $C_0 = 0$. Then (30) is satisfied with $\widetilde{C}_0 = 0$ and we can apply Theorem 5.3 with $k = 0$, $C_q = \widetilde{C}_q$ and $a = |f_e|^2$ if $1 < q \leq p_*$; and with $k = 0$, $C_q = (\widetilde{C}_q)^{1/2}$ and $a = |f_e|$ if $p_* < q < p^*$. Thus the existence of global solutions is assured for sufficiently small u_0 and f_e in the sense of Theorem 5.3.

6.2.2 The lower semicontinuous case

Let $-\infty \leq r_0 < 0 < r_1 \leq +\infty$ and take $f^+,\ f^- \in \mathcal{C}_q$ such that

$$f^-(u) < f^+(u) \quad \forall u \in (r_0, r_1)$$

and define

$$G_0(u) = \begin{cases} \{f^+(u)\} & \text{if } \ r_1 \leq u \text{ (when } r_1 < +\infty), \\ \{f^-(u)\} & \text{if } \ u \leq r_0 \text{ (when } -\infty < r_0), \\ [f^-(u), f^+(u)] \cap \mathbb{Q}_n & \text{if } \ u \in (r_0, r_1), \end{cases} \tag{32}$$

where $\mathbb{Q}_n := \{q \in \mathbb{Q} : 10^n q \in \mathbb{Z}\}$ with n sufficiently large so that $[f^-(u), f^+(u)] \cap \mathbb{Q}_n \neq \emptyset$ for all $u \in (r_0, r_1)$.

Then it is easy to see that $G(t, x, u) = G_0(u) + f_e(t, x)$ is a closed but not convex valued lower semicontinuous function satisfying $(\mathbf{H}_G^2)$, and that there is no continuous section of $G_0(\cdot)$. Furthermore, since $f^{\pm} \in \mathcal{C}_q$, (30) is also satisfied.

Thus the same assertions on the existence of local and global solutions as those in the previous case hold.

6.3 The case where $D(\phi)$ is precompact

Here we consider the case where $D(\phi)$ is precompact, i.e.,

$$D(\phi) = \left\{u \in \mathbb{R}^1 ; \phi(u) < +\infty\right\} \subset [a, b] \ \text{ with } \ -\infty < a < 0 < b < +\infty.$$

Typical examples are given by

(i)

(**1**)

$$\phi\left(u\right)=I_{[a,b]}\left(u\right)=\begin{cases}0 & \text{if } u\in[a,b]\,,\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$\partial\phi\left(u\right)=\partial I_{[a,b]}\left(u\right)=\begin{cases}\{0\} & \text{if } \ u\in(a,b)\,,\\ (-\infty,0] & \text{if } \ u=a,\\ [0,+\infty) & \text{if } \ u=b,\\ \emptyset & \text{if } \ u\notin[a,b]\,;\end{cases}$$

(**2**)

$$\phi\left(u\right)=\phi_h\left(u\right)=\begin{cases}h\left(u\right) & \text{if } u\in(a,b)\,,\\ +\infty & \text{otherwise,}\end{cases}$$

where $h\in C^1\left((a,b)\,;\mathbb{R}^1\right)$ is convex and satisfies

$$\lim_{u\to a+0}h\left(u\right)=\lim_{u\to b-0}h\left(u\right)=+\infty.$$

Then we have

$$\partial\phi\left(u\right)=\partial\phi_h\left(u\right)=\begin{cases}h'\left(u\right) & \text{if } \ u\in(a,b)\,,\\ \emptyset & \text{if } \ u\notin(a,b)\,.\end{cases}$$

We again define $G_0(u)$ by (29) (resp. (32)) and $G\left(t,x,u\right)$ by (28). Assume that $f_i^{\pm}$ (resp. $f^{\pm}$) belong to $C(\mathbb{R}^1;\mathbb{R}^1)$. Then since $D\left(\partial I_{[a,b]}\right)=[a,b]$ and $D\left(\partial\phi_h\right)=(a,b)$, we can verify $(\mathbf{GC})_*$ with $k=0$, $C_q=0$ (for any $q\in[2,p)$) and $a(t,x)=C_f+f_e(t,x)$, where $C_f=\max\left\{\,|f_i^{\pm}(r)|\,;\ a\le r\le b,\ i=1,2\,\right\}$ (resp. $C_f=\max\left\{\,|f^{\pm}(r)|\,;\, a\le r\le b\,\right\}$).

Hence, for every $u_0\in D\left(\varphi\right)$ and $f_e\in L^2_{loc}\left([0,\infty);L^2(\Omega)\right)$, Theorem 5.2 assures the existence of global solutions of (1) for the case where G_0 is u.s.c (resp. l.s.c).

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