Multivalued ordinary differential equation governed by hypergraph Laplacian*

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1 Introduction

The *(weighted) hypergraph* is defined as a triplet $G = (V, E, w)$ of

- a finite set $V = \{v_1, \ldots, v_n\}$ (vertex set),
- a family $E \subset 2^V$ of subsets with more than one element of V, that is, $\#e \geq 2$ for every $e \in E$ (set of hyperedges),
- a function $w : E \to (0, \infty)$ (edge weight).

This can be interpreted as a model of a network structure in which vertices $v_1, \ldots, v_n \in V$ are connected by each hyperedge $e \in E$ (see Figure 1).

As for the case where G is a usual graph (i.e., every $e \in E$ satisfies $\#e = 2$), then an operator called "graph Laplacian" can be defined as the matrix of order $n = \#V$, which describes the random walk movement of particles on the graph. It is well known that the network structure of the graph can be investigated through the study of eigenvalues of the graph Laplacian, which is called "spectral graph theory" established in the 1980s. This theory has been applied to the algorithm of measuring the importance of website, which is called PageRank, and the Cheeger type inequality, which is related to the cluster analysis (see, e.g., [3, 4, 5, 7] and references therein).

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*Joint work with Masahiro Ikeda (Riken/Keio University).

Keywords: Hypergraph, hypergraph Laplacian, subdifferential, nonlinear evolution equation, p-Laplacian.

In order to develop the spectral graph theory to more general networks, Prof. Yuichi Yoshida introduced an operator called the hypergraph Laplacian $L_{G,p}$ in [19]. As seen in the next section, $L_{G,p}$ is a nonlinear and multivalued operator defined as the subdifferential of some convex function. Therefore, we can apply the abstract theory for the nonlinear evolution equation to this operator and the differential equation governed by $L_{G,p}$.

In this paper, we aim to study the hypergraph Laplacian $L_{G,p}$ more precisely beyond the facts known as the abstract results. In this next section, we state the definition of the hypergraph Laplacian and some facts which can be derived from the general theory. In Section 3, as for our main assertion, we introduce an inequality which holds for the hypergraph Laplacian $L_{G,p}$ and quite resembles the Poincaré–Wirtinger inequality in PDE. We next consider the evolution equation $x'(t) + L_{G,p}(x(t)) \ni h(t)$. This ODE has been applied to study the Cheeger like inequality and the PageRank of network represented by hypergraph (see, e.g., [10, 12, 18]). However, due to the complexity of structure, it seems that the argument for usual graphs, namely, the case where Laplacian is a matrix, is almost broken and the details of the ODE have not been well discussed yet. By using the Poincaré–Wirtinger type inequality, in the final section, we show some results for this equation other than the solvability, which have been already assured by the abstract theory.

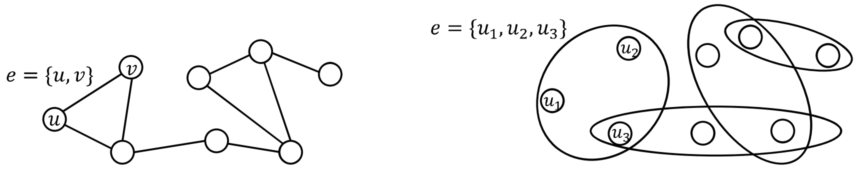

Figure 1: If $e \in E$ consists of two elements, then $e = \{u, v\}$ can be regarded as a segment connecting two vertices u, v. Hence every $e \in E$ is a binary set, then G represents a network composed of points and connecting lines (left figure, called *usual graph*). The hypergraph is a generalization of the usual graph which allows the grouping of multiple members (right figure). The weight $w(e)$ represents the multiplicity of connecting lines $e \in E$ or the degree of ease of a heat/particle flow across the pass $e \in E$.

2 Definition and Properties of Hypergraph Laplacian

2.1 Preliminary

We first review some well-known facts of maximal monotone operators and subdifferential operators for later use (details and their proof can be found in, e.g., [1, 2, 17]). Let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$. A (possibly) multivalued operator $A : H \to 2^H$ (the power set of H) is said to be monotone if $(\eta_1 - \eta_2, \xi_1 - \xi_2) \geq 0$ holds for any $\xi_j \in D(A)$ (the domain of A) and $\eta_j \in A\xi_j$ $(j = 1, 2)$. Moreover, a monotone operator A is said to be maximal monotone if there is no monotone operator which contains A properly. If A is maximal monotone,

- $A\xi$ forms a closed convex subset in H for every $\xi \in D(A)$.
- If $\eta_m \in A\xi_m$, $\xi_m \to \xi$ strongly in H, and $\eta_m \rightharpoonup \eta$ weakly in H as $m \to \infty$, then $\xi \in D(A)$ and $\eta \in A\xi$ hold (i.e., the maximal monotone operator is demiclosed).

It is well known that the subdifferential operator of a proper lower semi-continuous convex function is always maximal monotone. Here the subdifferential of a proper (i.e., $g \not\equiv +\infty$) lower semi-continuous and convex functional $g : H \to (-\infty, +\infty]$ is defined by

(1) $$\partial g : \xi \mapsto \{\eta \in H; \quad (\eta, z - \xi) \leq g(z) - g(\xi) \quad \forall z \in H\}.$$

By the definition, we also find that

(2) $$0 \in \partial g(\xi) \quad \Leftrightarrow \quad g(\xi) \leq g(z) \ \ \forall z \in H, \quad \text{i.e.,} \quad g(\xi) = \min_{z \in H} g(z).$$

There are many studies which are concerned with the theory of solvability of evolution equations governed by the subdifferential operator, which is the so-called Kōmura–Brézis theory. For instance, the existence of a unique solution to the Cauchy problem for a basic equation has been assured as follows (see [14] and [2, Theorem 3.6–3.7]):

Proposition 1. Let $g : H \to (-\infty, +\infty]$ be a proper lower semi-continuous convex function and assume that $\xi_0 \in D(g) := \{z \in H; g(z) < \infty\}$ and $h \in L^2(0, T; H)$. Then

(3) $$\begin{cases} \xi'(t) + \partial g(\xi(t)) \ni h(t) & t \in (0, T), \\ \xi(0) = \xi_0, \end{cases}$$

possesses a unique solution satisfying $\xi \in W^{1,2}(0, T; H)$. Moreover, if $t_0 \in [0, T)$ is a right-Lebesgue point of h (i.e., $\exists h(t_0 + 0) := \lim_{\tau \to +0} \frac{1}{\tau} \int_{t_0}^{t_0+\tau} h(s)ds$), the solution is right-differentiable at t_0 and its right-derivative satisfies

(4) $$\frac{d^+\xi}{dt}(t_0) = \left(h(t_0 + 0) - \partial g(\xi(t_0))\right)^\circ,$$

where $C^\circ := \operatorname{argmin}_{z\in C} \|z\| = \operatorname{Proj}_C 0$ for a closed convex set $C \subset H$.

When $g_\mu : H \to \mathbb{R}$ $(\mu = 1, \ldots, m)$ are lower semi-continuous convex functions, their maximum envelope $g(\xi) := \max_{\mu=1,\ldots,m} g_\mu(\xi)$ is also convex and lower semi-continuous on H, and then the subdifferential of g can be define. In the case where H is the finite dimensional space, we have the following maximum rule of subdifferential (or the so-called Danskin-Bertseka's Theorem, see, e.g., [15, Proposition 2.54]):

Proposition 2. Let H be a finite dimensional space, $g_\mu : H \to \mathbb{R}$ $(\mu = 1, 2, \ldots, m)$ be convex functions satisfying $D(g_\mu) = H$, and $g(\xi) := \max_{\mu=1,\ldots,m} g_\mu(\xi)$. Then for every $\xi \in H$, the subdifferential of g can be represented by

$$\partial g(\xi) = \partial\left(\max_{\mu=1,\ldots,m} g_\mu(\xi)\right) = \operatorname{conv}\left(\bigcup_{\nu\in N(\xi)} \partial g_\nu(\xi)\right),$$

where

$$N(\xi) := \left\{\nu \in \{1, 2, \ldots, m\};\ \ g_\nu(\xi) = \max_{\mu=1,\ldots,m} g_\mu(\xi)\right\}.$$

2.2 Definition

For simplicity, we suppose that the hypergraph $G = (V, E, w)$ is *connected* throughout this paper (general case is discussed in [13]). Namely, assume that for every $u, v \in V$ there exist some $u_1, \ldots, u_{N-1} \in V$ and $e_1, e_2, \ldots, e_N \in E$ s.t. $u_{j-1}, u_j \in e_j$ for any $j = 1, 2, \ldots, N$, where $u_0 = u$ and $u_N = v$ (see Figure 2).

In this paper, we consider the operators and the differential equations on $\mathbb{R}^V$, which stands for the family of mappings $x : V \to \mathbb{R}$. Obviously, we can identify $\mathbb{R}^V$ with the n-dimensional Euclidean space $\mathbb{R}^n$ by letting $x_i := x(v_i)$ and $x \sim (x_1, \ldots, x_n)$. Hence $\mathbb{R}^V$ can be regarded as a real Hilbert space with the following standard inner product and norm:

$$x \cdot y := \sum_{v\in V} x(v)y(v) = \sum_{i=1}^{n} x_i y_j, \qquad \|x\| := \sqrt{x \cdot x} \qquad x, y \in \mathbb{R}^V.$$

For later use, we here define $1_S : V \to \mathbb{R}$ with $S \subset V$ by

$$1_S(v) := \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{if } v \in V \setminus S. \end{cases}$$

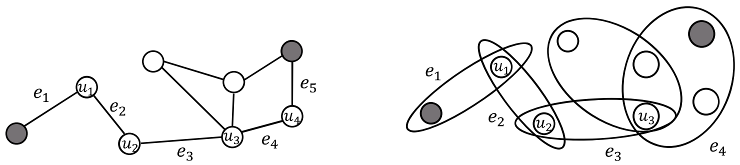

Figure 2: When G is connected, we can select suitable relay points $u_1, u_2, \ldots$ and bridges $e_1, e_2, \ldots$ which connect the two vertices chosen arbitrarily. This also implies that there is no isolated vertex nor island.

When $S = \{v\}$, we might abbreviate $1_{\{v\}}$ as 1_v. Note that $1_{v_i} \in \mathbb{R}^V$ can be identified with the i-th unit vector of the canonical basis of $\mathbb{R}^n$ and $1_V \sim (1, \ldots, 1)$. Moreover, let $B_e \subset \mathbb{R}^V$ be a subset defined with respect to each hyperedge $e \in E$ by

$$
\begin{aligned}
B_e &:= \operatorname{conv}\{1_u - 1_v \in \mathbb{R}^V;\ u, v \in e\} \\
&= \operatorname{conv}\left\{ (\ldots, 0, \overset{\overset{i}{\vee}}{1}, 0, \ldots, 0, \overset{\overset{j}{\vee}}{-1}, 0, \ldots) \in \mathbb{R}^n;\ i, j \text{ s.t. } v_i, v_j \in e \right\},
\end{aligned} \tag{5}
$$

which is called the *base polytope* for $e \in E$. Here $\operatorname{conv} Q$ denotes the convex hull of $Q \subset \mathbb{R}^V$.

We here consider

$$
f_e(x) := \max_{u,v \in e}(x(u) - x(v)) = \max_{u,v \in e} |x(u) - x(v)| = \max_{\substack{i,j \text{ s.t.} \\ v_i, v_j \in e}} |x_i - x_j| \qquad x \in \mathbb{R}^V, \tag{6}
$$

where $e \in E$. By (5), we also have $f_e(x) = \max_{b \in B_e} b \cdot x$. Clearly, $f_e : \mathbb{R}^V \to \mathbb{R}$ is continuous and convex, then f_e is subdifferentiable at every $x \in \mathbb{R}^V$. Since $f_e(x)$ is defined as the maximum envelope of $g_{ij}(x) := x_i - x_j$ (i, j s.t. $v_i, v_j \in e$) and $\partial g_{ij}(x) = 1_{v_i} - 1_{v_j}$, we can derive from Proposition 2

$$
\partial f_e(x) = \operatorname*{argmax}_{b \in B_e} b \cdot x = \left\{ b_e \in B_e;\ \ b_e \cdot x = \max_{b \in B_e} b \cdot x \right\}. \tag{7}
$$

Obviously, $b_e \cdot x = f_e(x)$ holds for every $x \in \mathbb{R}^V$ and $b_e \in \partial f_e(x)$.

As seen in (1), the subgradient of convex function possibly returns set-value at some $x \in \mathbb{R}^V$ where the functional is non-smooth. We here check the case where $\partial f_e(x)$ is a singleton or a set-value. Let $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{V\}$, i.e., the hyperedge includes all vertices of V. Remark that $f_e = f_V = \max_{i,j=1,\ldots,4} |x_i - x_j|$.

(Ex.1) Let $x = (x_1, x_2, x_3, x_4) = (2, 1, 1, -2)$. Since $f_e(x) = x_1 - x_4$, we have

$$
\partial f_e(x) = (1, 0, 0, -1) = 1_{v_1} - 1_{v_4}.
$$

Namely, $\partial f_e(x)$ coincides with the derivative of $x \mapsto x_1 - x_4$ and becomes a singleton.

(Ex.2) Let $x = (x_1, x_2, x_3, x_4) = (2, 2, 0, -1)$. Since $f_e(x) = x_1 - x_4 = x_2 - x_4$, we have

$$\begin{aligned}\partial f_e(x) &= \{(\lambda(1_{v_1} - 1_{v_4}) + (1-\lambda)(1_{v_2} - 1_{v_4});\ \lambda \in [0,1]\} \\ &= \{(\lambda, (1-\lambda), 0, -1)\,;\ \lambda \in [0,1]\}.\end{aligned}$$

(Ex.3) Let $x = (x_1, x_2, x_3, x_4) = (1, 1, -1, -1)$. Since $f_e(x) = x_1 - x_4 = x_2 - x_4 = x_1 - x_3 = x_2 - x_3$, then $\partial f_e(x)$ coincides with the convex combination of $1_{v_1} - 1_{v_4} = (1, 0, 0, -1)$, $1_{v_2} - 1_{v_4} = (0, 1, 0, -1)$, $1_{v_1} - 1_{v_3} = (1, 0, -1, 0)$, and $1_{v_2} - 1_{v_3} = (0, 1, -1, 0)$, i.e.,

$$\partial f_e(x) = \{(\lambda, (1-\lambda), -\mu, -(1-\mu))\,;\ \lambda, \mu \in [0,1]\}\,.$$

These examples imply that $\partial f_e(x)$ becomes multi-valued if $v \mapsto x(v)$ takes the maximum or minimum value at several vertices v, that is to say, the components of the vector $x = (x_1, \ldots, x_n)$ take maximum or minimum values for multiple coordinates.

By using f_e defined for each $e \in E$, we set the following continuous convex function on $\mathbb{R}^V$:

$$\varphi_{G,p}(x) := \frac{1}{p} \sum_{e \in E} w(e)(f_e(x))^p \qquad p \in [1, \infty). \tag{8}$$

Thanks to the subdifferential formula of the composition of functionals (see, [6, Corollary 3.5] and also [13, Proposition 2.2]), we can calculate the subgradient of $\varphi_{G,p}$ like the standard chain rule of differential:

$$\begin{aligned}\partial\varphi_{G,p}(x) &= \sum_{e \in E} w(e)(f_e(x))^{p-1} \partial f_e(x) \\ &= \left\{ \sum_{e \in E} w(e)(f_e(x))^{p-1} b_e; \quad b_e \in \operatorname*{argmax}_{b \in B_e} b \cdot x \right\}.\end{aligned} \tag{9}$$

Definition 1. The *hypergraph (p-)Laplacian* $L_{G,p} : \mathbb{R}^V \to 2^{\mathbb{R}^V}$ on the hypergraph $G = (V, E, w)$ with $p \in [1, \infty)$ is defined by $L_{G,p} := \partial\varphi_{G,p}$.

Remark 1. If $p > 1$ and G is a usual graph, i.e., each $e \in E$ contains two elements, $L_{G,p}(x)$ becomes a single-valued operator. Indeed, since $f_e(x) = |x(v_i) - x(v_j)|$ when $e = \{v_i, v_j\}$, we get

$$\varphi_{G,p}(x) = \frac{1}{2p} \sum_{i,j=1}^{n} w_{ij} |x_i - x_j|^p,$$

where $x(v_i)$ is abbreviated to x_i and

$$w_{ij} := \begin{cases} w(\{v_i, v_j\}) & \text{if} \quad \{v_i, v_j\} \in E, \quad \text{i.e., } v_i \text{ and } v_j \text{ are connected,} \\ 0 & \text{if} \quad \{v_i, v_j\} \notin E, \quad \text{i.e., } v_i \text{ and } v_j \text{ are disconnected.} \end{cases}$$

Clearly, this functional is differentiable in the classical sense except for $p = 1$ and its subgradient coincides with its derivative (see [1, Ch.1.2]). Especially, calculating partial derivatives for the case where $p = 2$, we have

$$\begin{aligned} \partial_{x_i} \varphi_{G,2}(x) &= \sum_{j=1}^{n} w_{ij}(x_i - x_j) = d_i x_i - \sum_{j=1}^{n} w_{ij} x_j \\ &= (-w_{i1}, \ldots, d_i - w_{ii}, \ldots, -w_{in}) \cdot x, \end{aligned}$$

where $d_i := \sum_{j=1}^{n} w_{ij}$ denotes the (weighted) number of vertex connected to v_i. Hence $L_{G,2} = \partial \varphi_{G,2}$ coincides with the classical graph Laplacian matrix for the usual graph, that is, $L_{G,2} = D - A$, where $D := \mathrm{diag}(d_1, \ldots, d_n)$ and $A := (w_{ij})$ are the square matrices of order $n = \#V$ called the (weighted) degree matrix and the (weighted) adjacency matrix, respectively.

On the other hand, when G is a hypergraph, $L_{G,p}(x)$ possibly returns a set-value on $\bigcup_{e \in E} \bigcup_{u,v \in e} \{x \in \mathbb{R}^V;\ x(u) = x(v)\}$ (union of hyperplanes) by the singularity of derivative of the max-function even if $p > 1$.

Remark 2. From the point of view of the graph theory and the discrete convex analysis, the function $f_e : \mathbb{R}^V \to \mathbb{R}$ is derived from the Lovász extension of the following set-function on 2^V, that is, the Choquet integral with respect to the following non-additive measure over V (see, e.g, [11, §6.3]):

$$F_e(S) = \begin{cases} 1 & \text{if } e \cap S, \quad e \cap S^c \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

which is called the cut function with the hyperedge $e \in E$. In [19], a generalization of the graph Laplacian (called the *submodular Laplacian*) is introduced as the subdifferential of the energy functional consisting of the Lovász extension of the general submodular set-function.

2.3 Basic Tools

We here consider the minimizers of f_e and $\varphi_{G,p}$. Remark that $f_e(x), \varphi_{G,p}(x) \geq 0$ hold for any $x \in \mathbb{R}^V$. By the definition (6), we obtain

$$f_e(x) = \max_{u,v \in e} |x(u) - x(v)| = 0 \quad \Leftrightarrow \quad x(u) = x(v) \ \ \forall u, v \in e.$$

Namely, x is the minimizer of f_e if and only if the elements of x at vertices contained by $e \in E$ take the same value. This fact immediately yields that

$$
\begin{aligned}
&\varphi_{G,p}(x) = \min_{y\in\mathbb{R}^V} \varphi_{G,p}(y) = 0 \quad \Leftrightarrow \quad f_e(x) = 0 \ \ \forall e \in E \\
&\quad \Leftrightarrow \quad x(u) = x(v) \ \ \forall u,v \in e \ \ \forall e \in E \quad \Leftrightarrow \quad x(u) = x(v) \ \ \forall u,v \in V \\
&\quad \Leftrightarrow \quad \exists c \in \mathbb{R} \ \text{ s.t. } x = c1_V = (c,\dots,c),
\end{aligned} \tag{10}
$$

since the hypergraph $G = (V,E,w)$ is assumed to be connected. Moreover, we obtain the following.

Theorem 2.1. Let $p \geq 1$. Then $x \in \mathbb{R}^V$ satisfies $0 \in L_{G,p}(x)$ if and only if $x = c1_V$ with some $c \in \mathbb{R}$. Furthermore, for every $x \in \mathbb{R}^V$ and $c \in \mathbb{R}$, it holds that

$$
\varphi_{G,p}(x + c1_V) = \varphi_{G,p}(x), \qquad L_{G,p}(x + c1_V) = L_{G,p}(x). \tag{11}
$$

Proof. The first result is derived from (2) and (10) directly. We next check (11). Since $(1_u - 1_v) \cdot 1_V = 1 - 1 = 0$ holds for any $u,v \in V$, we obtain $b \cdot 1_V = 0$ for any $b \in B_e = \mathrm{conv}\{1_u - 1_v;\ \ u,v \in e\}$. This leads to

$$
f_e(x + c1_V) = \max_{b\in B_e} b \cdot (x + c1_V) = \max_{b\in B_e} b \cdot x = f_e(x)
$$

for every $e \in E$, $x \in \mathbb{R}^V$, and $c \in \mathbb{R}$. By (7), we also have

$$
\begin{aligned}
b_e \in \partial f_e(x + c1_V) \quad &\Leftrightarrow \quad b_e \cdot (x + c1_V) = \max_{b\in B_e} b \cdot (x + c1_V) \\
&\Leftrightarrow \quad b_e \cdot x = \max_{b\in B_e} b \cdot x \quad \Leftrightarrow \quad b_e \in \partial f_e(x).
\end{aligned}
$$

Therefore, these and the definition (8)(9) entail (11). □

Remark 3. Theorem 2.1 implies the lack of coerciveness of $\varphi_{G,p}$. Indeed,

$$
\frac{\varphi_{G,p}(x + c1_V)}{\|x + c1_V\|} = \frac{\varphi_{G,p}(x)}{\|x + c1_V\|} \to 0 \ \text{ as } \ |c| \to \infty.
$$

3 Poincaré–Wirtinger Type Inequality

By the general theory of gradient flows, the solution to the evolution equation $\xi'(t) + \partial g(\xi(t)) \ni 0$ acts toward the minimizer of the energy functional g and the limit of solution $\lim_{t\to\infty} \xi(t)$ attains the minimum of g. According to this fact, we can expect that the hypergraph Laplacian $L_{G,p}$ has the effect of homogenizing the values of x and then the solution might move toward the mean value of the initial state.

Based on this, we here define the averaging of x by

$$\overline{x} := \left(\frac{1}{n} \sum_{v \in V} x(v) \right) 1_V = \left(\frac{1}{n} \sum_{i=1}^{n} x_i, \dots, \frac{1}{n} \sum_{i=1}^{n} x_i \right), \tag{12}$$

that is, a vector whose elements are the mean value of x. Then we can obtain the following inequality.

Theorem 3.1. Let $p \geq 1$. Then every $x \in \mathbb{R}^V$ and $y \in L_{G,p}(x)$ satisfy

$$\|x - \overline{x}\|^p \leq p\ \Gamma_{G,p} \varphi_{G,p}(x) = \Gamma_{G,p}\ x \cdot y, \tag{13}$$

where

$$\Gamma_{G,p} := \frac{n^p (N^*)^{p-1}}{\min_{e \in E} w(e)} \tag{14}$$

and N^* is the "diameter" of G, i.e.,

$$N^* := \max_{u,v \in V} \mathrm{dist}(u,v),$$
$$\mathrm{dist}(u,v) := \min \left\{ N;\ \begin{matrix} \exists u_1, \dots, u_{N-1} \in V, \quad \exists e_1, \dots, e_N \in E \text{ s.t.} \\ u_{i-1}, u_i \in e_i\ \forall i = 1, 2, \dots, N \quad (u_0 = u,\ u_N = v). \end{matrix} \right\}.$$

Proof. Since $b_e \cdot x = f_e(x)$ holds for every $x \in \mathbb{R}^V$ and $b_e \in \partial f_e(x)$, the representation of $L_{G,p}$ implies $x \cdot y = \sum_{e \in E} w(e)(f_e(x))^p = p\varphi_{G,p}(x)$ (recall (9)). In order to show $\|x - \overline{x}\|^p \leq p\ \Gamma_{G,p} \varphi_{G,p}(x)$, fix $u, v \in V$ arbitrarily and select $u_1, \dots, u_{N-1} \in V$ and $e_1, \dots, e_N \in E$ such that $u_{i-1}, u_i \in e_i$ for any $i = 1, 2, \dots, N$, where $u_0 = u$ and $u_N = v$ (recall that G is assumed to be connected and note that $N \leq N^*$). Hence by using Hölder's inequality, we have

$$\begin{aligned} |x(u) - x(v)| &\leq \sum_{i=1}^{N} |x(u_{i-1}) - x(u_i)| \leq \sum_{i=1}^{N} f_{e_i}(x) \\ &\leq \frac{(N^*)^{1/p'}}{\min_{e \in E} w(e)^{1/p}} \left(\sum_{e \in E} w(e)(f_e(x))^p \right)^{1/p} = \frac{(N^*)^{1/p'}}{\min_{e \in E} w(e)^{1/p}} \left(p\varphi_{G,p}(x) \right)^{1/p}, \end{aligned}$$

where $p' = p/(p-1)$ is the Hölder conjugate exponent. Recalling (12), we obtain

$$|x(u) - \overline{x}(u)| \leq \frac{1}{n} \sum_{v \in V} |x(u) - x(v)| \leq \frac{(N^*)^{1/p'}}{\min_{e \in E} w(e)^{1/p}} \left(p\varphi_{G,p}(x) \right)^{1/p} \quad \forall u \in V.$$

Therefore, by the general inequality $\|z\| \leq \sum_{i=1}^{n} |z_i|$, we can derive (13) with (14). □

One might find some similarities between (13) and the Poincaré–Wirtinger type inequality in PDE (see, e.g, [8, §5.8.1 Theorem 1]):

$$\left\| u - \frac{1}{|\Omega|}\int_\Omega u dx \right\|_{L^p(\Omega)}^p \le \gamma \|\nabla u\|_{L^p(\Omega)}^p = \gamma \int_\Omega u \left(-\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right)\right) dx \qquad \forall u \in W^{1,p}(\Omega).$$

For the hypergraph Laplacian, we also obtain the inverse inequality:

Theorem 3.2. Let $p \ge 1$. Then every $x \in \mathbb{R}^V$ and $y \in L_{G,p}(x)$ satisfy

$$\|x - \overline{x}\|^p \ge p\ \gamma_{G,p}\varphi_{G,p}(x) = \gamma_{G,p}\ x \cdot y, \tag{15}$$

where

$$\gamma_{G,p} := \frac{1}{n^{p/2}\sum_{e\in E} w(e)}. \tag{16}$$

Proof. Since $\overline{x}(u) = \overline{x}(v) = \frac{1}{n}\sum_{i=1}^n x_i$ holds for any $u, v \in V$, we have

$$\begin{aligned} f_e(x) &= \max_{u,v\in e} |x(u) - x(v)| = \max_{u,v\in e} |x(u) - \overline{x}(u) + \overline{x}(v) - x(v)| \\ &\le \sum_{v\in V} |x(v) - \overline{x}(v)| \le \sqrt{n}\|x - \overline{x}\|. \end{aligned}$$

Hence we have

$$p\varphi_{G,p}(x) \le n^{p/2}\left(\sum_{e\in E} w(e)\right)\|x - \overline{x}\|^p$$

which yields (15) with (16). □

Remark 4. One of the differences between the hypergraph Laplacian and p-Laplacian in PDE is that $\partial\varphi_{G,p}$ is not strongly monotone. Indeed, let $\#V = 4$, $E = \{V\}$, $w \equiv 1$, and

$$x = (x_1, x_2, x_3, x_4) := (1, a_1, b_1, -1), \qquad y = (y_1, y_2, y_3, y_4) := (1, a_2, b_2, -1).$$

By letting $|a_j|, |b_j| < 1$ $(j = 1, 2)$, we have $\operatorname{argmax}_{v\in V} x(v) = \operatorname{argmax}_{v\in V} y(v) = v_1$ and $\operatorname{argmin}_{v\in V} x(v) = \operatorname{argmin}_{v\in V} y(v) = v_4$. Then $f_e(x) = f_e(y) = (1 - (-1)) = 2$ and

$$L_{G,p}(x) = L_{G,p}(y) = (2^{p-1}, 0, 0, -2^{p-1}).$$

Hence we obtain $(L_{G,p}(x) - L_{G,p}(y)) \cdot (x - y) = 0$ although $a_j, b_j \in (-1, 1)$ can be chosen arbitrarily so that $x \ne y$ and $\overline{x} = \overline{y}$. That is to say, $(L_{G,p}(x) - L_{G,p}(y)) \cdot (x - y)$ can not be bounded from below by $\|x - y\|$ nor $\|(x - \overline{x}) - (y - \overline{y})\|$.

4 Evolution Equation with Hypergraph Laplacian

4.1 Cauchy Problem

Using the Poincaré type inequalities provided in the previous section, we here consider the following Cauchy problem of a multi-valued nonlinear ordinary differential equation associated with the hypergraph Laplacian:

$$\begin{cases} x'(t) + L_{G,p}(x(t)) \ni h(t) & t \in (0,T), \\ x(0) = x_0, \end{cases} \tag{17}$$

where $x : [0,T] \to \mathbb{R}^V$ is an unknown function and $h : [0,T] \to \mathbb{R}^V$ is a given external force. Since $L_{G,p}$ coincides with the subdifferential $\partial\varphi_{G,p}$, the Kōmura–Brézis theory (Proposition 1) is applicable to (17) and we can assure that for any given data $x_0 \in D(\varphi_{G,p}) = \mathbb{R}^V$ and $h \in L^2(0,T;\mathbb{R}^V)$ there exists a unique solution $x \in W^{1,2}(0,T;\mathbb{R}^V)$. We here aim to discuss the time global behavior of solution to (17) more precisely as a result which can not be discussed only by the abstract theory.

We test the equation (17) by $1_V = (1,\dots,1)$. Note that $x \cdot 1_V = \sum_{i=1}^n x_i$ holds for any $x \in \mathbb{R}^V$ and $b \cdot 1_V = 0$ for any $b \in B_e = \mathrm{conv}\{1_u - 1_v;\ \ u,v \in e\}$ (recall our proof for Theorem 2.1). Therefore, we have $y \cdot 1_V = 0$ for every $y \in L_{G,p}(x)$, which implies

$$\frac{d}{dt}\left(\frac{1}{n}\sum_{i=1}^n x_i(t)\right) = \frac{1}{n}\sum_{i=1}^n h_i(t),$$

that is, for every $t \in [0,T]$,

$$\overline{x}(t) = \overline{x_0} + \int_0^t \overline{h}(s)ds. \tag{18}$$

Especially, $\overline{x}(t) = \overline{x_0}$ holds for any $t \ge 0$ if $\overline{h} \equiv 0$.

By this "mass" conservation law of (17), we obtain the following decay estimate of solution tending to the mean value of initial state.

Theorem 4.1. Let x be a solution to (17) with $h \equiv 0$ and define $X(t) := \|x(t) - \overline{x_0}\|$. Then for every $t \ge 0$,

$$\left(X(0)^{2-p} - \frac{2-p}{\gamma_{G,p}}t\right)_+^{1/(2-p)} \le X(t) \le \left(X(0)^{2-p} - \frac{2-p}{\Gamma_{G,p}}t\right)_+^{1/(2-p)} \qquad \text{if } 1 \le p < 2,$$

$$X(0)\exp\left(-\gamma_{G,p}^{-1}t\right) \le X(t) \le X(0)\exp\left(-\Gamma_{G,p}^{-1}t\right) \qquad \text{if } p = 2,$$

$$\left(\frac{1}{X(0)^{p-2}} + \frac{p-2}{\gamma_{G,p}}t\right)^{-1/(p-2)} \le X(t) \le \left(\frac{1}{X(0)^{p-2}} + \frac{p-2}{\Gamma_{G,p}}t\right)^{-1/(p-2)} \qquad \text{if } p > 2,$$

where $(s)_+ := \max\{s, 0\}$ and $\Gamma_{G,p}, \gamma_{G,p}$ are constants defined by (14), (16), respectively.

Proof. Since $\overline{x_0} = c1_V$ with $c = \frac{1}{n}\sum_{v\in V} x_0(v)$, we have for every $y(t) \in L_{G,p}(x(t))$

$$y(t) \cdot (x(t) - \overline{x_0}) = p\varphi_{G,p}(x(t)) - c(y(t) \cdot 1_V) = p\varphi_{G,p}(x(t)).$$

Hence multiplying (17) by $x(t) - \overline{x_0}$, we obtain the following identity:

$$\frac{1}{2}\frac{d}{dt}\|x(t) - \overline{x_0}\|^2 + p\varphi_{G,p}(x(t)) = 0. \tag{19}$$

Then applying Theorem 3.1 and 3.2, we deduce

$$-2\gamma_{G,p}^{-1}\|x(t) - \overline{x}(t)\|^p \le \frac{d}{dt}\|x(t) - \overline{x_0}\|^2 \le -2\Gamma_{G,p}^{-1}\|x(t) - \overline{x}(t)\|^p, \tag{20}$$

which together with $\overline{x}(t) = \overline{x_0}$ leads to the inequalities in Theorem 4.1. □

Remark 5. By Remark 4, $\partial\varphi_{G,p}$ is not strongly monotone and then it is difficult to show the asymptotic behavior of two different solutions $x(t)$ and $y(t)$ by the standard method via a priori estimate of $x(t) - y(t)$. However, by virtue of Theorem 4.1 and the triangle inequality, we easily obtain the following.

Corollary 1. Let x, y be the solutions to (17) with $h \equiv 0$ and the initial data x_0, y_0, respectively. If $\overline{x_0} = \overline{y_0}$, then we have for every $t \ge 0$

$$\|x(t) - y(t)\| \le \begin{cases} \left(X_0^{2-p} - \dfrac{2-p}{\Gamma_{G,p}}t\right)_+^{1/(2-p)} + \left(Y_0^{2-p} - \dfrac{2-p}{\Gamma_{G,p}}t\right)_+^{1/(2-p)} & \text{if } 1 \le p < 2,\\ (X_0 + Y_0)\exp\left(-\Gamma_{G,p}^{-1}t\right) & \text{if } p = 2,\\ \left(\dfrac{1}{X_0^{p-2}} + \dfrac{p-2}{\Gamma_{G,p}}t\right)^{-1/(p-2)} + \left(\dfrac{1}{Y_0^{p-2}} + \dfrac{p-2}{\Gamma_{G,p}}t\right)^{-1/(p-2)} & \text{if } p > 2,\end{cases}$$

where $X_0 := \|x_0 - \overline{x_0}\|$ and $Y_0 := \|y_0 - \overline{y_0}\|$.

Remark 6. Multi-valued operators in the evolution equation possibly cause the "jump" of the derivative and then the solution to set-valued differential equations might not belong to C^1-class in general. For (17), we can construct an example of solution whose derivative is not continuous.

Let $\#V = 4$, $E = \{V\}$, $w \equiv 1$, and $p = 2$. Then the solution to (17) with $h \equiv 0$ and $x_0 = (2, 1, -1, 2)$ is

$$x_1(t) = \begin{cases} 2e^{-2t} & \text{if } t \le \frac{1}{2}\log 2,\\ \sqrt{2}e^{-t} & \text{if } t \ge \frac{1}{2}\log 2,\end{cases} \qquad x_4(t) = -x_1(t),$$

$$x_2(t) = \begin{cases} 1 & \text{if } t \le \frac{1}{2}\log 2,\\ \sqrt{2}e^{-t} & \text{if } t \ge \frac{1}{2}\log 2,\end{cases} \qquad x_3(t) = -x_2(t).$$

Indeed, when $0 \le t < t_0 := \frac{1}{2}\log 2$, we have by $x_1(t) > x_2(t) > x_3(t) > x_4(t)$

$$\begin{aligned} f_e(x(t)) &= x_1(t) - x_4(t) = 4e^{-2t}, \\ \partial\varphi_{G,2}(x(t)) &= f_e(x(t))(1,0,0,-1) = (4e^{-2t},0,0,-4e^{-2t}) \end{aligned}$$

(recall Ex.1 of §2.2), which satisfies $x'(t) = (-4e^{-2t},0,0,4e^{-2t}) = -\partial\varphi_{G,2}(x(t))$. If $t > t_0$, we have by $x_1(t) = x_2(t) > x_3(t) = x_4(t)$ (see Ex.3 of §2.2)

$$\begin{aligned} f_e(x(t)) &= x_1(t) - x_4(t) = x_2(t) - x_3(t) = 2\sqrt{2}e^{-t}, \\ \partial\varphi_{G,2}(x(t)) &= \left\{2\sqrt{2}e^{-t}\,(\lambda,(1-\lambda),-\mu,-(1-\mu))\,;\ \lambda,\mu\in[0,1]\right\} \\ &\ni \left(\sqrt{2}e^{-t},\sqrt{2}e^{-t},-\sqrt{2}e^{-t},-\sqrt{2}e^{-t}\right) = -x'(t). \end{aligned}$$

Hence $x(t)$ fulfills $-x'(t) \in \partial\varphi_{G,2}(x(t))$ except for $t = t_0$. Moreover, we can easily check that

$$\begin{aligned} (-\partial\varphi_{G,2}(x(t)))^\circ &= -\left(\partial\varphi_{G,2}(x(t))\right)^\circ \\ &= \begin{cases} (-4e^{-2t},0,0,4e^{-2t}) & \text{if} \quad 0 \le t < t_0, \\ (-\sqrt{2}e^{-t},-\sqrt{2}e^{-t},\sqrt{2}e^{-t},\sqrt{2}e^{-t}) & \text{if} \quad t \ge t_0, \end{cases} \end{aligned}$$

coincides with the right derivative of solution $\frac{d^+x}{dt}(t)$ for every $t \ge 0$.

By this example, one might perceive that the solutions to (17) first behave to bring the maximum value and the minimum value in $e \in E$ (i.e., $x_1(t)$ and $x_4(t)$) closer and the other components with middle value ($x_2(t)$ and $x_3(t)$) halt. After the components with the maximum or minimum value touch others, these components simultaneously act and the jump of time-derivative occurs.

4.2 Periodic Problem

Next we consider the following time-periodic problem:

$$\text{(21)} \qquad \begin{cases} x'(t) + L_{G,p}(x(t)) \ni h(t) & t \in (0,T), \\ x(0) = x(T). \end{cases}$$

As for the known results, [2, Corollaier 3.4] assures the solvability of the time-periodic problem of nonlinear evolution governed by the subdifferential operator with the coerciveness. For the case where the subdifferential is not coercive, [9, Théorème 3] guarantees the existence of a periodic solution to $\xi'(t) + \partial g(\xi(t)) \ni h(t)$ under the assumption that

$\frac{1}{T}\int_0^T h(t)dt$ belongs to the interior of the range of ∂g. However, the fact that $y \cdot 1_V = 0$ for every $y \in L_{G,p}(x)$ implies that $R(L_{G,p}) \subset \{c1_V \in \mathbb{R}^V;\ c \in \mathbb{R}\}^\perp = \{x \in \mathbb{R}^V;\ \overline{x} = 0\}$, which has no interior point. Hence, to the best of our knowledge, there are very few abstract theory which can be applied to our problem (21).

Although (21) cannot be solved by known results, one might perceive that the hypergraph Laplacian has properties similar to those of the Neumann-Laplacian in PDE so far, for instance, the lack of coerciveness, the mass conservation law, and the convergence of solution toward the mean value of initial state. So we can expect that a solution to (21) can be constructed by employing a technique for parabolic equations governed by the Neumann Laplacian (see, e.g., [16]). Note that

$$\int_0^T \overline{h}(t)dt = 0 \tag{22}$$

is obtained as a necessary condition of the existence of solutions satisfying $x(0) = x(T)$ by (18) with $t = T$.

Theorem 4.2. Let $h \in L^{\bar{p}'}(0,T;\mathbb{R}^V)$ with $\bar{p}' := \max\{2, p'\}$ and assume (22). Then (21) possesses at least one solution $x \in W^{1,2}(0,T;\mathbb{R}^V)$.

Proof. We set the following approximation problem with the parameter $\varepsilon > 0$:

$$\begin{cases} x'_\varepsilon(t) + \varepsilon x_\varepsilon(t) + L_{G,p}(x_\varepsilon(t)) \ni h(t) & t \in (0,T), \\ x_\varepsilon(0) = x_\varepsilon(T). \end{cases} \tag{23}$$

Since

$$\varphi^\varepsilon_{G,p}(x) := \frac{\varepsilon}{2}\|x\|^2 + \varphi_{G,p}(x)$$

satisfies the coerciveness and $\partial\varphi^\varepsilon_{G,p}(x) = \varepsilon x + L_{G,p}(x)$, (23) possesses a unique periodic solution $x_\varepsilon \in W^{1,2}(0,T;\mathbb{R}^V)$ for any given $h \in L^2(0,T;\mathbb{R}^V)$ (see [2, Corollaire 3.4]).

Multiplying (23) by 1_V, integrating over $[0,T]$, and using the condition $x_\varepsilon(0) = x_\varepsilon(T)$ and (22), we get

$$\int_0^T \left(\sum_{i=1}^n x_{\varepsilon i}(t)\right) dt = 0,$$

which implies that there exists some $t_\varepsilon \in [0,T]$ such that $\sum_{i=1}^n x_{\varepsilon i}(t_\varepsilon) = 0$, that is, $\overline{x_\varepsilon}(t_\varepsilon) = 0$. Testing (23) by 1_V again and integrating over $[t_\varepsilon, t]$ ($t \in [t_\varepsilon, t_\varepsilon + T]$), we have

$$\left(\sum_{i=1}^n x_{\varepsilon i}(t)\right) = \int_{t_\varepsilon}^t e^{-\varepsilon(t-s)}\left(\sum_{i=1}^n h_i(s)\right) ds \quad \Rightarrow \quad \overline{x_\varepsilon}(t) = \int_{t_\varepsilon}^t e^{-\varepsilon(t-s)}\overline{h}(s)ds,$$

which leads to

$$\sup_{0\le t\le T}\|\overline{x_\varepsilon}(t)\| \le \int_0^T \|\overline{h}(s)\|ds. \tag{24}$$

Multiplying (23) by x_ε and integrating over $[0,T]$, we have

$$\begin{aligned}&\varepsilon\int_0^T \|x_\varepsilon(t)\|^2dt + p\int_0^T \varphi_{G,p}(x_\varepsilon(t))dt\\ &\le \left(\int_0^T \|h(t)\|^{p'}dt\right)^{1/p'}\left[\left(\int_0^T \|x_\varepsilon(t)-\overline{x_\varepsilon}(t)\|^p dt\right)^{1/p} + \left(\int_0^T \|\overline{x_\varepsilon}(t)\|^p dt\right)^{1/p}\right].\end{aligned}$$

Let C denote a general constant independent of $\varepsilon > 0$. Then by (13) and (24),

$$\varepsilon\int_0^T \|x_\varepsilon(t)\|^2dt + \int_0^T \|x_\varepsilon(t)\|^p dt \le C. \tag{25}$$

Let $t_0 \in [0,T]$ attain the minimum of $t \mapsto \|x_\varepsilon(t)\|$. Clearly (25) implies $\|x_\varepsilon(t_0)\| \le C$. Testing (23) by x'_ε, we get

$$\int_0^T \|x'_\varepsilon(t)\|^2dt \le \int_0^T \|h(t)\|^2dt. \tag{26}$$

We here use the chain rule for the subdifferential (see [2, Lemme 3.3]):

$$y_\varepsilon(t)\cdot x'_\varepsilon(t) = \frac{d}{dt}\varphi_{G,p}(x_\varepsilon(t)) \quad \text{a.e.} \quad t\in[0,T],$$

where $y_\varepsilon : [0,T]\to\mathbb{R}^V$ is the section of $L_{G,p}(x_\varepsilon)$ satisfying (23), i.e., $x'_\varepsilon(t)+\varepsilon x_\varepsilon(t)+y_\varepsilon(t) = h(t)$ and $y_\varepsilon(t)\in L_{G,p}(x_\varepsilon(t))$ for a.e. $t\in(0,T)$. Since $\|x_\varepsilon(t_0)\|\le C$, (26) yields

$$\sup_{0\le t\le T}\|x_\varepsilon(t)\| \le C \tag{27}$$

and

$$\int_0^T \|y_\varepsilon(t)\|^2dt \le C. \tag{28}$$

By (26)(27)(28), we can discuss the standard argument of convergence of solutions and equation as $\varepsilon\to 0$, whence it follows Theorem 4.2. □

Remark 7. The uniqueness of time-periodic solution dose not hold in general. Indeed, let $\#V = 4$, $E = \{V\}$, $w\equiv 1$, $p = 2$, $\alpha,\beta > 0$ and

$$h(t) = \begin{pmatrix} 2\alpha\exp\left(2(t-\frac{T}{2})\right)+2\beta \\ 0 \\ 0 \\ -2\alpha\exp\left(2(t-\frac{T}{2})\right)-2\beta \end{pmatrix},$$

then

$$x(t) = \begin{pmatrix} \alpha \cosh\left(2(t - \frac{T}{2})\right) + \beta \\ a \\ b \\ -\alpha \cosh\left(2(t - \frac{T}{2})\right) - \beta \end{pmatrix}$$

becomes a solution to (21) for arbitrary fixed $a, b \in (-\alpha - \beta, \alpha + \beta)$.

By virtue of [9, Théorème 5], however, we can see the following fact.

Theorem 4.3. Let $x, y \in W^{1,2}(0, T; \mathbb{R}^V)$ be two solutions to (21) with the same given h satisfying (22), then there exists some constant $z \in \mathbb{R}^V$ such that $x = y + z$.

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