QUASICONVEXITY PRESERVING PROPERTY FOR FIRST ORDER NONLOCAL EVOLUTION EQUATIONS

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Abstract. This note is a companion of our earlier paper [Kagaya-Liu-Mitake, 2023] to study the quasiconvexity preserving property of positive, spatially coercive viscosity solutions to a class of first order evolution equations with monotone nonlocal terms. We show that if the initial value is quasiconvex, the viscosity solution to the Cauchy problem stays quasiconvex in space for all time. In contrast to our results in [Kagaya-Liu-Mitake, 2023], we focus only on the first order case, but slightly change our assumptions to allow more general dependence of the operator on the nonlocal term.

1. Introduction

This note accompanies our published paper [17] on quasiconvexity of solutions to fully nonlinear nonlocal evolution equations, focusing only on the first order case with slight adaptations. The equation we are concerned with is

$$\begin{cases} u_t + F(u, \nabla u, \{u(\cdot, t) < u(x, t)\}) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \quad (1.1) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \quad (1.2) \end{cases}$$

where $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a unknown function, and u_t and ∇u denote the time derivative and the spatial gradient of u, respectively. Here the initial condition $u_0 : \mathbb{R}^n \to \mathbb{R}$ is in $UC(\mathbb{R}^n)$, where $UC(\mathbb{R}^n)$ stands for the set of uniformly continuous functions on $\mathbb{R}^n$, $F : \mathbb{R} \times \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}$ is a given continuous function and $\mathcal{B}$ represents the collection of all measurable sets in $\mathbb{R}^n$. More precise assumptions on F will be given later. Recall that in [17] we assume that F depends on the intersection $\{u(\cdot, t) < u(x, t)\} \cap K$ for a given compact set $K \subset \mathbb{R}^n$ to restrict the nonlocal effect in a bounded region. In this work, we do not impose such a constraint on F but include a slightly more general assumption on the operator.

We aim to show the preservation of spatial quasiconvexity of viscosity solutions to (1.1) and (1.2). Here, a function $u \in C(\mathbb{R}^n \times [0, \infty))$ is said to be spatially quasiconvex if all sublevel sets of $u(\cdot, t)$ are convex in $\mathbb{R}^n$, or equivalently,

$$u(\lambda y + (1 - \lambda)z, t) \leq \max\{u(y, t), u(z, t)\}$$

holds for all $y, z \in \mathbb{R}^n$, $t \geq 0$ and $\lambda \in (0, 1)$. We refer to the related results in [6], where a general class of set evolutions with nonlocal terms is shown to preserve the convexity of the initial set. Such nonlocal evolutions can be reformulated via the so-called level set method as geometric equations, which in our context requires F to satisfy the homogeneity condition

$$F(r_1, cp, A) = cF(r_2, p, A) \quad \text{for all } c > 0,\ r_1, r_2 \in \mathbb{R},\ p \in \mathbb{R}^n,\ A \in \mathcal{B}. \tag{1.3}$$

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To fulfill the condition (1.3), F needs to be independent of the unknown u. We refer to [11] for more details about the level set formulation. For classical solutions of elliptic problems without nonlocal terms, quasiconvexity results can be found in [8, 9].

It is natural to discuss the preservation of quasiconvexity when the nonlocal operator F is not geometric. For broader applications, we are particularly interested in the case when F depends on u. Our assumptions on the operator F are as follows. Below, $m(A)$ represents the n-dimensional Lebesgue measure of $A \in \mathcal{B}$.

(F1) F is proper; namely, for any $p \in \mathbb{R}^n$ and $A \in \mathcal{B}$,

$$F(r_1, p, A) \le F(r_2, p, A)$$

holds for all $r_2 \ge r_1$.

(F2) F is locally bounded in the sense that for each $R > 0$, there holds

$$\sup\{|F(r, p, A)| : r \in \mathbb{R}, |p| \le R, A \in \mathcal{B}\} < \infty.$$

(F3) F is continuous in $\mathbb{R} \times \mathbb{R}^n \times \mathcal{B}$ with the topology of $\mathcal{B}$ given by $d(A_1, A_2) = m(A_1 \triangle A_2)$, where $A_1 \triangle A_2$ stands for the symmetric difference of A_1 and A_2, that is $A_1 \triangle A_2 := (A_1 \cup A_2) \setminus (A_1 \cap A_2)$ for all $A_1, A_2 \in \mathcal{B}$. Moreover, for any $R > 0$, there exists a modulus of continuity ω_R such that

$$F(r, p_1, A_1) - F(r, p_2, A_2) \le \omega_R\left(|p_1 - p_2| + m(A_1 \triangle A_2)\right) \tag{1.4}$$

for all $r \in \mathbb{R}$, $p_1, p_2 \in \mathbb{R}^n$ with $|p_1|, |p_2| \le R$ and $A_1, A_2 \in \mathcal{B}$.

(F4) F is monotone with respect to the set argument; namely,

$$F(r, p, A_1) \le F(r, p, A_2)$$

holds for all $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, $A_1, A_2 \in \mathcal{B}$ with $A_1 \subset A_2$.

(F5) F is stable with respect to the set argument in the sense that, for any $R > 0$,

$$\sup\{|F(r, p, A \cap B_\rho(0)) - F(r, p, A)| : r \in \mathbb{R}, |p| \le R, A \in \mathcal{B}\} \to 0 \quad \text{as } \rho \to \infty,$$

where $B_\rho(0)$ denotes the open ball centered at 0 with radius $\rho > 0$.

(F6) There exists $V \in C(\mathbb{R})$ such that

$$\sup_{r \in \mathbb{R}, A \in \mathcal{B}} |F(r, p, A) - V(r)| \to 0 \quad \text{as } p \to 0.$$

We stress that the monotonicity (F4) plays an important role in our analysis, especially for the comparison principle and therefore the uniqueness of viscosity solutions. By adapting the arguments in [12] to nonlocal problems, we provide a comparison theorem, Theorem 2.1, for possibly unbounded solutions satisfying growth condition (2.1). See [6, 22, 10, 23] for comparison results for monotone evolution equations in different settings. On the other hand, in the non-monotone case, one cannot expect the comparison principle to hold and alternative methods are needed for uniqueness of solutions and other related properties (see [1, 4, 5, 20]).

In our main result (Theorem 1.1) below, we only consider viscosity solutions to (1.1) and (1.2) that are uniformly positive and coercive in space, that is,

$$u \ge c_0 \quad \text{in } \mathbb{R}^n \times [0, \infty), \quad \text{for some } c_0 > 0, \text{ and} \tag{1.5}$$

$$\inf_{|x| \ge R,\ 0 \le t \le T} u(x, t) \to \infty \quad \text{as } R \to \infty \text{ for any } T \ge 0. \tag{1.6}$$

Under a further assumption that $u_0 \in UC(\mathbb{R}^n)$, we obtain the existence and uniqueness of solutions $u \in C(\mathbb{R}^n \times [0,\infty))$ of (1.1) (1.2), satisfying (1.5) and (1.6). We can also show that

$$|u(x,t) - u(y,t)| \leq \omega_0(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } t \geq 0, \tag{1.7}$$

where ω_0 denotes the modulus of continuity of u_0. See Theorem 2.3 for details.

We impose a key concavity condition on F via a transformed operator G_β with $0 < \beta < 1$ defined by

$$G_\beta(r,p,A) = \frac{1}{1-\beta} r^\beta F\left(r^{1-\beta}, (1-\beta)r^{-\beta}p, A\right) \quad \text{for } r > 0,\ p \in \mathbb{R}^n \text{ and } A \in \mathcal{B}. \tag{1.8}$$

(F7) When $\beta < 1$ is sufficiently close to 1,

$$r \mapsto G_\beta(r,p,A) \quad \text{is concave in } [c_0,\infty) \times \mathbb{S}^n \tag{1.9}$$

holds for any $p \in \mathbb{R}^n$ and $A \in \mathcal{B}$.

Let us now state our main result.

Theorem 1.1 (Quasiconvexity preserving property). *Assume* (F1)–(F7). *Let* $u_0 \in UC(\mathbb{R}^n)$. *Let* $u \in C(\mathbb{R}^n \times [0,\infty))$ *be the unique viscosity solution of* (1.1) *and* (1.2) *satisfying* (1.5), (1.6) *and* (1.7). *If* u_0 *is quasiconvex in* $\mathbb{R}^n$*, that is,* $\{u_0 < h\}$ *is convex for all* $h \in \mathbb{R}$*, then* $u(\cdot,t)$ *is quasiconvex in* $\mathbb{R}^n$ *for all* $t \geq 0$.

Our result is applicable to first order nonlinear equations including the level set equations for nonlocal geometric evolutions. As pointed out previously, it applies also to equations that are not geometric. The assumptions in Theorem 1.1 allow the operator F to depend on u. In [17], we provide a similar result for general second order nonlocal parabolic equations under a generalized version of (F7) (see (F7) in [17]).

In contrast to the set-theoretic arguments in [6], we apply the PDE-based approach in [17], which is inspired by the classical convexity/concavity results on various elliptic and parabolic equations including [21, 18, 19, 12, 2] etc. In addition, we also refer to [14, 15, 13, 16] for power convexity/concavity of solutions. See [17] for a more detailed introduction on this topic and references. Our strategy, which develops the idea in [18, 2, 13], is to show the quasiconvex envelope of a solution is a supersolution of the equation and then use the comparison principle to conclude the proof.

For a fixed $\lambda \in (0,1)$ and a given positive viscosity solution u of (1.1), in order to prove that the spatially quasiconvex envelope $u_{\star,\lambda}$, defined by

$$u_{\star,\lambda}(x,t) = \inf\left\{ \max\{u(y,t), u(z,t)\} : x = \lambda y + (1-\lambda)z \right\}, \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{1.10}$$

is a viscosity supersolution, we approximate $u_{\star,\lambda}$ by the power convex envelope $u_{q,\lambda}$, given by

$$u_{q,\lambda}(x,t) = \inf\left\{ (\lambda u(y,t)^q + (1-\lambda)u(z,t)^q)^{\frac{1}{q}} : \lambda y + (1-\lambda)z = x \right\} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{1.11}$$

as the exponent $q \to \infty$. The concavity condition (F5), with the choice $\beta = 1 - 1/q$, connects $u_{q,\lambda}$ to the supersolution property of (1.1) for $q > 1$ arbitrarily large. Then sending $q \to \infty$, we can obtain the supersolution property for $u_{\star,\lambda}$.

The rest of the paper is organized in the following way. In Section 2 we recall the definition and some basic properties of viscosity solutions of (1.1) and provide a comparison principle for our later applications. Section 3 is devoted to the proof of our main result, Theorem 1.1.

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2. Comparison Principle

In this section, we present a comparison principle for (1.1). We first recall the definition of viscosity solutions to (1.1). For a set $Q \subset \mathbb{R}^n \times [0,\infty)$, we denote by $USC\,(Q)$ and $LSC\,(Q)$, respectively, the set of the upper and lower semicontinuous functions in Q.

Definition 1 (Viscosity solutions)**.** (i) *A function $u \in USC\,(\mathbb{R}^n \times (0,\infty))$ is called a viscosity subsolution of* (1.1) *if whenever there exist $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ and $\varphi \in C^2(\mathbb{R}^n \times (0,\infty))$ such that $u-\varphi$ attains a local maximum at (x_0,t_0),*

$$\varphi_t(x_0,t_0) + F(u(x_0,t_0), \nabla\varphi(x_0,t_0), \{u(\cdot,t_0) < u(x_0,t_0)\}) \leq 0.$$

(ii) *A function $u \in LSC\,(\mathbb{R}^n \times (0,\infty))$ is called a viscosity supersolution of* (1.1) *if whenever there exist $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ and $\varphi \in C^2(\mathbb{R}^n \times (0,\infty))$ such that $u-\varphi$ attains a minimum at (x_0,t_0),*

$$\varphi_t(x_0,t_0) + F(u(x_0,t_0), \nabla\varphi(x_0,t_0), \{u(\cdot,t_0) \leq u(x_0,t_0)\}) \geq 0.$$

(iii) *A function $u \in C(\mathbb{R}^n \times (0,\infty))$ is called a viscosity solution of* (1.1) *if it is both a viscosity subsolution and a viscosity supersolution.*

One can use semijets instead of the test functions to rewrite the definition of viscosity solutions. See [7] for details. In the sequel, we are always concerned with viscosity solutions in this paper, and the term “viscosity” is omitted henceforth.

Theorem 2.1 (Comparison principle)**.** *Assume that* (F1)–(F6) *hold. Let $u \in USC\,(\mathbb{R}^n \times [0,\infty))$ and $v \in LSC\,(\mathbb{R}^n \times [0,\infty))$ be, respectively, a subsolution and a supersolution to* (1.1)*. Assume in addition that for any $T>0$, there exists $L_T>0$ such that*

$$u(x,t) \leq L_T(|x|+1), \quad v(x,t) \geq -L_T(|x|+1) \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T]. \tag{2.1}$$

If there exists a modulus of continuity ω_0 such that

$$u(x,0) - v(y,0) \leq \omega_0(|x-y|) \quad \text{for all } x,y \in \mathbb{R}^n, \tag{2.2}$$

then $u \leq v$ holds in $\mathbb{R}^n \times [0,\infty)$.

This comparison principle for the first order case is similar to [17, Theorem 3.1]. The main difference is that we additionally impose (F5) so that we can allow F to depend on the level sets of u without any restrictions. The following result is an adaptation of [12, Proposition 2.3].

Proposition 2.2 (Growth estimate)**.** *Assume that (F2) holds. Let $u \in USC\,(\mathbb{R}^n \times [0,\infty))$ and $v \in LSC\,(\mathbb{R}^n \times [0,\infty))$ be, respectively, a subsolution and a supersolution to* (1.1)*. For any fixed $T>0$, assume that* (2.1) *holds for some $L_T>0$. Then for any $L>L_T$ large, there exists $M>0$ such that*

$$u(x,t) - v(y,t) \leq L|x-y| + M(1+t) \quad \text{for all } x,y \in \mathbb{R}^n \text{ and } t \in [0,T).$$

Proof. Note that (2.2) yields the existence of $C_0 > 0$ such that

$$u(x,0) - v(y,0) \le C_0(|x-y|+1) \quad \text{for all } x, y \in \mathbb{R}^n. \tag{2.3}$$

Take $L > \max\{L_T, C_0\}$. Our goal is to show that

$$u(x,t) - v(y,t) - \psi(x,y) - M(1+t) \le 0 \quad \text{for } x, y \in \mathbb{R}^n,\ t \in [0,T) \tag{2.4}$$

for $M > 0$ sufficiently large, where $\psi(x,y) = L(|x-y|^2+1)^{\frac{1}{2}}$.

Suppose that (2.4) fails to hold for any arbitrarily large $M > 0$. Then, we may assume that $M > C_0$, and there exist $\hat{x}, \hat{y} \in \mathbb{R}^n, \hat{t} \in [0,T)$ such that

$$u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) - \psi(\hat{x},\hat{y}) - M(1+\hat{t}) > 0. \tag{2.5}$$

Set, for $\varepsilon, \lambda > 0$ small and $R > 0$ large,

$$\Psi_\varepsilon(x,t,y,s) = u(x,t) - v(y,s) - \psi(x,y) - L(g_R(x)+g_R(y)) - \frac{(t-s)^2}{2\varepsilon} - M(1+t) - \frac{\lambda}{T-t},$$

where $g_R \in C^2(\mathbb{R}^n)$ is a nonnegative function such that $g_R(x) = 0$ for $|x| < R$, $g_R(x)/|x| \to 1$ as $|x| \to \infty$, and $|\nabla g_R(x)|$ is bounded uniformly in $x \in \mathbb{R}^n$ and $R > 0$. It follows from (2.5) that $\Psi_\varepsilon(\hat{x},\hat{t},\hat{y},\hat{t}) > 0$ if we take $R > |\hat{x}|, |\hat{y}|$ and $\lambda > 0$ small depending only on M.

By (2.1), we see that Ψ_ε attains a positive maximum in $(\mathbb{R}^2 \times [0,T))^2$ at $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ for $R > 0$ and $M > 0$ large and for $\varepsilon > 0$ small. In fact, x_ε and y_ε are bounded uniformly with respect to ε. Moreover, we have $t_\varepsilon, s_\varepsilon \to t_0$ for some $t_0 \in [0,T)$ as $\varepsilon \to 0$. In view of the upper semicontinuity of u and lower semicontinuity of v as well as (2.3), we deduce that $t_0 \ne 0$ and thus $t_\varepsilon, s_\varepsilon > 0$ for all $\varepsilon > 0$ small.

Since u and v are, respectively, a subsolution and a supersolution of (1.1), we obtain

$$\begin{aligned}
&\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + M + \frac{\lambda}{(T-t_\varepsilon)^2} + F(u(x_\varepsilon,t_\varepsilon), p_1, \{u(\cdot,t_\varepsilon) < u(x_\varepsilon,t_\varepsilon)\}) \le 0,\\
&\frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + F(v(y_\varepsilon,s_\varepsilon), p_2, \{v(\cdot,t_\varepsilon) \le v(y_\varepsilon,t_\varepsilon)\}) \ge 0,
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= L(|x_\varepsilon - y_\varepsilon|^2+1)^{-\frac{1}{2}}(x_\varepsilon - y_\varepsilon) + L\nabla g_R(x_\varepsilon),\\
p_2 &= L(|x_\varepsilon - y_\varepsilon|^2+1)^{-\frac{1}{2}}(x_\varepsilon - y_\varepsilon) - L\nabla g_R(y_\varepsilon).
\end{aligned}$$

Since the boundedness of p_1, p_2, X_1, X_2 depends only on L, taking the difference between the viscosity inequalities above and applying (F2), we have $C_L > 0$ such that $M \le C_L$, which is a contradiction to the arbitrariness of $M > 0$. □

Let us now prove Theorem 2.1.

Proof of Theorem 2.1. Assume by contradiction that $\sup_{\mathbb{R}^n \times [0,T)}(u-v) =: \theta > 0$. Then, there exists $\lambda > 0$ such that

$$\sup_{(x,t)\in\mathbb{R}^n\times[0,T)} \left\{ u(x,t) - v(x,t) - \frac{\lambda}{T-t} \right\} > \frac{3\theta}{4}.$$

There exists $(x_1,t_1) \in \mathbb{R}^n \times [0,T)$ such that $u(x_1,t_1) - v(x_1,t_1) - \lambda/(T-t_1) > \theta/2$. Noting that $\sup_{\mathbb{R}^n}(u(\cdot,0) - v(\cdot,0)) \le 0$, we have $t_1 > 0$.

Define

$$\Phi(x,y,t) := u(x,t) - v(y,t) - \frac{|x-y|^4}{\varepsilon^4} - \alpha(|x|^2+|y|^2) - \frac{\lambda}{T-t}$$

for $\varepsilon, \alpha > 0, \lambda > 0$. It is then clear that there exists $\alpha_0 > 0$ small such that

$$\sup_{(x,y,t)\in\mathbb{R}^{2n}\times[0,T)} \Phi(x,y,t) > \frac{\theta}{4}$$

for all $0 < \alpha < \alpha_0$ and $\varepsilon > 0$ small. The growth condition (2.1) implies that Φ attains a maximum at some $(x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}) \in \mathbb{R}^n \times [0,T)$. We write $(\tilde{x}, \tilde{y}, \tilde{t})$ for $(x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha}, t_{\varepsilon,\alpha})$ by abuse of notations. It follows that

$$\frac{|\tilde{x}-\tilde{y}|^4}{\varepsilon^4} + \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) \le u(\tilde{x},\tilde{t}) - v(\tilde{y},\tilde{t}) - u(x_1,t_1) + v(x_1,t_1) + 2\alpha|x_1|^2 + \frac{\lambda}{T-t_1} - \frac{\lambda}{T-\tilde{t}}.$$

In view of Proposition 2.2, we have

$$\begin{aligned}\frac{|\tilde{x}-\tilde{y}|^4}{\varepsilon^4} + \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) &\le L(|\tilde{x}-\tilde{y}|+1) + M(\tilde{t}+1)\\ &\quad - u(x_1,t_1) + v(x_1,t_1) + 2\alpha|x_1|^2 + \frac{\lambda}{T-t_1} - \frac{\lambda}{T-\tilde{t}}.\end{aligned}$$

It follows that

$$\frac{|\tilde{x}-\tilde{y}|^4}{\varepsilon^4} - L|\tilde{x}-\tilde{y}| + \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) \le C$$

for some $C \ge 0$ which is independent of ε, α, which implies that

$$\alpha(|\tilde{x}| + |\tilde{y}|) \to 0 \text{ as } \alpha \to 0 \text{ for any } \varepsilon > 0, \quad \text{and} \quad \sup_{0<\alpha<\alpha_0} |\tilde{x}-\tilde{y}| \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, there exists $\varepsilon_0 > 0$ such that $\omega_0(|\tilde{x}-\tilde{y}|) \le \theta/4$ uniformly for all $0 < \varepsilon < \varepsilon_0$ and $0 < \alpha < \alpha_0$, where ω_0 is the modulus of continuity appearing in (2.2).

On the other hand, we have

$$u(\tilde{x},\tilde{t}) - v(\tilde{y},\tilde{t}) \ge \Phi(\tilde{x},\tilde{y},\tilde{t}) > \frac{\theta}{4}.$$

It follows that $\tilde{t} > 0$ for any $0 < \alpha < \alpha_0$ and $0 < \varepsilon < \varepsilon_0$. In what follows, we fix $0 < \varepsilon < \varepsilon_0$ small enough so that $\tilde{t} > 0$. We discuss two cases:

Case 1. $\liminf_{\alpha\to 0} |\tilde{x}-\tilde{y}| > 0$,

Case 2. $\liminf_{\alpha\to 0} |\tilde{x}-\tilde{y}| = 0$, i.e., $\exists \alpha_i \to 0$ such that $\lim_{\alpha_i\to 0} |\tilde{x}-\tilde{y}| = 0$.

Let us consider Case 1 first. Noticing that $\Phi(x,x,\tilde{t}) \le \Phi(\tilde{x},\tilde{y},\tilde{t})$ for all $x \in \mathbb{R}^n$, we have

$$u(x,\tilde{t}) - u(\tilde{x},\tilde{t}) \le v(x,\tilde{t}) - v(\tilde{y},\tilde{t}) - \frac{|\tilde{x}-\tilde{y}|^4}{\varepsilon^4} + 2\alpha|x|^2 - \alpha(|\tilde{x}|^2 + |\tilde{y}|^2). \tag{2.6}$$

Note that, for any $\rho > 0$ large,

$$\liminf_{\alpha\to 0}\left(-\frac{|\tilde{x}-\tilde{y}|^4}{\varepsilon^4} + 2\alpha \max_{x\in B_\rho(0)} |x|^2 - \alpha(|\tilde{x}|^2 + |\tilde{y}|^2)\right) \le \liminf_{\alpha\to 0}\left(-\frac{|\tilde{x}-\tilde{y}|^4}{\varepsilon^4} + 2\alpha\rho^2\right) < 0,$$

which implies that for all $x \in B_\rho(0)$ and $\alpha > 0$ small, depending on ρ,

$$-\frac{|\tilde{x}-\tilde{y}|^4}{\varepsilon^4} + 2\alpha|x|^2 - \alpha(|\tilde{x}|^2 + |\tilde{y}|^2) < 0.$$

By (2.6), for such α we thus have $u(x,\tilde{t}) - u(\tilde{x},\tilde{t}) < v(x,\tilde{t}) - v(\tilde{y},\tilde{t})$ for all $x \in B_\rho(0)$, which implies $V_\alpha[\tilde{y},\tilde{t}] \cap B_\rho(0) \subset U_\alpha[\tilde{x},\tilde{t}] \cap B_\rho(0)$, where we take

$$U_\alpha[\tilde{x},\tilde{t}] := \{u(\cdot,\tilde{t}) < u(\tilde{x},\tilde{t})\}, \quad V_\alpha[\tilde{y},\tilde{t}] := \{v(\cdot,\tilde{t}) \le v(\tilde{y},\tilde{t})\}.$$

It follows from (F1) and (F4) that, for all $p \in \mathbb{R}^n$,

$$F(v(\tilde{y},\tilde{t}), p, V_\alpha[\tilde{y},\tilde{t}] \cap B_\rho(0)) \leq F(u(\tilde{x},\tilde{t}), p, U_\alpha[\tilde{x},\tilde{t}] \cap B_\rho(0)). \tag{2.7}$$

Moreover, noticing that $p_0 := 4|\tilde{x}-\tilde{y}|^2(\tilde{x}-\tilde{y})/\varepsilon^4$ is bounded away from 0 uniformly in α, we have

$$p_1 := p_0 + 2\alpha\tilde{y} \neq 0, \quad p_2 := p_0 - 2\alpha\tilde{x} \neq 0 \tag{2.8}$$

for all $\alpha > 0$ small. Applying (F5) with $R = |p_0| + 1$, we have $I_u(\rho), I_v(\rho) \to 0$ as $\rho \to \infty$, where

$$I_u(\rho) := \sup_{\alpha\in(0,\alpha_0)} |F(u(\tilde{x},\tilde{t}), p_1, U_\alpha[\tilde{x},\tilde{t}]) - F(u(\tilde{x},\tilde{t}), p_1, U_\alpha[\tilde{x},\tilde{t}] \cap B_\rho(0))|,$$
$$I_v(\rho) := \sup_{\alpha\in(0,\alpha_0)} |F(v(\tilde{y},\tilde{t}), p_2, V_\alpha[\tilde{y},\tilde{t}]) - F(v(\tilde{y},\tilde{t}), p_2, V_\alpha[\tilde{y},\tilde{t}] \cap B_\rho(0))|.$$

Since u and v, respectively, are a viscosity subsolution and supersolution to (1.1), by the Crandall-Ishii lemma [7] we get

$$\begin{aligned} &h + F(u(\tilde{x},\tilde{t}), p_1, U_\alpha[\tilde{x},\tilde{t}]) \leq 0, \\ &k + F(v(\tilde{y},\tilde{t}), p_2, V_\alpha[\tilde{y},\tilde{t}]) \geq 0, \end{aligned} \tag{2.9}$$

where $h, k \in \mathbb{R}$ satisfy $h - k = \lambda/(T-\tilde{t})^2 \geq \lambda/T^2$. Taking the difference of both inequalities, we have

$$h - k \leq F(v(\tilde{y},\tilde{t}), p_2, V_\alpha[\tilde{y},\tilde{t}] \cap B_\rho(0)) - F(u(\tilde{x},\tilde{t}), p_1, U_\alpha[\tilde{x},\tilde{t}] \cap B_\rho(0)) + I_u(\rho) + I_v(\rho),$$

which, by (2.7) and (F3), yields

$$h - k \leq \omega_R(|p_1 - p_2|) + I_u(\rho) + I_v(\rho)$$

with $R = |p_0| + 1$. Sending $\alpha \to 0$ and $\rho \to \infty$, we deduce $h - k \leq 0$, which is a contradiction.

Let us turn to Case 2. In this case, we have as $\alpha_i \to 0$, $p_0 = 4|\tilde{x}-\tilde{y}|^2(\tilde{x}-\tilde{y})/\varepsilon^4 \to 0$ and thus $p_1, p_2 \to 0$, where p_1, p_2 are given as in (2.8) above. Then we can adopt the definition of subsolutions and supersolutions again to get (2.9) with $h - k = \lambda/(T-\tilde{t})^2$. Letting $\alpha_i \to 0$ and applying (F1) and (F6) we are led to

$$\frac{\lambda}{T^2} \leq \frac{\lambda}{(T-\tilde{t})^2} = h - k \leq V(v(\tilde{y},\tilde{t})) - V(u(\tilde{x},\tilde{t})) < 0,$$

which is obviously a contradiction. $\square$

Assuming that u_0 is uniformly continuous and there exists a subsolution satisfying (I) below, we can prove the existence of a unique viscosity solution that satisfies (1.5), (1.6) and (1.7).

(I) There exists a function $\phi \in C(\mathbb{R}^n \times [0,\infty))$ such that
- (i) $\phi(\cdot,t) \in UC(\mathbb{R}^n)$ for any $t \geq 0$,
- (ii) $u_0 \geq \phi(\cdot,0)$ in $\mathbb{R}^n$,
- (iii) $\phi \geq c_0$ in $\mathbb{R}^n \times [0,\infty)$ for some $c_0 > 0$.
- (iv) ϕ is coercive in space, that is,

$$\inf_{|x|\geq R,\ t\leq T} \phi(x,t) \to \infty \quad \text{as } R \to \infty \text{ for any } T \geq 0.$$

- (v) ϕ is a viscosity subsolution of (1.1).

Theorem 2.3 (Existence)**.** *Assume that* (F1)–(F6) *hold. Let* $u_0 \in UC(\mathbb{R}^n)$*. Assume in addition that* (I) *holds. Then there exists a unique solution* u *of* (1.1) *and* (1.2) *that satisfies* (1.5), (1.6) *and* (1.7)*.*

We omit the detailed proof here, since it is based on the stability result [22, (P2)] and the standard Perron's method (see [7, 11] for instance) adapted to (1.1). The uniform continuity (1.7) can also be shown by comparing the solution with its spatial translations; see similar arguments in the proofs of [12, Corollary 2.11] and [11, Theorem 3.5.1].

3. Quasiconvexity preserving

This section is devoted to proving our main result, Theorem 1.1. Fix arbitrarily $\lambda \in (0,1)$. For $u \in C(\mathbb{R}^n \times [0,\infty))$, let $u_{\star,\lambda}$ be given by (1.10). Our goal is to show that

$$u_{\star,\lambda}(x,t) = u(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{3.1}$$

Theorem 1.1 follows immediately, since u is quasiconvex in space if and only if (3.1) holds for all $\lambda \in (0,1)$. By the definition of $u_{\star,\lambda}$, it is clear that $u_{\star,\lambda} \leq u$ in $\mathbb{R}^n \times [0,\infty)$. It thus suffices to prove the reverse inequality. To this end, we use the power convex envelope function $u_{q,\lambda}$ ($q > 1$) given by (1.11) to approximate $u_{\star,\lambda}$. In fact, we have

$$u_{q,\lambda} \to u_{\star,\lambda} \quad \text{locally uniformly in } \mathbb{R}^n \times [0,\infty). \tag{3.2}$$

See [17, Proposition 4.1] for a more precise statement and proof of this convergence result.

We next show a key ingredient to prove (3.1), which stems from the idea in [2] to prove convexity of solutions to fully nonlinear equations by using its convex envelope. Such an idea is later developed in [13] to show a power-type convexity or concavity with a finite exponent. We here makes a further step, studying the limit case as the exponent tends to ∞.

Lemma 3.1. *Assume that* (F1)–(F7) *hold. Let $u \in C(\mathbb{R}^n \times [0,\infty))$ be a supersolution of* (1.1) *satisfying* (1.5) *and* (1.6). *Let $\lambda \in (0,1)$ and $u_{\star,\lambda}$ be the function defined by* (1.10). *Then $u_{\star,\lambda}$ is a supersolution of* (1.1).

Proof. For simplicity of notation, we write $w_\star = u_{\star,\lambda}$ and $w_q = u_{q,\lambda}$. Let us first show that $w_\star \in \mathrm{LSC}(\mathbb{R}^n \times [0,\infty))$. For an arbitrary $(x_0,t_0) \in \mathbb{R}^n \times [0,\infty)$, let (x_j,t_j) be a sequence satisfying

$$(x_j,t_j) \to (x_0,t_0), \quad w_\star(x_j,t_j) \to \liminf_{(x,t)\to(x_0,t_0)} w_\star(x,t) \quad \text{as } j \to \infty.$$

Due to the coercivity (1.6), there exist $y_j, z_j \in \mathbb{R}^n$ such that

$$x_j = \lambda y_j + (1-\lambda) z_j, \quad w_\star(x_j,t_j) = \min\{u(y_j,t_j), u(z_j,t_j)\}. \tag{3.3}$$

Since x_j is a bounded sequence, if either of the sequences y_j, z_j is unbounded, so does the other. Thus we can choose a subsequence such that $|y_j|, |z_j| \to \infty$, for which by (1.6) again we have

$$u(y_j,t_j) \to \infty, \quad u(z_j,t_j) \to \infty \quad \text{as } j \to \infty.$$

It follows from (3.3) that $w_\star(x_j,t_j) \to \infty$ as $j \to \infty$, which is a contradiction to the fact that $w_\star \leq u$ in $\mathbb{R}^n \times [0,\infty)$. Therefore, it is sufficient to assume that y_j and z_j are bounded sequences. Choosing subsequences y_j and z_j converging to y_0 and z_0 in $\mathbb{R}^n$ respectively, we can take the limit of (3.3) to obtain

$$\liminf_{(x,t)\to(x_0,t_0)} w_\star(x,t) = \min\{u(y_0,t_0), u(z_0,t_0)\} \geq w_\star(x_0,t_0).$$

Hence, $w_\star \in \mathrm{LSC}(\mathbb{R}^n \times [0,\infty))$. The lower semicontinuity of w_q can be proved similarly.

Let us next proceed to show that $w_\star$ satisfies the supersolution property. Suppose that there exist $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ and $\varphi \in C^2(\mathbb{R}^n \times (0,\infty))$ such that $w_\star - \varphi$ attains a strict minimum at (x_0,t_0). Without loss of generality we may assume $\varphi > 0$ in $\mathbb{R}^n \times (0,\infty)$.

In light of (3.2), there exists a sequence, indexed by q, of $(x_q, t_q) \in \mathbb{R}^n \times (0,\infty)$ such that $w_q - \varphi$ attains a strict minimum at (x_q, t_q) and

$$(x_q, t_q) \to (x_0, t_0), \quad w_q(x_q, t_q) \to w_\star(x_0, t_0) \quad \text{as} \quad q \to \infty.$$

Due to (1.6), there exist $y_q,\ z_q \in \mathbb{R}^n$ such that

$$x_q = \lambda y_q + (1-\lambda) z_q, \quad w_q(x_q, t_q) = \left(\lambda u(y_q, t_q)^q + (1-\lambda) u(z_q, t_q)^q\right)^{\frac{1}{q}}. \tag{3.4}$$

Shifting φ so that $\varphi(x_q, t_q) = w_q(x_q, t_q)$ and letting $v := u^q$ and $\psi := \varphi^q$, we see that

$$(y, z, t) \mapsto \lambda v(y, t) + (1-\lambda) v(z, t) - \psi(\lambda y + (1-\lambda) z, t)$$

takes a minimum at $(y_q, z_q, t_q) \in \mathbb{R}^n \times \mathbb{R}^n \times (0, \infty)$.

Using the Crandall-Ishii lemma [7], we have $(h_q, \eta_q), (k_q, \zeta_q) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$\lambda h_q + (1-\lambda) k_q = \psi_t(x_q, t_q), \quad \eta_q = \zeta_q = \nabla \psi(x_q, t_q), \tag{3.5}$$

and, by the assumption that u is a supersolution of (1.1),

$$\begin{aligned} &h_q + G_\beta(a_q, \eta_q, \{v(\cdot, t_q) \le v(y_q, t_q)\}) \ge 0, \\ &k_q + G_\beta(b_q, \zeta_q, \{v(\cdot, t_q) \le v(z_q, t_q)\}) \ge 0, \end{aligned} \tag{3.6}$$

where $a_q = v(y_q, t_q), b_q = v(z_q, t_q)$, $\beta = 1 - \frac{1}{q}$ and G_β is the transformed operator given by (1.8). Let us divide our argument into the following two cases:

- Case 1. $\nabla \varphi(x_0, t_0) \ne 0$,
- Case 2. $\nabla \varphi(x_0, t_0) = 0$, and $\nabla^2 \varphi(x_0, t_0) = 0$.

We write $\xi_q = \nabla \psi(x_q, t_q)$ for simplicity of notation.

In Case 1, by (3.5), we have $\nabla \psi(x_q, t_q) = \eta_q = \zeta_q \ne 0$ when $q > 1$ is sufficiently large. Multiplying the first inequality in (3.6) by λ and the second by $1 - \lambda$ and then adding them up, by (3.5) we are led to

$$\begin{aligned} &\psi_t(x_q, t_q) + \lambda G_\beta(a_q, \xi_q, W_\star[x_0, t_0]\}) + (1-\lambda) G_\beta(b_q, \xi_q, W_\star[x_0, t_0])) \\ &\quad \ge \lambda \left(G_\beta(a_q, \xi_q, W_\star[x_0, t_0]) - G_\beta(a_q, \xi_q, U[y_q, t_q])\right) \\ &\quad\quad + (1-\lambda) \left(G_\beta(b_q, \xi_q, W_\star[x_0, t_0]) - G_\beta(b_q, \xi_q, U[z_q, t_q])\right), \end{aligned}$$

where we denote, for any $(x, t) \in \mathbb{R}^n \times (0, \infty)$,

$$\begin{aligned} &W_\star[x, t] := \{w_\star(\cdot, t) \le w_\star(x, t)\}, \\ &U[x, t] := \{u(\cdot, t) \le u(x, t)\} = \{v(\cdot, t) \le v(x, t)\}. \end{aligned}$$

Noticing that $\lambda a_q + (1-\lambda) b_q = w_q(x_q, t_q)^q$ and applying (F7), we then get

$$\begin{aligned} &\psi_t(x_q, t_q) + G_\beta\left(w_q(x_q, t_q)^q, \nabla \psi(x_q, t_q), W_\star[x_0, t_0]\right) \\ &\quad \ge \lambda \left(G_\beta(a_q, \xi_q, W_\star[x_0, t_0]) - G_\beta(a_q, \xi_q, U[y_q, t_q])\right) \\ &\quad\quad + (1-\lambda) \left(G_\beta(b_q, \xi_q, W_\star[x_0, t_0]) - G_\beta(b_q, \xi_q, U[z_q, t_q])\right). \end{aligned}$$

Rewriting this relation in terms of the operator F, we are led to

$$\varphi_t(x_q, t_q) + F(w_q(x_q, t_q), \nabla \varphi(x_q, t_q), W_\star[x_0, t_0]) \ge \lambda \frac{u(y_q, t_q)^{q-1}}{\varphi(x_q, t_q)^{q-1}} D_{1,q} + (1-\lambda) \frac{u(z_q, t_q)^{q-1}}{\varphi(x_q, t_q)^{q-1}} D_{2,q}, \tag{3.7}$$

where $C_q = C\varphi(x_q, t_q)^{1-q}/q$, and

$$\begin{aligned} &D_{1,q} = F(u(y_q, t_q), \nabla \varphi(x_q, t_q), W_\star[x_0, t_0]) - F(u(y_q, t_q), \nabla \varphi(x_q, t_q), U[y_q, t_q]), \\ &D_{2,q} = F(u(z_q, t_q), \nabla \varphi(x_q, t_q), W_\star[x_0, t_0]) - F(u(z_q, t_q), \nabla \varphi(x_q, t_q), U[z_q, t_q]). \end{aligned}$$

The assumption that $u \geq c_0$ yields

$$\begin{aligned}\frac{\lambda u(y_q,t_q)^{q-1}+(1-\lambda)u(z_q,t_q)^{q-1}}{\varphi(x_q,t_q)^{q-1}} &= \frac{\lambda u(y_q,t_q)^{q-1}+(1-\lambda)u(z_q,t_q)^{q-1}}{w_q(x_q,t_q)^{q-1}}\\ \leq \frac{\lambda u(y_q,t_q)^{q}+(1-\lambda)u(z_q,t_q)^{q}}{c_0 w_q(x_q,t_q)^{q-1}} &= \frac{w_q(x_q,t_q)^q}{c_0 w_q(x_q,t_q)^{q-1}} = \frac{w_q(x_q,t_q)}{c_0},\end{aligned}$$

which implies

$$\sup_{q>1}\left\{\frac{\lambda u(y_q,t_q)^{q-1}+(1-\lambda)u(z_q,t_q)^{q-1}}{\varphi(x_q,t_q)^{q-1}}\right\} < \infty. \tag{3.8}$$

Let us proceed to estimate $D_{1,q}, D_{2,q}$ in (3.7). Note that by (3.4)

$$\limsup_{q\to\infty} u(y_q,t_q) \leq \limsup_{q\to\infty} w_q(x_q,t_q) = w_\star(x_0,t_0).$$

Also, it is easily seen that

$$w_\star(\cdot,t_0) = u_{\star,\lambda}(\cdot,t_0) \leq u(\cdot,t_0) \leq \operatorname*{lim\,inf_*}_{q\to\infty} u(\cdot,t_q) \quad \text{in } \mathbb{R}^n.$$

Since (1.6) implies that $U[y_q,t_q]$ are bounded uniformly in q, by [17, Lemma 2.2] we have

$$m(U[y_q,t_q]\setminus W_\star[x_0,t_0]) \to 0 \quad \text{as } q\to\infty.$$

Noticing that (F4) yields

$$F(u(y_q,t_q),\nabla\varphi(x_q,t_q),W_\star[x_0,t_0]) \geq F(u(y_q,t_q),\nabla\varphi(x_q,t_q),W_\star[x_0,t_0]\cap U[y_q,t_q]),$$

in view of (1.4) in (F3), we deduce that

$$D_{1,q} \geq -\omega_R\Big(m\big(U[y_q,t_q]\setminus W_\star[x_0,t_0]\big)\Big)$$

for $q>1$ sufficiently large, where $R = |\nabla\varphi(x_0,t_0)|+1$. Similarly, we have

$$D_{2,q} \geq -\omega_R\Big(m\big(U[z_q,t_q]\setminus W_\star[x_0,t_0]\big)\Big)$$

with $m(U[z_q,t_q]\setminus W_\star[x_0,t_0]) \to 0$ as $q\to\infty$. Hence, thanks to (3.8), sending $\varepsilon\to 0$ and then $q\to\infty$ in (3.7), we get

$$\varphi_t(x_0,t_0) + F(w_\star(x_0,t_0),\nabla\varphi(x_0,t_0),\{u_{\star,\lambda}(\cdot,t_0)\leq u_{\star,\lambda}(x_0,t_0)\}) \geq 0.$$

Let us next turn to Case 2. If $\nabla\varphi(x_q,t_q)\neq 0$ along a subsequence, then passing to the limit of (3.7) as $\varepsilon\to 0$ and $q\to\infty$ via the subsequence, by (F6) we get the desired relation

$$\varphi_t(x_0,t_0) + V(w_\star(x_0,t_0)) \geq 0. \tag{3.9}$$

It remains to consider the case when $\nabla\varphi(x_q,t_q)=0$ for all $q>1$ large. Adopting the definition of supersolutions, we have

$$\begin{aligned} h_q + G_\beta(v(y_q,t_q),0,\{v(\cdot,t_q)\leq v(y_q,t_q)\}) \geq 0,\\ k_q + G_\beta(v(z_q,t_q),0,\{v(\cdot,t_q)\leq v(z_q,t_q)\}) \geq 0,\end{aligned}$$

which by (F6) yields

$$(1-\beta)h_q + v(y_q,t_q)^\beta V(v(y_q,t_q)^{1-\beta}) \geq 0, \quad (1-\beta)k_q + v(z_q,t_q)^\beta V(v(z_q,t_q)^{1-\beta}) \geq 0.$$

It follows from (F7) that

$$\frac{1}{q}(\lambda h_q + (1-\lambda)k_q) + w_q(x_q,t_q)^{q-1}V(w_q(x_q,t_q)) \geq 0$$

which, together with the relation $w_q(x_q,t_q)=\varphi(x_q,t_q)$, yields,

$$\frac{1}{q}\varphi(x_q,t_q)^{1-q}(\lambda h_q+(1-\lambda)k_q)+V(w_q(x_q,t_q))\geq 0$$

Noticing that, due to (3.5),

$$\varphi_t(x_q,t_q)=\frac{1}{q}\varphi(x_q,t_q)^{1-q}\psi_t(x_q,t_q)=\frac{1}{q}\varphi(x_q,t_q)^{1-q}(\lambda h_q+(1-\lambda)k_q),$$

we obtain

$$\varphi_t(x_q,t_q)+V(w_q(x_q,t_q))\geq 0. \tag{3.10}$$

In view of (3.8), letting $q\to\infty$ in (3.10), we again end up with (3.9). $\square$

Proof of Theorem 1.1. By the quasiconvexity of u_0, we have $u_{\star,\lambda}(\cdot,0)=u_0$ in $\mathbb{R}^n$. Moreover, since $c_0\leq u_{\star,\lambda}\leq u$ holds in $\mathbb{R}^n\times[0,\infty)$, $u_{\star,\lambda}$ obviously satisfies the growth condition (2.1). Note that $u_{\star,\lambda}$ also satisfies (1.6). Indeed, for any $R>0$ and $(x,t)\in\mathbb{R}^n\times[0,\infty)$ with $|x|\geq R$, if $\lambda y+(1-\lambda)z=x$ for $y,z\in\mathbb{R}^n$, then either $|y|\geq R$ or $|z|\geq R$ holds and therefore

$$\max\{u(y,t),u(z,t)\}\geq \inf_{\mathbb{R}^n\setminus B_R(0)} u(\cdot,t).$$

By (1.10) we thus have $u_{\star,\lambda}(x,t)\geq \inf_{\mathbb{R}^n\setminus B_R(0)} u(\cdot,t)$ for all $(x,t)\in\mathbb{R}^n\times[0,\infty)$ fulfilling $|x|\geq \mathbb{R}$. We can immediately use the coercivity of u in space to obtain the coercivity of $u_{\star,\lambda}$.

The relation (3.1) is then an immediate consequence of Lemma 3.1 and the comparison principle, Theorem 2.1. Noticing that (3.1) implies the quasiconvexity of u in space, we complete the proof of Theorem 1.1. $\square$

We conclude the note with a concrete example:

$$u_t+V(u)+|\nabla u|\left(W(u)+Q(\{u(\cdot,t)<u(x,t)\})\right)=0,\quad \text{in } \mathbb{R}^n\times(0,\infty),$$

where $V,W\in C^2(\mathbb{R})$ are given bounded nondecreasing functions and Q is a finite measure in $\mathbb{R}^n$ that is absolutely continuous with respect to m. Then there exists $M_Q>0$ such that $Q(A)\leq M_Q$ for all $A\in\mathcal{B}$. This equation in general does not satisfy the geometricity condition (1.3). The operator F is

$$F(r,p,A)=V(r)+|p|W(r)+|p|Q(A).$$

We easily see that (F1)–(F6) hold in this case. Let us verify that F satisfies (F7) under further assumptions on V,W. Indeed, the operator G_β as in (1.8) is

$$G_\beta(r,p,A)=\frac{1}{1-\beta}r^\beta V(r^{1-\beta})+|p|W(r^{1-\beta})+|p|Q(A).$$

It is clear that, for $0<\beta<1$ close to 1, the condition (1.9) holds provided that $r\mapsto r^\beta V(r^{1-\beta})$ and $r\mapsto W(r^{1-\beta})$ is concave in $[c_0,\infty)$.

Moreover, it is possible to construct ϕ satisfying (I) if $V(c_0)\leq 0$ and $u_0\in UC(\mathbb{R}^n)$ satisfies

$$\lim_{R\to\infty}\inf_{|x|\geq R}\frac{u_0(x)}{|x|}>0,\qquad \inf_{x\in\mathbb{R}^n}u_0(x)\geq c_0$$

for some $c_0>0$. In fact, we can take $\phi(x,t):=\max\{m|x|-Ct-M,c_0\}$ with $m>0$ small, $M>0$ large so that $u_0\geq\phi$ in $\mathbb{R}^n$, and

$$C:=\sup_{r\in\mathbb{R}}\{V(r)+mW(r)\}+mM_Q.$$

We omit the detailed verification here.

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