$H^2(ds)$-Sobolev gradient flow for the modified elastic energy

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1 Introduction

The paper is devoted to a $H^2(ds)$-Sobolev gradient flow for a functional defined on closed curves. We shall announce a result ([20]) which is a joint work with P. Schrader of Murdoch University.

In this paper, we consider a gradient flow for the modified elastic energy defined on closed curves:

$$\mathcal{E}_\lambda(\gamma) := E(\gamma) + \lambda^2 L(\gamma)$$

with

$$E(\gamma) := \frac{1}{2}\int_\gamma |\kappa|^2\, ds, \qquad L(\gamma) := \int_\gamma ds,$$

where $\gamma : \mathbb{R}/2\pi\mathbb{Z} := S^1 \to \mathbb{R}^n$, $n \geq 1$, $\lambda \neq 0$, and s and κ denote the arc length parameter and the curvature of γ, respectively. The functional E is well-known as the elastic energy or the Euler–Bernoulli bending energy, and $L(\gamma)$ denotes the length of γ. The critical points of E with length constraint is called *elastica*. One of tool of analysis on elastica is to construct gradient flows towards elastica. In 1985, taking advantage of the fact that the energy E can be regarded as the Dirichlet energy of the tangent vector of the curves, J. Langer and D. A. Singer [9, 10] considered a H^1-gradient flow for E, which is a second order parabolic equation with a nonlocal term. The work by [9] was extended into L^2-gradient flows for E and have been studied by many researchers (e.g., see [2, 3, 5, 8, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24, 25] and references therein). The purpose of this paper is to give a new gradient trajectories to elastica.

In this paper we consider the Cauchy problem on the $H^2(ds)$-gradient flow for the functional $\mathcal{E}_\lambda$ defined on curves in $\mathbb{R}^n$:

$$\text{(GF)} \qquad \begin{cases} \partial_t \gamma = -\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma), \\ \gamma(\cdot, 0) = \gamma_0(\cdot). \end{cases}$$

Here $\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma)$ denotes the $H^2(ds)$-gradient of $\mathcal{E}_\lambda$ at γ, which is defined in Section 2. We consider initial curves in the class

$$\mathcal{I}^2(S^1,\mathbb{R}^n) := \{\gamma \in H^2(S^1,\mathbb{R}^n) \mid |\gamma'(u)| > 0 \text{ in } S^1\},$$

which is the set of all regular closed curves in $H^2(S^1,\mathbb{R}^n)$. The main result of this paper is stated as follows:

Theorem 1.1. *Let $\gamma_0 \in \mathcal{I}^2(S^1,\mathbb{R}^n)$. Then problem* (GF) *possesses a unique global-in-time solution $\gamma \in C^1([0,\infty),\mathcal{I}^2(S^1,\mathbb{R}^n))$. Moreover, the solution γ converges to an elastica as $t \to \infty$ in the $H^2(ds)$-topology.*

For gradient flows for the modified elastic energy, it is now a standard result that the flow has a unique global-in-time solution and that the solution converges to an elastica along a time sequence, i.e., the solution sub-converges to an elastica as $t \to \infty$. The point is how to extend sub-convergence to full limit convergence. In general, L^2-gradient flow and H^1-gradient flow for the modified elastic energy converge to an elastica as $t \to \infty$ under a translation or reparametrization. On the other hand, Theorem 1.1 asserts that the solution of (GF) converges to an elastica as $t \to \infty$ without any additional modification. This is one of the contributions of Theorem 1.1.

2 Formulation and preliminary

In this section, first we define the $H^2(ds)$-gradient flow for the functional $\mathcal{E}_\lambda$.

For $\gamma \in \mathcal{I}^2(S^1,\mathbb{R}^n)$, we define the $H^2(ds)$-inner product by

$$\langle u, v\rangle_{H^2(ds)} := \int_0^{L(\gamma)} \sum_{j=0}^{2} \partial_s^j u(s)\cdot \partial_s^j v(s)\, ds, \quad u,v \in \mathcal{I}^2(S^1,\mathbb{R}^n),$$

where s denote the arc length parameter of γ. We denote by $\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma)$ the $H^2(ds)$-gradient of $\mathcal{E}_\lambda$ at γ, which is defined by

$$\frac{d}{d\varepsilon}\mathcal{E}_\lambda(\gamma+\varepsilon\varphi)\Big|_{\varepsilon=0} = \langle \nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma), \varphi\rangle_{H^2(ds)} \quad \text{for all} \quad \varphi \in H^2(S^1,\mathbb{R}^n).$$

Since

$$\frac{d}{d\varepsilon}\mathcal{E}_\lambda(\gamma+\varepsilon\varphi)\Big|_{\varepsilon=0} = \int_0^{L(\gamma)} \nabla_{L^2(ds)}\mathcal{E}_\lambda(\gamma)\cdot\varphi\, ds$$

with

$$\nabla_{L^2(ds)}\mathcal{E}_\lambda(\gamma) = 2\partial_s^4\gamma + 3\partial_s(\kappa^2\partial_s\gamma) - \lambda^2\partial_s^2\gamma,$$

the $H^2(ds)$-gradient $\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma)$ is given by the solution of

$$\partial_s^4\Phi - \partial_s^2\Phi + \Phi = \nabla_{L^2(ds)}\mathcal{E}_\lambda(\gamma)$$

with C^2-periodic boundary condition. Let $G = G(s, \tilde{s})$ be the Green function, i.e., the solution to

$$\partial_s^4 G(s, \tilde{s}) - \partial_s^2 G(s, \tilde{s}) + G(s, \tilde{s}) = \delta(s, \tilde{s})$$

which is C^2-periodic, where δ denotes the Dirac delta function. The precise form of G is written as follows:

$$G(x, y; \gamma) = \frac{A(L(\gamma) - |x-y|, |x-y|)}{\beta(L(\gamma))}, \quad 0 \le x, y \le L(\gamma),$$

where

$$\begin{aligned} A(x_1, x_2) &= \sinh \frac{\sqrt{3}x_1}{2} \cos \frac{x_2}{2} + \sinh \frac{\sqrt{3}x_2}{2} \cos \frac{x_1}{2} \\ &\quad + \sqrt{3} \cosh \frac{\sqrt{3}x_1}{2} \sin \frac{x_2}{2} + \sqrt{3} \cosh \frac{\sqrt{3}x_2}{2} \sin \frac{x_1}{2}, \\ \beta(\ell) &= 2\sqrt{3}\Big(\cosh \frac{\sqrt{3}\ell}{2} - \cos \frac{\ell}{2}\Big). \end{aligned}$$

Then the $H^2(ds)$-gradient of $\mathcal{E}_\lambda$ is derived as follows:

$$\begin{aligned} \nabla_{H^2(ds)} \mathcal{E}_\lambda(\gamma) &= \int_0^{L(\gamma)} G(s, \tilde{s}) \nabla_{L^2(ds)} \mathcal{E}_\lambda(\gamma)(\tilde{s})\, d\tilde{s} \\ &= 2\gamma(s) - \int_0^{L(\gamma)} \big[2G(s, \tilde{s})\gamma(\tilde{s}) + G_{\tilde{s}}(s, \tilde{s})\gamma_{\tilde{s}}(\tilde{s})\big(3\kappa(\tilde{s})^2 + 2 - \lambda^2\big)\big]\, d\tilde{s}, \end{aligned}$$

and then the $H^2(ds)$-gradient flow for $\mathcal{E}_\lambda$ is written as

$$\partial_t \gamma(s, t) = -2\gamma(s, t) + \int_0^{L(\gamma)} \big[2G(s, \tilde{s})\gamma(\tilde{s}, t) + G_{\tilde{s}}(s, \tilde{s})\gamma_{\tilde{s}}(\tilde{s}, t)\big(3\kappa(\tilde{s}, t)^2 + 2 - \lambda^2\big)\big]\, d\tilde{s}.$$

We define the $H^2(ds)$-Riemannian distance on $\mathcal{I}^2(S^1, \mathbb{R}^2)$ as follows:

$$\operatorname{dist}(\alpha, \beta) := \inf_{p \in P} \int_0^1 \|p'(t)\|_{H^2(ds_p)}\, dt, \quad \alpha,\, \beta \in \mathcal{I}^2(S^1, \mathbb{R}^2),$$

where s_p denotes the arc length parameter of p, and

$$P := \{p \in C^1([0,1], \mathcal{I}^2(S^1, \mathbb{R}^n)) \mid p(0) = \alpha,\, p(1) = \beta\}.$$

By [7, Theorem 1.9.5], since $H^2(ds)$ is a strong Riemannian metric, the distance function defines a metric on $\mathcal{I}^2(S^1, \mathbb{R}^n)$ whose topology coincides with the H^2-topology.

Lemma 2.1 ([1], Lemma 4.2)**.** *Let $B_r^{\mathrm{dist}}(\gamma_0)$ be the open ball with radius $r > 0$ with respect to the $H^2(ds)$-Riemannian distance.*

(i) *Given $\gamma_0 \in \mathcal{I}^2(S^1, \mathbb{R}^n)$ there exist $r > 0$ and $C > 0$ such that*

$$\operatorname{dist}(\gamma_1, \gamma_2) \le C\|\gamma_1 - \gamma_2\|_{H^2}$$

for all $\gamma_1, \gamma_2 \in B_r^{\mathrm{dist}}(\gamma_0)$.

(ii) *Given* $B_r^{\mathrm{dist}}(\gamma_0) \subset \mathcal{I}^2(S^1, \mathbb{R}^n)$ *there exists* $C > 0$ *such that*

$$\|\gamma_1 - \gamma_2\|_{H^2} \le C \mathrm{dist}(\gamma_1, \gamma_2)$$

for all $\gamma_1, \gamma_2 \in B_r^{\mathrm{dist}}(\gamma_0)$.

It is known that the metric space $(\mathcal{I}^2(S^1, \mathbb{R}^n), \mathrm{dist})$ possesses the completeness. The completeness plays an important role in the proof of Theorem 1.1.

Proposition 2.1 ([1], Theorem 4.3)**.** *The space* $(\mathcal{I}^2(S^1, \mathbb{R}^n), \mathrm{dist})$ *is a complete metric space.*

3 Proof of Theorem 1.1

We start with the existence of local-in-time solutions of problem (GF). The $H^2(ds)$-gradient flow for $\mathcal{E}_\lambda$ can be regarded as an ODE in $H^2(S^1, \mathbb{R}^n)$. In fact, the $H^2(ds)$-gradient flow is written as

$$\begin{aligned} \partial_t \gamma(u,t) = -2\gamma(u,t) + \int_0^1 \Big[& 2G(s, \tilde{s}; \gamma)\gamma(\tilde{u},t)|\partial_{\tilde{u}}\gamma(\tilde{u},t)| \\ & + \frac{1}{|\partial_{\tilde{u}}\gamma(\tilde{u},t)|}\partial_{\tilde{u}} G(s, \tilde{s}; \gamma)\partial_{\tilde{u}}\gamma(\tilde{u},t)\big(3\kappa(\tilde{u},t)^2 + 2 - \lambda^2\big)\Big]\, d\tilde{u} \\ =: F(\gamma), \end{aligned}$$

where

$$s = \int_0^u |\partial_\xi \gamma(\xi, t)|\, d\xi, \qquad \tilde{s} = \int_0^{\tilde{u}} |\partial_\xi \gamma(\xi, t)|\, d\xi.$$

Thus the existence of local-in-time solutions of (GF) is proved by the generalized Picard–Lindelöf Theorem (e.g., see [26, Theorem 3.A]). In fact, we can verify:

Lemma 3.1. *Let* $\gamma_0 \in \mathcal{I}^2(S^1, \mathbb{R}^n)$ *and* $b = \frac{1}{2}\min_{u \in S^1} |\gamma_0'(u)|$*. Then there exists a positive constant* C *depending on* γ_0 *such that*

$$\|F(\gamma)\|_{H^2} \le C, \quad \|DF_\gamma\|_{(H^2)^*} \le C,$$

for all $\gamma \in H^2(S^1, \mathbb{R}^n)$ *with* $\|\gamma - \gamma_0\|_{H^2} < b/C_S$*, where* C_S *denotes the Sobolev constant of the imbedding* $H^1(S^1) \subset C^{\frac{1}{2}}(S^1)$.

Then we have:

Proposition 3.1. *Let* $\gamma_0 \in \mathcal{I}^2(S^1, \mathbb{R}^n)$*. Then there exists* $T > 0$ *such that problem* (GF) *possesses a unique solution in* $C^1([0,T), \mathcal{I}^2(S^1, \mathbb{R}^n))$.

On the proof of the existence of global-in-time solutions, the following lemma plays an important role:

Lemma 3.2. *Assume that* $\gamma \in C^1((a,b),\mathcal{I}^2(S^1,\mathbb{R}^n))$ *satisfies*

$$\int_a^b \|\partial_t \gamma\|_{H^2(ds)}\, dt < \infty. \tag{3.1}$$

Then $\lim_{t\uparrow b}\gamma(t)$ *exists in* $(\mathcal{I}^2(S^1,\mathbb{R}^n),\mathrm{dist})$.

Proof. Fix a monotone increasing sequence $\{t_j\} \subset (a,b)$ such that $t_j \to b$ as $j \to \infty$ arbitrarily. We claim that $\{\gamma(t_j)\}$ is Cauchy in $(\mathcal{I}^2(S^1,\mathbb{R}^n),\mathrm{dist})$. Suppose not, there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ we find $j > k > N$ satisfying $\mathrm{dist}(\gamma(t_j),\gamma(t_k)) > \varepsilon$. Since

$$\mathrm{dist}(\gamma(t_j),\gamma(t_k)) \le \int_{t_k}^{t_j} \|\partial_t \gamma(t)\|_{H^2(ds)}\, dt,$$

this clearly contradicts the assumption (3.1). Then, it follows from Proposition 2.1 that $\gamma(t_j)$ converges to some γ_b as $j \to \infty$ in $(\mathcal{I}^2(S^1,\mathbb{R}^n),\mathrm{dist})$. We note that the limit γ_b is unique. In fact, if we find a sequence $\{\tilde{t}_j\} \subset (a,b)$ such that $\gamma(\tilde{t}_j) \to \tilde{\gamma}_b$ as $j \to \infty$ in $(\mathcal{I}^2(S^1,\mathbb{R}^n),\mathrm{dist})$, taking $\{\bar{t}_j\}$ to be the ordered union of $\{t_j\}$ and $\{\tilde{t}_j\}$, we have $\gamma_b = \tilde{\gamma}_b$. Since $\{t_j\} \subset (a,b)$ is arbitrary, we obtain the conclusion. $\square$

Then we have:

Proposition 3.2. *Let* $\gamma_0 \in \mathcal{I}^2(S^1,\mathbb{R}^n)$. *Then problem* (GF) *possesses a unique global-in-time solution* $\gamma \in C^1([0,\infty),\mathcal{I}^2(S^1,\mathbb{R}^n))$.

Proof. Suppose that $T_{\max} < \infty$. Since γ satisfies the $H^2(ds)$-gradient flow for $\mathcal{E}_\lambda$, we have

$$\mathcal{E}_\lambda(\gamma(t)) - \mathcal{E}_\lambda(\gamma_0) = \int_0^t \frac{d}{d\tau}\mathcal{E}_\lambda(\gamma(\tau))\, d\tau = -\int_0^t \|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma(\tau))\|^2_{H^2(ds)}\, d\tau,$$

and then

$$\int_0^t \|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma(\tau))\|^2_{H^2(ds)}\, d\tau \le \mathcal{E}_\lambda(\gamma_0).$$

This together with Hölder's inequality implies that

$$\int_0^{T_{\max}} \|\partial_t\gamma(\tau)\|_{H^2(ds)}\, d\tau \int_0^{T_{\max}} \|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma(\tau))\|_{H^2(ds)}\, d\tau \le \sqrt{T_{\max}}\sqrt{\mathcal{E}_\lambda(\gamma_0)}. \tag{3.2}$$

Combining (3.2) with Lemma 3.2, we find a curve $\gamma_* \in \mathcal{I}^2(S^1,\mathbb{R}^n)$ such that

$$\gamma(\cdot,t) \to \gamma_* \quad \text{as} \quad t \uparrow T_{\max} \quad \text{in} \quad (\mathcal{I}^2(S^1,\mathbb{R}^n),\mathrm{dist}).$$

Then we deduce from Proposition 3.1 that the solution $\gamma : S^1 \times [0,T_{\max}) \to \mathbb{R}^n$ can be extended. This clearly contradicts the definition of $T_{\max}$. $\square$

We turn to the proof of full limit convergence of global-in-time solutions to elastica. If one can verify that

$$\int_0^\infty \|\partial_t\gamma(\tau)\|_{H^2(ds)}\, d\tau < \infty, \tag{3.3}$$

then Lemma 3.2 implies the full limit convergence of solutions of (GF). By the gradient structure of the $H^2(ds)$-gradient flow, as in the proof of Proposition 3.2, it is easy to show that

$$\int_0^\infty \|\partial_t\gamma(\tau)\|^2_{H^2(ds)}\, d\tau = \int_0^\infty \|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma(\tau)\|^2_{H^2(ds)}\, d\tau < \mathcal{E}_\lambda(\gamma_0) < \infty. \tag{3.4}$$

However, the L^2-integrability does not imply the full limit convergence. One of tool to extend the L^2-integrability into the L^1-integrability (3.3) is Łojasiewicz–Simon's gradient inequality. Although the Łojasiewicz–Simon gradient inequality for the L^2-gradient flow for E or $\mathcal{E}_\lambda$ has been proved (e.g., see [4, 15]), Łojasiewicz–Simon's gradient inequality for the $H^2(ds)$-gradient flow for $\mathcal{E}_\lambda$ is one of contributions of the paper ([20]).

Theorem 3.1. *Let $\sigma \in \mathcal{I}^2(S^1, \mathbb{R}^n)$ be a stationary point of $\mathcal{E}_\lambda$. Then there exist constants $Z \in (0,\infty)$, $\delta \in (0,1]$, and $\theta \in [\frac{1}{2}, 1)$ such that if $\gamma \in \mathcal{I}^2(S^1, \mathbb{R}^n)$ with $\|\gamma - \sigma\|_{H^2} < \delta$ then*

$$\|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma)\|_{H^2(ds)} \geq Z|\mathcal{E}_\lambda(\gamma) - \mathcal{E}_\lambda(\sigma)|^\theta.$$

We prove Theorem 3.1 along the strategy given by [6]. More precisely, we will verify that

(i) analyticity of $\mathcal{E}_\lambda$,

(ii) $d^2\mathcal{E}_\lambda$ is a Fredholm operator with the index 0.

Similarly to [4] we can verify condition (i). However, a difficulty arises from condition (ii). Indeed, if $\varphi \in \mathrm{Ker}(d^2\mathcal{E}_\lambda)$, then any reparametrization of φ also belongs to the space $\mathrm{Ker}(d^2\mathcal{E}_\lambda)$. For, problem (GF) and functional $\mathcal{E}_\lambda$ are invariant under any reparametrization. Therefore, in order to prove Theorem 3.1, first we fix a suitable parametrization. Let $H^1_{zm}(S^1, \mathbb{R}^n) := \{\alpha \in H^1(S^1, \mathbb{R}^n) \mid \int_{S^1} \alpha\, du = 0\}$ and define

$$\Phi : \mathcal{I}^2(S^1, \mathbb{R}^n) \to H^1_{zm}(S^1, \mathbb{R}^n), \quad \Phi(\gamma) := |\gamma_u| - L(\gamma).$$

Then $\Omega := \Phi^{-1}(0)$ is the subset of $\mathcal{I}^2(S^1, \mathbb{R}^n)$ consisting of curves which are parametrized proportional to arc length. For the restricted functional $\mathcal{E}_\lambda|\Omega$ we have:

Proposition 3.3. *Let $\varsigma \in \Omega$ be a stationary point of $\mathcal{E}_\lambda$. Then there exist constants $Z \in (0,\infty)$, $\delta \in (0,1]$, and $\theta \in [\frac{1}{2}, 1)$ such that if $\alpha \in \Omega$ with $\|\alpha - \varsigma\|_{H^2} < \delta$ then*

$$\|d(\mathcal{E}_\lambda|\Omega)(\alpha)\|_{T_\alpha\Omega^*} \geq Z|\mathcal{E}_\lambda(\alpha) - \mathcal{E}_\lambda(\varsigma)|^\theta.$$

Since any $\gamma \in \Omega$ is parametrized by a fixed parameter, we can eliminate the difficulty on condition (ii). Then Proposition 3.3 can be proved along the strategy given by [6]. Combining the Łojasiewicz–Simon gradient inequality in Proposition 3.3 with the estimate

$$\|d(\mathcal{E}_\lambda|\Omega)(\alpha)\|_{T_\alpha\Omega^*} \leq \|d\mathcal{E}_\lambda(\alpha)\|_{H^{2*}} \leq C\|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma)\|_{H^2(ds)} \tag{3.5}$$

for any stationary point γ of $\mathcal{E}_\lambda$ in $\mathcal{I}^2(S^1,\mathbb{R}^n)$ and its arc length proportional reparametrization α, we obtain Theorem 3.1.

Finally, employing Theorem 3.1, we prove full limit convergence of solutions of (GF) to elastica. First we prove the subconvergence of the solution to an elastica γ_*. Then, applying Theorem 3.1, we obtain Łojasiewicz–Simon gradient inequality with respect to the stationary point γ_*. However, in order to apply the Łojasiewicz–Simon gradient inequality to the global-in-time solution of (GF), we have to verify that the global-in-time solution belongs to the H^2-neighborhood of γ_* for sufficiently large $t > 0$. To this aim, we prepare the following Palais–Smale type condition for $\mathcal{E}_\lambda|\Omega$.

Proposition 3.4. *Let $\{\alpha_j\}_j \subset \Omega$ be a sequence of curves such that $\mathcal{E}_\lambda(\alpha_j)$ and $\|\alpha_j\|_{L^2}$ are bounded, and $\|d\mathcal{E}_\lambda(\alpha_j)\| \to 0$ as $j \to \infty$. Then $\{\alpha_j\}_j$ has a subsequence converging in H^2.*

Then we have:

Theorem 3.2. *Let γ be a global-in-time solution to problem* (GF)*. Then there exists a stationary point $\gamma_\infty \in H^2(S^1,\mathbb{R}^n)$ such that*

$$\gamma(t) \to \gamma_\infty \quad in \quad H^2 \quad as \quad t \to \infty.$$

Proof. Let

$$\alpha(t) := P(\gamma(t)) - \frac{1}{L(\gamma(t))}\int_0^{L(\gamma(t))} \gamma(t)\, ds,$$

where $P(\gamma(t))$ is the arc length proportional reparametrization of $\gamma(t)$. From parametrization and translation invariance of the energy we have

$$\lambda^2 L(\alpha) < \mathcal{E}_\lambda(\alpha) = \mathcal{E}_\lambda(\gamma) \le \mathcal{E}_\lambda(\gamma_0).$$

Moreover, using the Poincaré–Wirtinger inequality, we see that $\|\alpha(t)\|_{L^2}$ is also bounded. From (3.3) there exists a monotone divergent sequence $\{t_j\}$ such that

$$\|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma(t))\|_{H^2(ds)} \to 0 \quad \text{as} \quad j \to \infty.$$

This together with (3.5) implies that

$$\|d\mathcal{E}_\lambda(\alpha(t_j))\|_{(H^2)^*} \to 0 \quad \text{as} \quad j \to \infty.$$

From now on we abbreviate $\alpha(t_j)$ to α_j. Since $\{\alpha_j\}$ satisfies the assumption in Proposition 3.4, there exists a subsequence, still denote $\{\alpha_j\}$, converging in H^2 to a stationary point α_∞. Now by Theorem 3.1 there are constants $Z > 0$, $\delta \in (0,1]$, and $\theta \in [\frac{1}{2}, 1)$ such that if $x \in \mathcal{I}^2(S^1,\mathbb{R}^n)$ with $\|x - \alpha_\infty\|_{H^2} < \delta$ then

$$\|\nabla_{H^2(ds)}\mathcal{E}_\lambda(x)\|_{H^2(ds)} \ge Z|\mathcal{E}_\lambda(x) - \mathcal{E}_\lambda(\alpha_\infty)|^\theta. \tag{3.6}$$

Since the $H^2(ds)$-Riemannian distance and the standard H^2 metric are equivalent, there exist $\tilde{\delta} > 0$, $r > 0$ such that

$$B_{\tilde{\delta}}^{H^2}(\alpha_\infty) \subset B_r^{\mathrm{dist}}(\alpha_\infty) \subset B_{\delta}^{H^2}(\alpha_\infty).$$

For any $i \in \mathbb{N}$ such that $\alpha_i \in B_{\tilde{\delta}}^{H^2}(\alpha_\infty)$ we let $\beta_i(t)$ be the $H^2(ds)$-gradient flow with initial data $\beta_i(t_i) = \alpha_i$. Then due to the uniqueness of the flow, for all $t > t_i$, $\beta_i(t)$ is a fixed (i.e. time independent) reparametrization and translation of $\gamma(t)$, namely

$$\beta_i(t) = \gamma(t) \circ \omega_{\gamma(t_i)}^{-1} - \frac{1}{L(\gamma(t_i))} \int_0^{L(\gamma(t_i))} \gamma(t_i)\, ds,$$

where

$$\omega_\gamma(u) := \frac{1}{L(\gamma)} \int_0^u |\gamma'(v)|\, dv.$$

Using the isometry property we have

$$\|\nabla_{H^2(ds)} \mathcal{E}_\lambda(\beta_i(t))\|_{H^2(ds)} = \|\nabla_{H^2(ds)} \mathcal{E}_\lambda(\gamma(t))\|_{H^2(ds)}. \tag{3.7}$$

It follows that the trajectories $\beta_i(t)$ and $\gamma(t)$ have the same $H^2(ds)$-length. Let $T_i > 0$ be the maximum time such that

$$\|\beta_i(t)\|_{H^2} < \tilde{\delta} \quad \text{for all} \quad t \in [t_i, T_i).$$

Define

$$H(t) := \big(\mathcal{E}_\lambda(\gamma(t)) - \mathcal{E}_\lambda(\alpha_\infty)\big)^{1-\theta}.$$

Then $H(t)$ is positive and monotonically decreasing because $\mathcal{E}_\lambda(\alpha) = \mathcal{E}_\lambda(\gamma)$. Since the Łojasiewicz–Simon gradient inequality (3.6) holds for $\beta_i(t)$ with $t \in [t_i, T_i)$, we observe from $\mathcal{E}_\lambda(\beta_i(t)) = \mathcal{E}_\lambda(\gamma(t))$ and (3.7) that

$$\begin{aligned} -H'(t) &= -(1-\theta)\big(\mathcal{E}_\lambda(\gamma(t)) - \mathcal{E}_\lambda(\alpha_\infty)\big)^{-\theta} \frac{d\mathcal{E}_\lambda(\gamma(t))}{dt} \\ &= (1-\theta)\big(\mathcal{E}_\lambda(\gamma(t)) - \mathcal{E}_\lambda(\alpha_\infty)\big)^{-\theta} \|\nabla_{H^2(ds)} \mathcal{E}_\lambda(\gamma(t))\|_{H^2(ds)}^2 \\ &\geq (1-\theta) Z \|\nabla_{H^2(ds)} \mathcal{E}_\lambda(\gamma(t))\|_{H^2(ds)}. \end{aligned}$$

Integrating the inequality over $[t_i, T_i)$ we get

$$(1-\theta) Z \int_{t_i}^{T_i} \|\nabla_{H^2(ds)} \mathcal{E}_\lambda(\gamma(t))\|_{H^2(ds)}\, dt \leq H(t_i) - H(T_i).$$

Now if we fix a $j \in \mathbb{N}$ such that $\|\alpha_j - \alpha_\infty\|_{H^2} < \tilde{\delta}$ and let $W := \bigcup_{i \geq j} [t_i, T_i)$, we have

$$\int_W \|\nabla_{H^2(ds)} \mathcal{E}_\lambda(\gamma(t))\|_{H^2(ds)}\, dt \leq \frac{H(t_i)}{(1-\theta) Z}. \tag{3.8}$$

In fact, there exists $N \in \mathbb{N}$ such that $\|\beta_N(t) - \alpha_\infty\|_{H^2} < \tilde{\delta}$ for all $t > t_N$. If not, then for each $i \in \mathbb{N}$ there exists $T_i > 0$ such that $\beta_i(T_i)$ is on the boundary of the ball $B_{\tilde{\delta}}^{H^2}(\alpha_\infty)$, and there exists a subsequence, still denoted $\{t_i\}$, such that the intersection $\bigcap_{i\geq j}[t_i, T_i)$ is empty. By the choice of $\tilde{\delta} > 0$, Lemma 2.1 applies and there is a $C > 0$, depending only on α_∞ and r, such that

$$\begin{aligned}\tilde{\delta} = \|\beta_i(T_i) - \alpha_\infty\|_{H^2} &\leq \|\beta_i(t_i) - \alpha_\infty\|_{H^2} + \|\beta_i(t_i) - \beta_i(T_i)\|_{H^2} \\ &\leq \|\alpha(t_i) - \alpha_\infty\|_{H^2} + C\mathrm{dist}(\beta_i(t_i), \beta_i(T_i)) \\ &\leq \|\alpha(t_i) - \alpha_\infty\|_{H^2} + C\int_{t_i}^{T_i} \|\partial_t \gamma(t)\|_{H^2(ds)}\, dt,\end{aligned}$$

where we have used (3.7). However, then the integral $\int_W \|\nabla_{H^2(ds)}\mathcal{E}_\lambda(\gamma(t))\|_{H^2(ds)}\, dt$ cannot be finite, contradicting (3.8). Thus there exists $N \in \mathbb{N}$ such that $\beta_N(t) \in B^{H^2}_{\tilde{\delta}}(\alpha_\infty)$ for all $t > t_N$ and therefore

$$\int_{t_N}^{\infty} \|\partial_t \gamma(t)\|_{H^2(ds)}\, dt < \infty,$$

that is, the $H^2(ds)$-length of $\gamma(t)$ is finite. Hence it follows from Lemma 3.2 that the flow converges in the $H^2(ds)$-distance, and therefore also in H^2, where we used Lemma 2.1. □

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