Introduction to Time-fractional Differential Equations: a sketch of theory

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Abstract

This article explains a partial shape of the complete theory for time-fractional differential equations, which is still under construction. First we define a fractional derivative with the order between 0 and 1 in suitable Sobolev spaces, and show some properties on fractional calculus. Then we establish theories for initial value problems and initial boundary value problems. Finally we discuss several remarkable properties for time-fractional differential equations.

The article mainly aims at demonstrating a sketch of the total theory covering from fractional calculus to linear or nonlinear time-fractional partial differential equations, and so this article is not a survey and refers to other works for more details.

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Part I

1 Motivation

Let $\Gamma(\cdot)$ denote the gamma function and let $0 < \alpha < 1$. Then, we define the Caputo derivative

$$d_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dv}{ds}(s) ds, \quad t > 0 \tag{1.1}$$

and the Riemann-Liouville derivative

$$D_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} v(s) ds, \quad t > 0, \tag{1.2}$$

provided that the right-hand sides are defined. The Caputo derivative $d_t^\alpha v(t)$ requires $v \in W^{1,1}(0,T)$ for example because we first calculate $\frac{dv}{ds}(s)$ in some sense. On the other hand, the Riemann-Liouville derivative $D_t^\alpha v$ is well-defined for $v \in W^{1,1}(0,T)$, but it is presently not clear how much we can relax such regularity of v for defining $d_t^\alpha v$.

This suggests that we should define the domains for fractional derivatives, that is, we should clarify for which functions we are considering a fractional derivative.

The above simple consideration implies that $W^{1,1}(0,T)$ is a possible domain, but L^1-structure is not a convenient choice at least the first step towards the theory. It is natural to choose domains within L^2-framework. Thus is $H^1(0,T)$ a good space? The answer is NO: if one interprets D_t^α or d_t^α as an α-times derivative, then the space $H^1(0,T)$ of 1-time differentiable functions is too restrictive but we can intuitively assume that a natural choice is like $H^\alpha(0,T)$.

This is a motivation for our choice of the domain of fractional derivative, and in Section 2 we will describe more details.

Moreover, we show some inconsistency if we do not suitably specify the domains of the fractional derivative, For it, it is sufficient to consider a very simple initial value problem:

$$d_t^\alpha u(t) = f(t), \quad 0 < t < T, \qquad u(0) = a. \tag{1.3}$$

Let $f \in L^2(0,T)$ and $a \in \mathbb{R}$ be given. Needless to say, for $\alpha = 1$, we have

$$u(t) = \int_0^t f(s)ds + a, \quad 0 < t < T.$$

If $\alpha = 1$ and u satisfies (1.3) with given $f \in L^2(0,T)$, then $u \in H^1(0,T)$, so that $u(0)$ can be defined in the sense of the trace. In other words, for $\alpha = 1$, for any $f \in L^2(0,T)$ and $a \in \mathbb{R}$, there exists a unique $u \in H^1(0,T)$ satisfying (1.3). The situation is different from the case of $0 < \alpha < 1$. In particular, let $0 < \alpha < \frac{1}{2}$ and

$$f(t) = t^{\delta - \frac{1}{2}}, \quad 0 < t < T,$$

where $\delta > 0$ is a constant. Then $f \in L^2(0,T)$. Moreover, we can formally apply the solution formula (e.g., Kilbas, Srivastava and Trujillo [24], p.141), and obtain

$$u(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\delta - \frac{1}{2}} ds = a + C_0 t^{\alpha + \delta - \frac{1}{2}}, \tag{1.4}$$

where we set

$$C_0 = \frac{\Gamma\left(\delta + \frac{1}{2}\right)}{\Gamma\left(\alpha + \delta + \frac{1}{2}\right)}.$$

Moreover $u(t)$ given by (1.4) cannot satisfy (1.3) if $0 < \alpha < \frac{1}{2}$ and $\delta > 0$ is small such that $\alpha + \delta - \frac{1}{2} < 0$. Indeed $\lim_{t \downarrow 0} u(t) = \infty$, and so the initial condition does not make any usual sense. Furthermore we formally calculate $d_t^\alpha t^{\alpha+\delta-\frac{1}{2}}$:

$$d_t^\alpha t^{\alpha+\delta-\frac{1}{2}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{d}{ds}(s^{\alpha+\delta-\frac{1}{2}})ds$$
$$=\frac{\alpha+\delta-\frac{1}{2}}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\alpha+\delta-\frac{3}{2}}ds.$$

However, since $\alpha+\delta-\frac{3}{2} < -1$, the integral does not exist. This means that formula (1.4) does not hold for $f \in L^2(0,T)$ in general, so that the initial value problem (1.3) associated with the Caputo derivative cannot be formulated for arbitrary data $f \in L^2(0,T)$.

The function space $L^2(0,T)$ is reasonable and convenient as data space. Hence it is natural to formulate the initial value problem and define a feasible fractional derivative for $f \in L^2(0,T)$ in order to establish a unified theory for fractional differential equations. Thus we construct the theory where the fractional derivatives should be included in $L^2(0,T)$. This is our main motivation, and we construct a seemingly different fractional derivative ∂_t^α although we will prove that it is essentially same as the closure operator of the Caputo derivative in ${}_0C^1[0,T]$ (see Theorem 3.4 in Section 3).

We conclude this section with the justification concerning the Laplace transform. Needless to say, the Laplace transform is important not only in the calculus but also for study of the abstract evolution equations (see Arendt, Batty, Hieber and Neubrander [2] for $\alpha = 1$), so that it is important to consider the fractional derivative consistently with the Laplace transform. In particular, the justification of the initial value in defining the Laplace transform is indispensable in order to use the Laplace transform

$$\widehat{u}(p) := \int_0^\infty e^{-pt}u(t)dt, \quad p > p_0 : \quad \text{some constant.}$$

The following formula

$$\widehat{d_t^\alpha u}(p) = p^\alpha \widehat{u}(p) - p^{\alpha-1}u(0)$$

is quite well-known but we have to justify the sense of $u(0)$ which requires a certain smoothness of u at $t=0$. Such regularity at $t=0$ is not well established for $u \in L^2(0,T)$.

Our formulation essentially relies on the property of the generalized fractional derivative ∂_t^α defined later in some Sobolev space. As for other approach, we can refer to Zacher [47]. These properties are feasible for the applications such as the clarification of the Sobolev regularity of solutions to initial-boundary value problems.

In this article, we consider only $0 < \alpha < 1$, although we can treat $\alpha > 1$ similarly.

2 Function spaces $H_\alpha(0,T)$ as domains of fractional derivatives

For $\beta > 0$, we define the Riemann-Liouville fractional integral operator J^β by

$$J^\beta u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s)ds, \quad 0 < t < T, \quad u \in L^2(0,T). \tag{2.1}$$

We write $J := J^1$ and $J^0 u(t) = Iu(t) := u(t)$ for $u \in L^2(0,T)$.

We note that $J^\beta L^2(0,T) \subset L^2(0,T)$, and we understand that J^β is an operator from $L^2(0,T)$ to $L^2(0,T)$. In order to define an adequate fractional derivative which is denoted by ∂_t^α, we should fulfill

(1) ∂_t^α should be well-defined in a subspace of the Sobolev space of order α.

(2) the norm equivalence between $\|\partial_t^\alpha u\|$ and some conventional norm of u such as the norm in a Sobolev space.

For them,

- We will interpret J^α as the fractional power of an operator

$$Ju(t) = \int_0^t u(s)ds \quad \text{with the domain } \mathcal{D}(J) = L^2(0,T).$$

- We define ∂_t^α as the inverse to J^α.

These issues are done respectively in Sections 2 and 3. The arguments in Section 2 and a part of Section 3 are based on Gorenflo, Luchko and Yamamoto [14], Gorenflo and Yamamoto [16], Kubica, Ryszewska and Yamamoto [28], Yamamoto [46].

By $L^2(0,T)$ and $H^\alpha(0,T)$ we mean the usual L^2-space and the fractional Sobolev space on the interval $(0,T)$ (see e.g., Adams [1], Chapter VII), respectively, and we define the norm in $H^\alpha(0,T)$ by

$$\|u\|_{H^\alpha(0,T)} := \left(\|u\|^2_{L^2(0,T)} + \int_0^T \int_0^T \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\alpha}} dtds \right)^{\frac{1}{2}}.$$

The L^2-norm and the scalar product in L^2 are denoted by $\|\cdot\| = \|\cdot\|_{L^2(0,T)}$ and $(\cdot,\cdot)$, respectively. By $\sim$ we denote a norm equivalence. Since J^α is injective in $L^2(0,T)$, by $J^{-\alpha}$ we denote the algebraic inverse to J^α.

In this section, we first define the range space $J^\alpha L^2(0,T)$ of the operator J^α in $L^2(0,T)$ and second relate it to the above fractional Sobolev spaces. For defining of the fundamental function spaces, we prepare two lemmata.

Lemma 2.1

$$D_t^\alpha J^\alpha u = u \quad \text{for all } u \in L^2(0,T).$$

Proof of Lemma 2.1. By the definition of D_t^α, exchanging the orders of the integrals, we have

$$\begin{aligned} D_t^\alpha J^\alpha u(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} J^\alpha u(s)ds \\ =&\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\left(\int_0^t (t-s)^{-\alpha}\left(\int_0^s (s-\xi)^{\alpha-1}u(\xi)d\xi\right)ds\right) \\ =&\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t\left(\int_\xi^t (t-s)^{-\alpha}(s-\xi)^{\alpha-1}ds\right)u(\xi)d\xi \\ =&\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t \Gamma(\alpha)\Gamma(1-\alpha)u(\xi)d\xi = \frac{d}{dt}\int_0^t u(\xi)d\xi = u(t). \end{aligned}$$

Thus the proof of Lemma 2.1 is complete. ■

Hence we can readily prove

Lemma 2.2

$J^\alpha : L^2(0,T) \longrightarrow L^2(0,T)$ is injective: if $u \in L^2(0,T)$ satisfies $J^\alpha u = 0$ in $(0,T)$, then $u = 0$ in $(0,T)$.

Indeed, if $J^\alpha u = 0$, then $u = D_t^\alpha J^\alpha u = 0$ in $(0,T)$.

Therefore, the inverse to J^α exists and is an operator from $L^2(0,T)$ to $L^2(0,T)$. By $J^{-\alpha}$, we denote the inverse operator which is understood algebraically. Then we can define

$$\begin{cases} H_\alpha(0,T) := J^\alpha L^2(0,T), \\ \|v\|_{H_\alpha(0,T)} := \|J^{-\alpha}v\|_{L^2(0,T)}, \quad \text{that is,} \quad \|J^\alpha u\|_{H_\alpha(0,T)} := \|u\|_{L^2(0,T)}. \end{cases} \tag{2.2}$$

Postponing the concrete characterization of $H_\alpha(0,T)$, we here prove

Lemma 2.3.

$H_\alpha(0,T)$ is a Banach space with the norm $\|\cdot\|_{H_\alpha(0,T)}$.

Proof of Lemma 2.3. We can readily prove that $H_\alpha(0,T)$ is a normed space. We will prove the completeness: let $\lim_{n,m\to\infty}\|u_n - u_m\|_{H_\alpha(0,T)} = 0$. Then the definition implies $\lim_{n,m\to\infty}\|J^{-\alpha}u_n - J^{-\alpha}u_m\|_{L^2(0,T)} = 0$. Since $L^2(0,T)$ is complete, we can find $v_0 \in L^2(0,T)$ such that

$$\lim_{n\to\infty}\|J^{-\alpha}u_n - v_0\|_{L^2(0,T)} = 0.$$

The definition of the norm yields

$$\|u_n - J^\alpha v_0\|_{H_\alpha(0,T)} = \|J^\alpha(J^{-\alpha}u_n - v_0)\|_{H_\alpha(0,T)} = \|J^{-\alpha}u_n - v_0\|_{L^2(0,T)}.$$

Therefore $\lim_{n\to\infty}\|u_n - J^\alpha v_0\|_{H_\alpha(0,T)} = 0$ and $J^\alpha v_0 \in H_\alpha(0,T)$. Thus the proof of Lemma 2.3 is complete. ■

Now we are ready to state the characterization of $H_\alpha(0,T)$.

Theorem 2.1.
Let $0 < \alpha < 1$.
(i)

$$H_\alpha(0,T) := \begin{cases} \{v \in H^\alpha(0,T);\, v(0) = 0\}, & \frac{1}{2} < \alpha \le 1, \\ \left\{v \in H^{\frac{1}{2}}(0,T);\, \int_0^T \frac{|v(t)|^2}{t} dt < \infty\right\}, & \alpha = \frac{1}{2}, \\ H^\alpha(0,T), \quad 0 < \alpha < \frac{1}{2} \end{cases}$$

with the following equivalent norm

$$\|v\|_{H_\alpha(0,T)} = \begin{cases} \|v\|_{H^\alpha(0,T)}, & 0 < \alpha \le 1,\ \alpha \ne \frac{1}{2}, \\ \left(\|v\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|v(t)|^2}{t} dt\right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}. \end{cases}$$

(ii) There exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1^{-1}\|J^\alpha u\|_{H_\alpha(0,T)} \le \|u\|_{L^2(0,T)} \le C_1\|J^\alpha u\|_{H_\alpha(0,T)} \quad \text{for all } u \in L^2(0,T) \tag{2.3}$$

and

$$C_2^{-1}\|J^{-\alpha}v\|_{L^2(0,T)} \le \|v\|_{H_\alpha(0,T)} \le C_2\|J^{-\alpha}v\|_{L^2(0,T)} \quad \text{for all } v \in H_\alpha(0,T). \tag{2.4}$$

(iii) $J^{-\alpha}J^\alpha u = u$ all $u \in L^2(0,T)$ and $J^\alpha J^{-\alpha} v = v$ for all $v \in H_\alpha(0,T)$.

The proof can be found in [14], [28].

We can prove

$$J^\alpha L^2(0,T) \subset H_1(0,T), \quad \alpha \ge 1.$$

However, here we limit the range of α to $0 < \alpha \le 1$ and we omit further characterization of $J^\alpha L^2(0,T)$.

The first equality in (iii) is directly seen by the definition, while the second equality in (iii) is verified as follows. For $u \in H_\alpha(0,T)$, the definition of $H_\alpha(0,T)$ yields the existence of $w \in L^2(0,T)$ satisfying $u = J^\alpha w$. Therefore $J^\alpha J^{-\alpha}u = J^\alpha J^{-\alpha}(J^\alpha w) = J^\alpha w$. Hence $J^\alpha J^{-\alpha}u = u$ for $u \in H_\alpha(0,T)$. ■

Henceforth we write for example (2.4) by

$$\|J^{-\alpha}v\|_{L^2(0,T)} \sim \|v\|_{H_\alpha(0,T)},$$

when there is no fear of confusion.

Remark. For $H_{\frac{1}{2}}(0,T)$, Lions and Magenes [29] use a different notation ${}_0H_0^{\frac{1}{2}}(0,T)$ (Remark 11.5 (p.68) in [29]). However we here use $H_{\frac{1}{2}}(0,T)$ as well as $H_\alpha(0,T)$, $0 < \alpha < 1$.

We conclude this section with two lemmata concerning $H_\alpha(0,T)$, which may be helpful for more understanding.

We introduce the following sets:

$${}_0W^{1,1}(0,T) = \{u \in W^{1,1}(0,T);\, u(0) = 0\} \tag{2.5}$$

and a subspace of it

$$W_\alpha(0,T) := \Bigg\{u \in W^{1,1}(0,T);\ \text{there exists a constant } C_u > 0 \text{ such that}$$

$$\left|\frac{du}{dt}(t)\right| \le C_u t^{\alpha-1} \text{ almost all } t, \quad u(0) = 0\Bigg\}. \tag{2.6}$$

Here $C_u > 0$ depends on a choice of u. For example, $t^\beta \in W_\alpha(0,T)$ for $\beta \ge \alpha$. We remark that ${}_0W^{1,1}(0,T)$ is a closed set in $W^{1,1}(0,T)$, while $W_\alpha(0,T)$ is not because of the condition on $\frac{du}{dt}(t)$.

Henceforth by $C > 0$, $C_1 > 0$, etc., we denote generic constants which are independent of functions under consideration but dependent on α, T, while C_u means that it depends on a function or a quantity u under consideration.

It is not always direct to verify whether a given function belongs to $H_\alpha(0,T)$ or not, but we can prove that the space $W_\alpha(0,T)$ is a convenient subspace of $H_\alpha(0,T)$. Indeed, by means of Theorem 2.1, we can prove verify

Lemma 2.4.

(i) $W_\alpha(0,T) \subset H_\alpha(0,T)$.

(ii) $D_t^\alpha u = d_t^\alpha u$ *and* $J^\alpha D_t^\alpha u = u$ *for* $u \in {}_0W^{1,1}(0,T)$.

Proof of Lemma 2.4. Part (ii) can be proved directly by the definition of D_t^α, d_t^α and J^α. Let $u \in W_\alpha(0,T)$ be arbitrarily given. It suffices to prove that we can find $w \in L^2(0,T)$ such that $u = J^\alpha w$. Thus, by means of part (ii), we see that $D_t^\alpha u$ is a candidate of such w. By part (ii), we have

$$D_t^\alpha u = d_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{du}{ds}(s) ds.$$

From $u \in W_\alpha(0,T)$ it follows that $\left|\frac{du}{ds}(s)\right| \le Cs^{\alpha-1}$, so that

$$|D_t^\alpha u(t)| \le \frac{C}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds \le \frac{C}{\Gamma(1-\alpha)} \Gamma(1-\alpha)\Gamma(\alpha)$$

for $0 < t < T$, which implies that $D_t^\alpha u \in L^\infty(0,T) \subset L^2(0,T)$. Setting $w = D_t^\alpha u \in L^2(0,T)$, in terms of part (ii), we obtain $u = J^\alpha D_t^\alpha u = J^\alpha w$, which means that for $u \in W_\alpha(0,T)$, we can find $w \in L^2(0,T)$ such that $u = J^\alpha w \in H_\alpha(0,T)$. Thus the proof of Lemma 2.4 is complete. ■

Next, thanks to Lemma 2.4, we can prove some density property, which allows us to apply a traditional density argument for functions in $H_\alpha(0,T)$ if necessary.

We set

$${}_0C^1[0,T] = \{\varphi \in C^1[0,T];\, \varphi(0) = 0\}.$$

By Lemma 2.4, we can have the algebraic inclusions: ${}_0C^1[0,T] \subset W_\alpha(0,T) \subset H_\alpha(0,T)$. Moreover we prove

Lemma 2.5.

$$\overline{{}_0C^1[0,T]}^{H_\alpha(0,T)} = H_\alpha(0,T).$$

Here and henceforth, $\overline{Z}^X$ denotes the closure of a subset $Z \subset X$ by the norm in X. Lemma 2.5 is useful, because, thanks to the lemma, in order to prove estimates in $H_\alpha(0,T)$, it usually suffices to prove them for ${}_0C^1[0,T]$.

Remark. By Theorem 11.1 (p.55) in [29] and the mollifier, we see that

$$\overline{{}_0C^1[0,T]}^{H^\alpha(0,T)} = \begin{cases} H^\alpha(0,T), & 0 < \alpha \le \frac{1}{2}, \\ H_\alpha(0,T), & \frac{1}{2} < \alpha < 1. \end{cases}$$

We should distinguish $H_\alpha(0,T)$ from $H^\alpha(0,T)$.

3 Definition of the fractional derivatives and properties

We understand that $J^{-\alpha}$ is an operator inverse to $J^\alpha : L^2(0,T) \longrightarrow L^2(0,T)$, and by J and J^{-1} we mean J^1 and $(J^1)^{-1}$ respectively.

Now we reach

Definition 3.1. For $0 \le \alpha \le 1$, we set

$$\partial_t^\alpha u := J^{-\alpha} u, \quad u \in H_\alpha(0,T).$$

Remark. In this article, we mainly consider ∂_t^α for the case of $0 < \alpha < 1$. For $\alpha > 1$, $\notin \mathbb{N}$, on the basis of ∂_t^α with $0 < \alpha < 1$, we can define as follows: Let $\alpha = m + \gamma$ with $m \in \mathbb{N}$ and $0 < \gamma < 1$. Then

$$\partial_t^\alpha u = \partial_t^\gamma \left(\frac{d^m u}{dt^m} \right)$$

with

$$\mathcal{D}(\partial_t^\alpha) = \left\{ u \in H^\alpha(0,T);\ \frac{d^m u}{dt^m} \in H_\gamma(0,T) \right\},$$

and we can argue the isomorphism and fractional differential equations in the same way, and we can consider ∂_t^α in the dual space of $H_\alpha(0,T)$ by the Gel'fand triple $H_\alpha(0,T) \subset L^2(0,T) \subset (H_\alpha(0,T))'$. However, we omit the details in this article and we can refer to Yamamoto [46].

By Theorem 2.1, we note that $H_\alpha(0,T) = J^\alpha L^2(0,T)$. Therefore ∂_t^α in $H_\alpha(0,T)$ is well-defined and $\partial_t^\alpha u \in L^2(0,T)$ for $u \in H_\alpha(0,T)$. Moreover $\partial_t^\alpha : H_\alpha(0,T) \longrightarrow L^2(0,T)$ is surjective. Indeed let $v \in L^2(0,T)$ be arbitrarily given. By Theorem 2.1, we have $\varphi := J^\alpha v \in H_\alpha(0,T)$ and so $\partial_t^\alpha \varphi = v$ by the definition, which means that $\partial_t^\alpha : H_\alpha(0,T) \longrightarrow L^2(0,T)$ is surjective.

On the other hand, we can prove

$$J^{-1} u = \frac{du}{dt}, \quad u \in H_1(0,T).$$

Indeed, setting $v = J^{-1}u$, we have $v \in L^2(0,T)$ by Theorem 2.1 and $u = Jv$, that is, $u(t) = \int_0^t v(s)ds$. By $v \in L^2(0,T)$, we see that u is absolutely continuous on $[0,T]$, and $\frac{du}{dt}(t) = v(t)$ for almost all $t \in (0,T)$, that is, $\frac{du}{dt}(t) = (J^{-1}u)(t)$ for almost all $t \in (0,T)$. Replacing α and β respectively by $1-\alpha$ and 1 in Theorem 2.3 (ii), we obtain

$$J^{-\alpha} = J^{(1-\alpha)-1} = J^{-1}J^{1-\alpha},$$

that is, $J^{-\alpha} = \frac{d}{dt}(J^{1-\alpha})$.

Thus, summing up, we can state

Theorem 3.1.

Let $0 < \alpha < 1$. Then ∂_t^α is an isomorphism between $H_\alpha(0,T)$ and $L^2(0,T)$. That is, $\partial_t^\alpha : H_\alpha(0,T) \longrightarrow L^2(0,T)$ is injective and surjective, and

$$\|\partial_t^\alpha u\|_{L^2(0,T)} \sim \|u\|_{H_\alpha(0,T)}. \tag{3.1}$$

Moreover

$$\partial_t^\alpha u = J^{-\alpha}u = \frac{d}{dt}(J^{1-\alpha}u) = D_t^\alpha u, \quad u \in H_\alpha(0,T) \tag{3.2}$$

and

$$\partial_t^\alpha u = D_t^\alpha u = d_t^\alpha u \quad \text{for } u \in H_\alpha(0,T) \cap {}_0W^{1,1}(0,T). \tag{3.3}$$

We can calculate $\partial_t^\alpha u$ concretely by means of $D_t^\alpha u$ for $u \in H_\alpha(0,T)$. Formula $\partial_t^\alpha u = \frac{d}{dt}(J^{1-\alpha}u)$ in (3.2) can correspond to the classical inversion for finding w solving $J^\alpha w = u$ (e.g., Gorenflo and Vessella [15]) for $u \in {}_0W^{1,1}(0,T)$, but our construction for ∂_t^α guarantees the formula for $u \in H_\alpha(0,T)$, which is a wider space than the set of all absolutely continuous functions on $[0,T]$.

Next we describe the fundamental formula on the Laplace transform of fractional derivatives and some successive derivative.

Theorem 3.2.

Let $u \in H_\alpha(0,T)$ with arbitrary $T > 0$. If $(\widehat{|\partial_t^\alpha u|})(p)$ exists for $p > p_0$: some positive constant, then $\widehat{u}(p)$ exists for $p > p_0$ and

$$\widehat{\partial_t^\alpha u}(p) = p^\alpha \widehat{u}(p) \quad \text{for } p > p_0.$$

Theorem 3.3.

Let $\alpha, \beta \geq 0$. Then

$$\partial_t^\alpha(\partial_t^\beta u) = \partial_t^{\alpha+\beta} u \quad \text{for all } u \in H_{\alpha+\beta}(0,T),$$

provided that $\alpha + \beta \leq 1$.

By an adequate but natural defintion for $\partial_t^\alpha u$ for $\alpha \geq 1$, Theorem 3.3 holds without the constraint $\alpha + \beta \leq 1$. We recall that $d_t^\alpha(d_t^\beta u) = d_t^{\alpha+\beta} u$ does not hold in general even for $u \in C^1[0,T]$. The natural successive derivative formula holds thanks to our choice of the domains $\mathcal{D}(\partial_t^\alpha)$.

We conclude this section with the equivalence of ∂_t^α with the closed extension of the Caputo derivative operator d_t^α. As for the closed extension and the closure of an operator, see e.g., Kato [22] (Chapter III, §5). We recall the classical Caputo derivative

$$d_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{du}{ds}(s) ds$$

and attach the domain $\mathcal{D}(d_t^\alpha) = {}_0C^1[0,T]$. We consider d_t^α as an operator from $\mathcal{D}(d_t^\alpha) = {}_0C^1[0,T] \subset L^2(0,T)$ to $L^2(0,T)$.

By $\overline{d_t^\alpha}$ we denote the closure of d_t^α with $\mathcal{D}(d_t^\alpha) = {}_0C^1[0,T]$. Then we prove

Theorem 3.4.
We have $\mathcal{D}(\overline{d_t^\alpha}) = H_\alpha(0,T)$, *and*

$$\overline{d_t^\alpha} = \partial_t^\alpha = D_t^\alpha \quad \text{on } H_\alpha(0,T).$$

This theorem means that our definition of ∂_t^α is consistent with the classical Caputo derivative by considering the closure of the operator.

4 Important functions in $H_\alpha(0,T)$

In this section, we introduce the Mittag-Leffler functions, and we consider functions which play an important role for fractional differential equations and show that they belong to $H_\alpha(0,T)$.

For $\alpha, \beta > 0$, we define

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \tag{4.1}$$

These functions are called the Mittag-Leffler functions. It is known that $E_{\alpha,\beta}(z)$ is an entire function in z with $\alpha, \beta > 0$. The Mittag-Leffler functions have been studied well (e.g., Kilbas, Srivastava and Trujillo [24], Podlubny [36]). Henceforth we recall that $W_\alpha(0,T)$ is defined by (2.6).

Proposition 4.1.

Let $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$. Then $E_{\alpha,1}(-\lambda t^\alpha) - 1 \in W_\alpha(0,T) \subset H_\alpha(0,T)$.

Proposition 4.2.
Let $0 < \alpha < 1$ and $f \in L^2(0,T)$. Then

$$(B_\lambda f)(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds \in H_\alpha(0,T) \tag{4.2}$$

and we can find a constant $C > 0$ such that $\|B_\lambda f\|_{H_\alpha(0,T)} \le C\|f\|_{L^2(0,T)}$ for all $f \in L^2(0,T)$.

In the proposition, it is essential for later arguments that the constant $C > 0$ is independent of $\lambda > 0$.

5 Initial value problems for linear time-fractional ordinary differential equations

Now we formulate an initial value problem for a linear fractional ordinary differential equation by:

$$\begin{cases} \partial_t^\alpha (u(t) - a) = p(t)u(t) + f(t), \quad 0 < t < T, \\ u - a \in H_\alpha(0,T). \end{cases} \tag{5.1}$$

We note that if $\frac{1}{2} < \alpha < 1$, then $H_\alpha(0,T) \subset C[0,T] \cap \{v \in H^\alpha(0,T);\, v(0) = 0\}$ by Theorem 2.1 (i), and so $u - a \in H_\alpha(0,T)$ yields $u(0) = a$, which can justify the initial condtion in the pointwise sense for $\frac{1}{2} < \alpha < 1$.

Formulation (5.1) brings the well-posedness uniformly for $\alpha \in (0,1)$ and arbitrary $f \in L^2(0,T)$. More precisely,

Theorem 5.1.
Let $p \in L^\infty(0,T)$ and $a \in \mathbb{R}$ be given. Then there exists a unique solution u to (5.1). Moreover for $0 < \alpha < 1$, we can choose a constant $C > 0$ such that

$$\|u - a\|_{H_\alpha(0,T)} \le C(|a| + \|f\|_{L^2(0,T)}). \tag{5.2}$$

Moreover, since $\|u - a\|_{H^\alpha(0,T)} \le \|u - a\|_{H_\alpha(0,T)}$ and

$$\|u - a\|_{H^\alpha(0,T)} \ge \|u\|_{H^\alpha(0,T)} - \|a\|_{H^\alpha(0,T)} = \|u\|_{H^\alpha(0,T)} - \sqrt{T}|a|,$$

using (5.2) and $\|a\|_{H^\alpha(0,T)} = \|a\|_{L^2(0,T)} = \sqrt{T}|a|$, we can obtain

$$\|u\|_{H^\alpha(0,T)} \le C(|a| + \|f\|_{L^2(0,T)}). \tag{5.3}$$

Here and henceforth $C > 0$ denotes generic constants which are independent of initial values and non-homogeneous terms.

In the pointwise sense, the unique existence of solutions to initial value problems for fractional ordinary differential equations with D_t^α and ∂_t^α, has been well studied (e.g., [24], [36]), but such pointwise formulations meet difficulty in several cases such as $f \notin L^\infty(0,T)$.

Proof of Theorem 5.1.

By Theorem 2.1, we can rewrite (5.1) by

$$J^{-\alpha}(u-a) = p(t)u(t) + f(t), \quad u - a \in H_\alpha(0,T),$$

which is equivalent to

$$u(t) = a + J^\alpha(pu)(t) + (J^\alpha f)(t)$$
$$= a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad 0 < t < T. \tag{5.4}$$

By Theorem 2.1 and $p \in L^\infty(0,T)$, we see that

$$\|J^\alpha(pu)\|_{H_\alpha(0,T)} \le C\|pu\|_{L^2(0,T)} \le C\|p\|_{L^\infty(0,T)}\|u\|_{L^2(0,T)},$$

which means that $J^\alpha(pu) : L^2(0,T) \longrightarrow H_\alpha(0,T)$ is bounded and so is a compact operator from $L^2(0,T)$ to itself, because the embedding $H_\alpha(0,T) \longrightarrow L^2(0,T)$ is compact. Now we assume that $u(t) = J^\alpha(pu)(t)$ for $0 < t < T$, that is,

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)u(s)ds, \quad 0 < t < T.$$

Then

$$|u(t)| \le C\int_0^t (t-s)^{\alpha-1}|u(s)|ds, \quad 0 < t < T.$$

Applying a general Gronwall inequality (e.g., Chapter 7 in Henry [17] or Appendix of this article), we obtain $u = 0$ in $(0,T)$.

Consequently the Fredholm alternative yields that there exists a unique solution $u \in L^2(0,T)$ satisfying (5.4). Moreover, since $u - a = J^\alpha(pu + f)$ in $L^2(0,T)$, we have that $u - a \in J^\alpha L^2(0,T) = H_\alpha(0,T)$. Thus the proof of the unique existence of u is complete.

Moreover (5.4) and $p \in L^\infty(0,T)$ yield

$$|u(t)| \le |a| + C\int_0^t (t-s)^{\alpha-1}|f(s)|ds + C\int_0^t (t-s)^{\alpha-1}|u(s)|ds, \quad 0<t<T.$$

We set

$$R(t) = |a| + C\int_0^t (t-s)^{\alpha-1}|f(s)|ds.$$

Applying the general Gronwall inequality, we have

$$|u(t)| \le CR(t) + C\int_0^t (t-s)^{\alpha-1}R(s)ds \tag{5.5}$$

$$\le C\left(|a| + \int_0^t (t-s)^{\alpha-1}|f(s)|ds\right) + C\int_0^t (t-s)^{\alpha-1}\left(|a| + \int_0^s (s-\xi)^{\alpha-1}|f(\xi)|d\xi\right)ds$$

for $0 \le t \le T$. We take the norms in $L^2(0,T)$. The Young inequality on the convolution yields

$$\left\|\int_0^t (t-s)^{\alpha-1}|f(s)|ds\right\|_{L^2(0,T)} \le \|t^{-\alpha}\|_{L^1(0,T)}\|f\|_{L^2(0,T)} \le \frac{T^{1-\alpha}}{1-\alpha}|f\|_{L^2(0,T)}.$$

Moreover

$$\int_0^t \left(\int_0^s (s-\xi)^{\alpha-1}|f(\xi)|d\xi\right)ds = \int_0^t \left(\int_\xi^t (s-\xi)^{\alpha-1}ds\right)|f(\xi)|d\xi = \int_0^t \frac{(t-\xi)^\alpha}{\alpha}|f(\xi)|d\xi,$$

and so

$$\left\|\int_0^t \left(\int_0^s (s-\xi)^{\alpha-1}|f(\xi)|d\xi\right)ds\right\|_{L^2(0,T)} \le \left\|\frac{t^\alpha}{\alpha}\right\|_{L^1(0,T)}\|f\|_{L^2(0,T)}$$

again by the Young inequality. By exchanging the orders of the integrals, we can similarly obtain

$$\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-\xi)^{\alpha-1}|f(\xi)|d\xi\right)ds = \int_0^t |f(\xi)|\left(\int_\xi^t (t-s)^{\alpha-1}(s-\xi)^{\alpha-1}ds\right)d\xi$$

$$=\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}\int_0^t (t-\xi)^{2\alpha-1}|f(\xi)|d\xi,$$

and

$$\left\|\int_0^t (t-s)^{\alpha-1}\left(\int_0^s (s-\xi)^{\alpha-1}|f(\xi)|d\xi\right)ds\right\|_{L^2(0,T)} \le \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}\left\|\int_0^t (t-\xi)^{2\alpha-1}|f(\xi)|d\xi\right\|_{L^2(0,T)}$$

$$\le C\|f\|_{L^2(0,T)}.$$

Consequently (5.5) implies $\|u\|_{L^2(0,T)} \le C(|a| + \|f\|_{L^2(0,T)})$. Hence the first equation in (5.1) yields

$$\|\partial_t^\alpha(u-a)\|_{L^2(0,T)} \le C(\|u\|_{L^2(0,T)} + \|f\|_{L^2(0,T)}) \le C(|a| + \|f\|_{L^2(0,T)}).$$

Thus the proof of (5.2) is completed. ■

In contrast with (5.1), the following is a conventional formulation for an initial value problem for an ordinary fractional differential equation:

$$\begin{cases} d_t^\alpha u = p(t)u + f(t), \quad 0 < t < T, \\ u(0) = a. \end{cases}$$

As is described in Section 1, even in the case of $p \equiv 0$, this formulation is not well-defined for general $f \in L^2(0,T)$, and the initial condition $u(0) = a$ may be inconsistent, while our formulation is well-posed for all $f \in L^2(0,T)$.

Moreover we can prove the following proposition, which clarifies the relation between the above conventional formulation and (5.1).

Proposition 5.1.
We consider two formulations for initial value problems:

$$\begin{cases} d_t^\alpha u = p(t)u + f(t), \quad 0 < t < T, \\ u(0) = a \end{cases} \tag{5.6}$$

and

$$\begin{cases} \partial_t^\alpha (u - a) = p(t)u + f(t), \quad 0 < t < T, \\ u - a \in H_\alpha(0,T). \end{cases} \tag{5.7}$$

Then
(i) Let $p \in L^\infty(0,T)$ and $f \in L^2(0,T)$. If $u \in W^{1,1}(0,T)$ satisfies (5.6), then u satisfies (5.7).
(ii) Let $p \in C^1[0,T]$ and $f \in W^{1,1}(0,T)$. Then the unique solution u to (5.7) is in $W^{1,1}(0,T)$ and satisfies (5.6).

Next we give a solution formula:

Proposition 5.2.
Let $f \in L^2(0,T)$. Then the solution $u - a \in H_\alpha(0,T)$ to (5.1) with $p(t) \equiv \lambda$: constant, is given by

$$\begin{aligned} u(t) = \, & aE_{\alpha,1}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s) ds \\ = \, & aE_{\alpha,1}(-\lambda t^\alpha) + (B_\lambda f)(t), \quad 0 < t < T. \end{aligned}$$

Here B_λ is defined by (4.2).

Part II

6 Initial boundary value problems for linear time-fractional diffusion equations

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary $\partial\Omega$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\partial_j = \frac{\partial}{\partial x_j}$, $\partial_j\partial_k = \frac{\partial^2}{\partial x_j \partial x_k}$ for $1 \le j, k \le n$, and let $(u, v) = (u, v)_{L^2(\Omega)}$ be the scalar product in $L^2(\Omega)$: $(u, v) = \int_\Omega u(x)v(x)dx$.

We first deal with the following initial boundary value problem for the time-fractional diffusion equation whose elliptic part is symmetric:

$$\begin{cases} \partial_t^\alpha(u(x,t) - a(x)) = \sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k u(x,t)) + c(x)u(x,t) + F(x,t), \\ \qquad\qquad\qquad x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega\times(0,T)} = 0, \end{cases} \tag{6.1}$$

where we assume

$$\begin{cases} a_{jk} = a_{kj} \in C^1(\overline{\Omega}), \quad j,k = 1, ..., n, \quad c \in C(\overline{\Omega}), \\ \text{there exists a constant } \nu_0 > 0 \text{ such that} \\ \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \ge \nu_0 \sum_{j=1}^n \xi_i^2, \quad x \in \overline{\Omega},\ \xi_1, ..., \xi_n \in \mathbb{R}, \end{cases} \tag{6.2}$$

and

$$c(x) \le 0 \quad \text{for } x \in \overline{\Omega}. \tag{6.3}$$

We define an operator in $L^2(\Omega)$ by

$$(Av)(x) = -\sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k v(x)), \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega). \tag{6.4}$$

Then, under the condition (6.2), the operator A is positive-definite and self-adjoint in $L^2(\Omega)$. Let $0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of A, where λ_k appears in the sequence as often as its multiplicity requires. Let $\varphi_k, k \in \mathbb{N}$ be the eigenfunction of A corresponding to the eigenvalue λ_k. It is known that $\lambda_k \to \infty$ as $k \to \infty$ and the eigenfunctions φ_k can be chosen to be orthonormal, i.e., $(\varphi_j, \varphi_k) = 1$ if $j = k$ and $(\varphi_j, \varphi_k) = 0$ if $j \neq k$. These eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}}$ build an orthonormal basis of $L^2(\Omega)$. Using the standard technique (see e.g., Pazy [35], Tanabe [41]), a fractional power A^γ of the operator A can be defined for any $\gamma \in \mathbb{R}$ and the inclusion $\mathcal{D}(A^\gamma) \subset H^{2\gamma}(\Omega)$ holds true for $\gamma \geq 0$.

By the Fourier method, applying Propositions 4.1 and 4.2, we can prove the unique existence of solution (see also Sakamoto and Yamamoto [38]):

Theorem 6.1

For $F \in L^2(0,T;L^2(\Omega))$ and $a \in H_0^1(\Omega)$, there exists a unique solution u to (6.1) satisfying

$$u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \quad u - a \in H_\alpha(0,T;L^2(\Omega)), \tag{6.5}$$

and we can find a constant $C > 0$ such that

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|u-a\|_{H_\alpha(0,T;L^2(\Omega))} \leq C(\|a\|_{H_0^1(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))}) \tag{6.6}$$

for all $a \in H_0^1(\Omega)$ and $F \in L^2(0,T;L^2(\Omega))$.

Moreover,

$$u(t) := u(x,t) = \sum_{k=1}^\infty E_{\alpha,1}(-\lambda_k t^\alpha)(a,\varphi_k)\varphi_k + \sum_{k=1}^\infty (B_{\lambda_k}(f(\cdot,t),\varphi_k))(t)\varphi_k, \tag{6.7}$$

where the series is convergent in $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ and $H_\alpha(0,T;L^2(\Omega)) + \{a\}$.

Next we describe a solution formula by a different way. For the statement, we introduce operators from $L^2(\Omega)$ to itself. For a function $a \in L^2(\Omega)$, let us define the operator

$$\begin{cases} S(t)a := \sum_{k=1}^\infty E_{\alpha,1}(-\lambda_k t^\alpha)(a,\varphi_k)\varphi_k, \\ K(t)a := \sum_{k=1}^\infty t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k t^\alpha)(a,\varphi_k)\varphi_k, \quad t > 0. \end{cases} \tag{6.8}$$

Then we can prove

Theorem 6.2.

(i) For $t > 0$, the operators $S(t) : L^2(\Omega) \longrightarrow L^2(\Omega)$ and $K(t) : L^2(\Omega) \longrightarrow L^2(\Omega)$ are bounded and moreover for any $0 < \gamma < 1$, there exists a constant $C_\gamma > 0$ such that

$$\|A^\gamma S(t)a\| \leq C_\gamma t^{-\alpha\gamma}\|a\|, \qquad t > 0$$

and

$$\|A^\gamma K(t)a\| \leq C_\gamma t^{\alpha(1-\gamma)-1}\|a\|, \qquad t > 0.$$

Moreover $A^\gamma S(t)a = S(t)A^\gamma a$ *and* $A^\gamma K(t)a = K(t)A^\gamma a$ *for* $a \in \mathcal{D}(A^\gamma)$ *and* $t > 0$.
(ii) For $a \in H_0^1(\Omega)$ *and* $F \in L^2(0,T;L^2(\Omega))$, *the solution* $u(t) := u(\cdot,t)$ *to (6.1) is given by*

$$u(t) = S(t)a + \int_0^t K(t-s)F(s)ds, \quad 0 < t < T. \tag{6.9}$$

(iii) For any $a \in L^2(\Omega)$, *we can extend* $S(t) : (0,T] \longrightarrow L^2(\Omega)$ *analytically to Re* $z > 0$.

The solution formula is useful for discussing a perturbed systems of (6.1). Now we consider a more general time-fractional diffusion equation with first-order terms, where A is defined by (6.4). Note that we do not assume (6.3): $c(x) \le 0$ for $x \in \overline{\Omega}$.

We consider

$$\begin{cases} \partial_t^\alpha (u(x,t) - a(x)) = -Au(x,t) + \sum_{j=1}^n b_j(x)\frac{\partial}{\partial x_j}u(x,t) + c(x)u(x,t) + F(x,t), \\ \qquad\qquad x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega\times(0,T)} = 0, \end{cases} \tag{6.10}$$

where $b_j \in L^\infty(\Omega)$, $1 \le j \le n$ and $c \in L^\infty(\Omega)$.

Then we can prove

Theorem 6.3

For $F \in L^2(0,T;L^2(\Omega))$ *and* $a \in H_0^1(\Omega)$, *there exists a unique solution* u *to (6.10) satisfying the regularity (6.5). Moreover, the same estimate as (6.6) holds.*

Proof.

Then according to Fujiwara [11], the inequalities

$$\begin{cases} \|v\|_{H^2(\Omega)} \le C\|Av\| \quad \text{for } v \in H^2(\Omega) \cap H_0^1(\Omega), \\ \mathcal{D}(A^{\frac{1}{2}}) = H_0^1(\Omega), \quad C^{-1}\|A^{\frac{1}{2}}v\| \le \|v\|_{H^1(\Omega)} \le C\|A^{\frac{1}{2}}v\|, \quad \text{for } v \in H_0^1(\Omega) \end{cases} \tag{6.11}$$

hold true. Now we interpret the function $\sum_{j=1}^n b_j(x)\partial_j u(x,t) + c(x)u(x,t)$ as a non-homogeneous term in the equation (6.1) and apply Theorem 6.2, so that we can represent a solution $u(t) := u(\cdot,t)$ to problem (6.10) in the form

$$u(t) = \int_0^t K(t-s)\left(\sum_{j=1}^n b_j\partial_j u(s) + cu(s)\right) ds + \int_0^t K(t-s)F(s)ds, \quad t > 0. \tag{6.12}$$

First we prove the uniqueness of the solution to (6.12) within the class (6.5). Assume $F = 0$ in (6.12). Then, since $u(\cdot,t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for $t > 0$, by Theorem 6.2 (i), we obtain

$$\|A^{\frac{1}{2}}u(t)\| \le C\left\|\int_0^t A^{\frac{1}{2}}K(t-s)\left(\sum_{j=1}^n b_j\partial_j u(s) + cu(s)\right) ds\right\|$$

$$\leq C\int_0^t (t-s)^{\frac{1}{2}\alpha-1}\left(\left\|\sum_{j=1}^n b_j\partial_j u(s)\right\| + \|cu(s)\|\right) ds.$$

Therefore (6.11) yields

$$\|u(t)\|_{H^1(\Omega)} \leq C\int_0^t (t-s)^{\frac{1}{2}\alpha-1}\|u(s)\|_{H^1(\Omega)}ds, \quad t>0.$$

The generalized Gronwall inequality (see e.g., Lemma 7.1.1 (p.188) in [17]) yields $u(t) = 0$, $0<t<T$ that completes the proof of the uniqueness of solution.

Next we will prove the existence of the solution. First an operator Q from $L^2(0,T;H^2(\Omega))$ to itself is introduced by

$$Qu(t) = \int_0^t K(t-s)\left(\sum_{j=1}^n b_j\partial_j u(s) + cu(s)\right) ds, \quad 0<t<T.$$

We set $G(t) = \int_0^t K(t-s)F(s)ds$. It is sufficient to prove that the equation $u = Qu+G(t)$ has a unique solution in $L^2(0,T;H_0^1(\Omega))$. Indeed, by Theorem 2.1, we can improve the regularity of u, and show that u is the solution to (6.10) within the class (6.5). Moreover, Theorem 6.1 yields

$$\|G\|_{L^2(0,T;H^2(\Omega))} + \|G\|_{H^\alpha(0,T;L^2(\Omega))} \leq C\|F\|_{L^2(0,T;L^2(\Omega))}. \tag{6.13}$$

The estimates (6.11) and Theorem 6.2 (i) lead to the inequality

$$\|A^{\frac{1}{2}}Qu(t)\| = \left\|\int_0^t A^{\frac{1}{2}}K(t-s)\left(\sum_{j=1}^n b_j\partial_j u(s)+cu(s)\right)ds\right\|$$

$$\leq C\int_0^t (t-s)^{\frac{1}{2}\alpha-1}\|A^{\frac{1}{2}}u(s)\|ds, \quad 0<t<T. \tag{6.14}$$

Applying (6.14), we obtain the following chain of the inequalities:

$$\|A^{\frac{1}{2}}Q^2u(t)\| = \|A^{\frac{1}{2}}Q(Qu(t))\| \leq C\int_0^t (t-s)^{\frac{1}{2}\alpha-1}\|A^{\frac{1}{2}}(Qu(s))\|ds$$

$$\leq C^2\int_0^t (t-s)^{\frac{1}{2}\alpha-1}\left(\int_0^s (s-\xi)^{\frac{1}{2}\alpha-1}\|A^{\frac{1}{2}}u(\xi)\|d\xi\right) ds$$

$$= C^2\int_0^t \left(\int_\xi^t (t-s)^{\frac{1}{2}\alpha-1}(s-\xi)^{\frac{1}{2}\alpha-1}ds\right)\|A^{\frac{1}{2}}u(\xi)\|d\xi$$

$$= \frac{\left(C\Gamma\left(\frac{1}{2}\alpha\right)\right)^2}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\|A^{\frac{1}{2}}u(\xi)\|d\xi.$$

Repeating the last estimation m-times, we obtain the inequality

$$\|A^{\frac{1}{2}}Q^m u(t)\| \leq \frac{\left(C\Gamma\left(\frac{1}{2}\alpha\right)\right)^m}{\Gamma\left(\frac{1}{2}m\alpha\right)}\int_0^t (t-s)^{\frac{m}{2}\alpha-1}\|A^{\frac{1}{2}}u(s)\|ds,$$

which is valid for $0 < t < T$, $m \in \mathbb{N}$. Now we choose $m \in \mathbb{N}$ such that $\frac{m}{2}\alpha - 1 > 0$ and set $C_m = \frac{\left(C\Gamma\left(\frac{1}{2}\alpha\right)\right)^m}{\Gamma\left(\frac{1}{2}m\alpha\right)}$. Then

$$\|Q^m u(t)\|_{H^1(\Omega)} \le C_m \int_0^t \max_{0\le t\le t}(t-s)^{\frac{m}{2}\alpha-1}\|u(s)\|_{H^1(\Omega)}ds$$

$$\le T^{\frac{m}{2}\alpha-1}C_m \int_0^t \|u(s)\|_{H^1(\Omega)}ds.$$

Hence, setting $\rho_m = T^{\frac{m}{2}\alpha-1}C_m$, we reach the estimate

$$\|Q^m u(t)\|^2_{H^1(\Omega)} \le \rho_m^2 \left(\int_0^t \|u(s)\|_{H^1(\Omega)}ds\right)^2 \le \rho_m^2 T^2 \int_0^T \|u(s)\|^2_{H^1(\Omega)}ds,$$

which implies the inequality

$$\int_0^T \|Q^m u(t)\|^2_{H^1(\Omega)}dt \le \rho_m^2 T^2 \int_0^T \|u(s)\|^2_{H^1(\Omega)}ds.$$

By the asymptotic behavior of the gamma function, it is easy to verify that

$$\lim_{m\to\infty}\rho_m = T^{-1}\lim_{m\to\infty}\frac{\left(T^{\frac{\alpha}{2}}C\Gamma\left(\frac{1}{2}\alpha\right)\right)^m}{\Gamma\left(\frac{1}{2}m\alpha\right)} = 0. \tag{6.15}$$

Hence $|T\rho_m| < 1$ for large $m \in \mathbb{N}$. Now we set $\widetilde{Q}u = Qu + G$. It follows from (6.15) that $\widetilde{Q}^m$ is a contraction from $L^2(0,T;H^1(\Omega))$ to itself. Hence the mapping $\widetilde{Q}^m$ has a unique fixed point $u_* \in L^2(0,T;H^1(\Omega))$, that is, $\widetilde{Q}^m u_* = u_*$. Then $\widetilde{Q}^{m+1}u_* = \widetilde{Q}(\widetilde{Q}^m u_*) = \widetilde{Q}u_*$, that is, $\widetilde{Q}^m(\widetilde{Q}u_*) = \widetilde{Q}u_*$, which means that the point $\widetilde{Q}u_*$ is also a fixed point of the mapping $\widetilde{Q}^m$. By the uniqueness of the fixed point of $\widetilde{Q}^m$, we finally see the equality $u_* = \widetilde{Q}u_* = Qu_* + G$. Thus the equation $u = Qu + G$ has a unique solution in $L^2(0,T;H_0^1(\Omega))$ and $\|u\|_{L^2(0,T;H^1(\Omega))} \le C\|G\|_{L^2(0,T;H^1(\Omega))}$. Therefore $\left\|\sum_{j=1}^n b_j\partial_j u + cu\right\|_{L^2(0,T;L^2(\Omega))} \le C\|F\|_{L^2(0,T;L^2(\Omega))}$ and so (6.6) in Theorem 6.1 yields the estimate

$$\begin{aligned}
&\left\|Q\left(\sum_{j=1}^n b_j\partial_j u + cu\right)\right\|_{L^2(0,T;H^2(\Omega))\cap H^\alpha(0,T;L^2(\Omega))}\\
=&\left\|\int_0^t K(t-s)\left(\sum_{j=1}^n b_j\partial_j u(s) + cu(s)\right)ds\right\|_{L^2(0,T;H^2(\Omega))\cap H^\alpha(0,T;L^2(\Omega))}\\
\le& C\left\|\sum_{j=1}^n b_j\partial_j u + cu\right\|_{L^2(0,T;L^2(\Omega))} \le C\|F\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

which proves Theorem 6.3. ■

7 Properties of solutions

In this section, we discuss some important properties for time-fractional diffusion equations. Some properties are similar to the case $\alpha = 1$: the parabolic equations, and others are remarkably distinct. In Subsections 7.1 and 7.2, we describe two distinct properties, which may provide mathematical accounts for the diffusion phenomena indicating some anomalies, compared with the classical diffusion. In Section 10-4, we will again mention such properties from the phenomenalismical viewpoints. Moreover in this section, we will discuss the properties related to comparison and the positivity of the solutions, which are similar to the case $\alpha = 1$.

7.1. Backward problem in time

Let the elliptic operator A be defined by (6.4) and let condition (6.2) be assumed. Moreover we assume that $b_j, c \in C^1(\overline{\Omega})$, $j = 1, ..., n$.

We consider a backward problem in time:

$$\begin{cases} \partial_t^\alpha u(x,t) = -Au(x,t), \quad x \in \Omega,\, 0 < t < T, \\ u|_{\partial\Omega} = 0, \\ u(\cdot, T) = b \end{cases} \tag{7.1}$$

with $b \in H^2(\Omega) \cap H_0^1(\Omega)$.

In the case $\alpha = 1$, the backward problem (7.1) is not well-posed, and in particular, the mapping $b \mapsto u(\cdot, 0)$ is not continuous from $H^m(\Omega)$ to $L^2(\Omega)$ for any $m \in \mathbb{N}$. However, the case $0 < \alpha < 1$ is drastically different. That is, we can prove

Theorem 7.1 (Floridia, Li and Yamamoto [6])

For each $b \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (7.1) such that $\partial_t^\alpha u \in C((0,T]; L^2(\Omega))$. Moreover we can choose constants $C_1, C_2 > 0$ depending on T such that

$$C_1 \|u(\cdot,0)\|_{L^2(\Omega)} \le \|u(\cdot,T)\|_{H^2(\Omega)} \le C_2 \|u(\cdot,0)\|_{L^2(\Omega)}. \tag{7.2}$$

The theorem implies that the time-fractional diffusion equation improves the regularity of the initial value by exactly 2 as the Sobolev space order, which means that the time-fractional diffusion equation with $0 < \alpha < 1$, has a much weaker smoothing property than $\alpha = 1$.

To the best knowledge of the authors, Sakamoto and Yamamoto [38] is the first work for the well-posedness of the backward problem in time with extra unnecessary assumption that $c \le 0$ in Ω. As for backward problems for time-fractional equations with symmetric A, we can refer to many works: Liu and Yamamoto [30], Tuan, Huynh, Ngoc, and Zhou [42].

By the same proof, we can obtain

Corollary 7.1.
In Theorem 7.1, for each distinct $T_1, T_2 > 0$, there exist contants $C_3 = C_3(T_1, T_2) > 0$ and $C_4 = C_4(T_1, T_2) > 0$ such that

$$C_3 \|u(\cdot, T_2)\|_{H^2(\Omega)} \le \|u(\cdot, T_1)\|_{H^2(\Omega)} \le C_4 \|u(\cdot, T_2)\|_{H^2(\Omega)}.$$

The backward problem is important also for case $1 < \alpha < 2$ and see Floridia and Yamamoto [8].

7.2. Asymptotic behavior of solution for large t

We consider an initial boundary value problem (6.1) with $F = 0$ where we assume (6.2) and (6.3):

$$c(x) \le 0, \quad x \in \Omega.$$

It is well-known that we can find a constant $C_5 > 0$ such that

$$\|u(\cdot, t)\| \le C_5 e^{-\lambda_1 t} \|u(\cdot, 0)\| \quad \text{if } \alpha = 1.$$

The asymptotic behavior of solution $u(\cdot, t)$ in the case $0 < \alpha < 1$ is remarkably different:

Theorem 7.2.
There exists a constant $C_6 > 0$ such that

$$\|u(\cdot, t)\| \le C_6 t^{-\alpha} \|u(\cdot, 0)\| \tag{7.3}$$

for each solution u to (6.1).

This means a much slower decay of solutions as $t \to \infty$ for the case $0 < \alpha < 1$.

The proof can be found in Sakamaoto and Yamamoto [38], which relies directly on the representation formula (6.7) and an estimate of the Mittag-Leffler function $E_{\alpha,1}(-\lambda_k t^\alpha)$. Moreover the decay rate $t^{-\alpha}$ is the best possible in a sense (e.g., Theorem 4.3 in [38]).

We note that the same decay was proved in the case where the coefficients of A depend on $(x, t) \in \Omega \times (0, \infty)$ (Chapter 5 of Kubica, Ryszewska and Yamamoto [28]) and in the case of $Av(x) = -\sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_j v(x)) - \sum_{j=1}^n b_j(x)\partial_j v(x) - c(x)v(x)$

with $c \leq 0$ in Ω (Gölgeleyen and Yamamoto [13]).

7.3. Comparison principle for a linear time-fractional diffusion equation

In what follows, let A be defined by (6.4), and let (6.2) be satisfied, and $b_j, c \in L^\infty(\Omega)$, $1 \leq j \leq n$. We emphasize that (6.3) is not assumed. By $u(F,a)$, we denote the solution to the problem (6.10) with the initial data a and the source function F. Then

Theorem 7.3.
Let $a \in H_0^1(\Omega)$ and $F \in L^2(\Omega \times (0,T))$ satisfy $F(x,t) \geq 0$ for $(x,t) \in \Omega \times (0,T)$ and $a(x) \geq 0$ for $x \in \Omega$. Then

$$u(F,a)(x,t) \geq 0 \quad \textit{for } (x,t) \in \Omega \times (0,T).$$

Corollary 7.2.
Let $a_1, a_2 \in H_0^1(\Omega)$ and $F_1, F_2 \in L^2(\Omega \times (0,T))$ satisfy $a_1(x) \geq a_2(x)$ for $x \in \Omega$ and $F_1(x,t) \geq F_2(x,t)$ for $(x,t) \in \Omega \times (0,T)$. Then

$$u(F_1,a_1)(x,t) \geq u(F_2,a_2)(x,t) \quad \textit{for } (x,t) \in \Omega \times (0,T).$$

As for more details, see surveys Luchko and Yamamoto [31, 32, 33].

7.4. Strict positivity of a solution

In Subsection 7.3, we show the non-negativity of solution if an initial value and a non-homogeneous term are non-negative in the domains under consideration. Here we discuss the strict positivity of the solution.

We consider an initial boundary value problem (6.1), where $c = 0$ in Ω and $F = 0$ in $\Omega \times (0,T)$, and we assume condition (6.2), but not (6.3). Then

Theorem 7.4 ([31])
Let an initial value $a \in L^2(\Omega)$ satisfy $a \geq 0$ and $a \not\equiv 0$ in Ω. We assume that u satisfies the first equation in (6.1), and belongs to $C((0,T];C(\overline{\Omega}))$ as well as (6.5). Then

$$u(x,t) > 0 \quad \textit{for } x \in \Omega \textit{ and } 0 < t \leq T.$$

The proof is based on a weak Harnack inquality below stated, and for the statement we introduce notations. Let $B(x_0,r) := \{x \in \mathbb{R}^n;\ |x - x_0| < r\}$ with $x_0 \in \Omega \subset \mathbb{R}^n$ and $r > 0$, and

$$Q_-(x_0,t_0,r) := B(x_0,\delta r) \times \left(t_0,\ t_0 + \delta\tau r^{\frac{2}{\alpha}}\right),$$

$$Q_+(x_0, t_0, r) := B(x_0, \delta r) \times \left(t_0 + (2-\delta)\tau r^{\frac{2}{\alpha}},\ t_0 + 2\tau r^{\frac{2}{\alpha}}\right)$$

with $\delta \in (0,1)$, $t_0 > 0$ and $r > 0, \tau > 0$, and by $|Q_-(x_0, t_0, r)|$, we denote the measure in $\mathbb{R}^n \times \mathbb{R}$.

Then we state the weak Harnack inequality for time-fractional diffusion equation (Zacher [48]).

Theorem 7.5

We assume that u satisfies the first equation in (6.1), and $C((0,T]; C(\overline{\Omega}))$ as well as (6.5). Let $0 < \delta < 1$ and $r > 1$ be fixed. For any $t_0 > 0$, $0 < p < \frac{2+n\alpha}{2+n\alpha-2\alpha}$ and $r > 0$ satisfying $t_0 + 2\tau r^{\frac{2}{\alpha}} < T$ and $B(x_0, 2r) \subset \Omega$, we have

$$\left(\frac{1}{|Q_-(x_0,t_0,r)|}\int_{Q_-(x_0,t_0,r)} |u(x,t)|^p dxdt\right)^{\frac{1}{p}} \le C \inf_{Q_+(x_0,t_0,r)} u.$$

Here the constant $C > 0$ depends on $a_{jk}, \delta, \tau, \alpha, n, p, r$.

Part III

8 Time local existence of solutions to initial boundary value problems for semilinear time-fractional diffusion equations

The arguments are based on Luchko and Yamamoto [32], [33]. Let $n = 1, 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary $\partial\Omega$.

We introduce some notations and results needed for further discussions. We define an elliptic operator A as follows:

$$\begin{cases} Av(x) := -\sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k v(x)) - c(x)v(x), \quad x \in \Omega, \\ \mathcal{D}(A) = \{v \in H^2(\Omega);\ \partial_{\nu_A} v = 0 \quad \text{on } \partial\Omega\}. \end{cases} \tag{8.1}$$

Here we assume

$$\begin{cases} a_{jk} = a_{kj} \in C^1(\overline{\Omega}), \quad j,k = 1, ..., n, \\ \text{there exists a constant } \mu_0 > 0 \text{ such that} \\ \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \ge \mu_0 \sum_{j=1}^n \xi_j^2, \quad x \in \overline{\Omega}\, \xi_1, ..., \xi_n \in \mathbb{R}, \\ c(x) < 0 \quad \text{for all } x \in \overline{\Omega}, \end{cases} \tag{8.2}$$

and we set

$$\partial_{\nu_A} u(x) = \sum_{j,k=1}^{n} a_{jk}(x)\nu_j(x)\partial_k u(x), \quad x \in \partial\Omega,$$

where $\nu(x) = (\nu_1(x), ..., \nu_n(x))$ is the unit outward normal vector to $\partial\Omega$ at $x \in \partial\Omega$. We can similarly consider more general A and other boundary conditions such as the Robin boundary condition.

Without fear of confusion, we use the same notations as in Section 6 in spite of the different boundary condition for A. Thus let the eigenvalues of A be numbered according to their multiplicities: $0 < \lambda_1 \le \lambda_2 \le \cdots$. Note that $\lambda_k \to \infty$ as $k \to \infty$. Let φ_k be an eigenvector corresponding to the eigenvalue λ_k such that $A\varphi_k = \lambda_k\varphi_k$ and $(\varphi_j, \varphi_k) = 0$ if $j \ne k$ and $(\varphi_j, \varphi_j) = 1$. Then the system $\{\varphi_k\}_{k\in\mathbb{N}}$ of the eigenvectors forms an orthonormal basis in $L^2(\Omega)$. By the assumption $c(x) < 0$ for $x \in \overline{\Omega}$ in (8.2), we can verify that $\lambda_k > 0$ for all $k \in \mathbb{N}$. For any $\gamma \ge 0$, we can define the fractional powers A^γ of the operator A by the following relation (see, e.g., [35]):

$$A^\gamma v = \sum_{k=1}^{\infty} \lambda_k^\gamma (v, \varphi_k)\varphi_k, \quad \mathcal{D}(A^\gamma) := \left\{ v \in L^2(\Omega); \sum_{k=1}^{\infty} \lambda_k^{2\gamma}(v, \varphi_k)^2 < \infty \right\}$$

and

$$\|v\|_{\mathcal{D}(A^\gamma)} := \left(\sum_{k=1}^{\infty} \lambda_k^{2\gamma}(v, \varphi_k)^2 \right)^{\frac{1}{2}}.$$

We also mention the important inclusion $\mathcal{D}(A^\gamma) \subset H^{2\gamma}(\Omega)$. We can similarly introduce the operators $S(t)$ and $K(t)$ to (6.8).

We mainly consider an initial boundary value problem for a semilinear time-fractional diffusion equation:

$$\begin{cases} \partial_t^\alpha (u(x,t) - a(x)) = -Au(x,t) + f(x, u(x,t), \nabla u(x,t)), \quad x \in \Omega,\ 0 < t < T, \\ \partial_{\nu_A} u = 0 \quad \text{on } \partial\Omega \times (0,T). \end{cases} \tag{8.3}$$

We can understand that $f(\cdot, u(\cdot,t), \nabla u(\cdot,t))$ is a function in Ω, and is determined by $u(\cdot,t) =: u(t)$. Then we can define

$$F(u(t)) = f(\cdot, u(\cdot,t), \nabla u(\cdot,t)),$$

which means that $F(u(t))$ is a mapping from $t \in [0,T]$ to a spatial function. Then we can rewrite (8.3) as

$$\begin{cases} \partial_t^\alpha (u(t) - a) = -Au(t) + F(u(t)), \quad 0 < t < T, \\ u(t) \in \mathcal{D}(A), \quad 0 < t < T. \end{cases} \tag{8.4}$$

Now we introduce conditions posed on the semilinear term F in (8.4). We choose and fix a constant γ satisfying

$$\frac{3}{4} < \gamma \le 1.$$

We assume that we can find a constant $\rho > 0$ such that there exists a constant $C_F = C_F(\rho) > 0$ satisfying

$$\begin{cases} \text{(i)} \quad \|f(v)\| \le C_F, \quad \|f(v_1) - f(v_2)\| \le C_F\|v_1 - v_2\|_{\mathcal{D}(A^\gamma)} \\ \text{if } \|v\|_{\mathcal{D}(A^\gamma)}, \|v_1\|_{\mathcal{D}(A^\gamma)}, \|v_2\|_{\mathcal{D}(A^\gamma)} \le \rho \\ \text{(ii) there exists a constant } \varepsilon \in \left(0, \frac{3}{4}\right) \text{ such that} \\ \|f(v)\|_{H^{2\varepsilon}(\Omega)} \le C_F(\rho) \quad \text{if } \|v\|_{\mathcal{D}(A^\gamma)} \le \rho. \end{cases} \tag{8.5}$$

Henceforth, by $C > 0$, $C_0, C_1 > 0$, etc., we denote generic constants, which are independent of the functions u, v, etc. under consideration, and we write C_F, $C(\rho)$ in the case we need to specify a dependence on related quantities. We note that condition (i) is necessary for the argument on the fixed point in the space $C([0,T];\mathcal{D}(A^\gamma))$, while condition (ii) guarantees more regularity of u.

Before we state the main results of this section, let us discuss two examples of the source functions, which satisfy the condition (8.5).

Example 1. For $f \in C^1(\mathbb{R})$, by setting $F(u) := f(u(x,t))$ for $(x,t) \in \Omega \times (0,T)$, we define $F : \mathcal{D}(A^\gamma) \longrightarrow L^2(\Omega)$, $\frac{3}{4} < \gamma < 1$. Then F satisfies (8.5).

Example 2. We set

$$f(x, v(x), \nabla v(x)) := \sum_{j=1}^{n} b_j(x) v(x) \partial_j v(x), \quad x \in \Omega,$$

where $b_j \in C^1(\overline{\Omega})$, $1 \le j \le n$. Then (8.5) is satisfied. In particular, a semilinear term of this type is contained in the time-fractional Burgers equation $\partial_t^\alpha u = \partial_x^2 u - u\partial_x u$.

Now we are ready to state the local unique existence of a solution to the initial-boundary value problem (8.4).

Theorem 8.1.

(i) Let a semilinear term F satisfy the condition (i) in (8.5) with $\rho > 0$ and $\|a\|_{\mathcal{D}(A^\gamma)} \le \rho$. Then there exists a constant $T = T(\rho) > 0$ such that there exists a unique solution $u \in C([0,T];\mathcal{D}(A^\gamma))$ to

$$u(t) = S(t)a + \int_0^t K(t-s)F(u(s))ds, \quad 0 < t < T.$$

(ii) Let a semilinear term F satisfy the conditions (i) and (ii) in (8.5) with $\rho > 0$ and $\|a\|_{\mathcal{D}(A^\gamma)} \le \rho$. Then there exists a constant $T = T(\rho) > 0$ such that the initial boundary value problem (8.4) possesses a unique solution $u = u_a(x,t)$ satisfying the inclusions

$$u_a \in L^2(0,T;H^2(\Omega)) \cap C([0,T];\mathcal{D}(A^\gamma)), \quad u_a - a \in H_\alpha(0,T;L^2(\Omega)). \tag{8.6}$$

Moreover, there exists a constant $C(\rho) > 0$, such that

$$\|u_a - u_b\|_{L^2(0,T;H^2(\Omega))} \le C\|a-b\|_{\mathcal{D}(A^\gamma)}, \tag{8.7}$$

provided that $\|a\|_{\mathcal{D}(A^\gamma)}, \|b\|_{\mathcal{D}(A^\gamma)} \le \rho$.

The results and the proof are similar to the partial differential equation of parabolic type which correspond to the case $\alpha = 1$ (see, e.g., Henry [17], Pazy [35], Yagi [44]).

Proof.

For the operators $S(t)$ and $K(t)$ defined similarly to (6.8), we can prove the same properties as Theorem 6.2.

For a fixed $\gamma \in \left(\frac{3}{4}, 1\right)$ in the condition (8.5) and a fixed initial value $a \in \mathcal{D}(A^\gamma)$, we define an operator $L : L^2(0,T;L^2(\Omega)) \longrightarrow L^2(0,T;L^2(\Omega))$ by

$$(Lu)(t) := S(t)a + \int_0^t K(t-s)F(u(s))ds, \quad 0 < t < T.$$

For the constant $\rho > 0$, we set

$$V := \{v \in C([0,T];\mathcal{D}(A^\gamma));\ \|u - S(\cdot)a\|_{C([0,T];\mathcal{D}(A^\gamma))} \le \rho\}. \tag{8.8}$$

Then we prove

Lemma 8.1.

Let $H \in C([0,T];L^2(\Omega))$. Then

$$\int_0^t K(t-s)H(s)ds \in C([0,T];\mathcal{D}(A^\gamma)).$$

Proof. Let $0 < \eta < t \le T$. We have the representation

$$\begin{aligned}&\int_0^t A^\gamma K(t-s)H(s)ds - \int_0^\eta A^\gamma K(\eta - s)H(s)ds\\ =&\int_0^t A^\gamma K(s)H(t-s)ds - \int_0^\eta A^\gamma K(s)H(\eta-s)ds\end{aligned}$$

$$
\begin{aligned}
&= \int_\eta^t A^\gamma K(s)H(t-s)ds + \int_0^\eta A^\gamma K(s)(H(t-s) - H(\eta-s))ds \\
&=: I_1 + I_2.
\end{aligned}
$$

For the first integral, by Theorem 6.2 (i) and $\gamma < 1$, we have the relations

$$
\begin{aligned}
\|I_1\| \le & C \int_\eta^t s^{\alpha(1-\gamma)-1} \max_{0\le s\le t} \|H(t-s)\| ds \\
\le & C\|H\|_{C([0,T];\mathcal{D}(A^\gamma))} \frac{t^{\alpha(1-\gamma)} - \eta^{\alpha(1-\gamma)}}{\alpha(1-\gamma)} \longrightarrow 0
\end{aligned}
$$

as $\eta \uparrow t$.

Next, we obtain the following norm estimates

$$
\begin{aligned}
\|I_2\| = & \left\| \int_0^\eta A^\gamma K(s)(H(t-s) - H(\eta-s))ds \right\| \\
\le & C \int_0^\eta s^{(1-\gamma)\alpha-1} \max_{0\le\eta\le t\le T} \|H(t-s) - H(\eta-s)\| ds.
\end{aligned}
$$

For $H \in C([0,T];L^2(\Omega))$, the function

$$
|s^{(1-\gamma)\alpha-1}| \max_{0\le\eta\le t\le T} \|H(t-s) - H(\eta-s)\|
$$

is an integrable function with respect to $s \in (0,\eta)$ and

$$
\lim_{\eta\uparrow t} s^{(1-\gamma)\alpha-1} \max_{0\le\eta\le t\le T} \|H(t-s) - H(\eta-s)\| = 0
$$

for almost all $s \in (0,\eta)$. Hence, the Lebesgue convergence theorem implies the relation $\lim_{\eta\uparrow t} \|I_2\| = 0$, which implies

$$
\int_0^t A^\gamma K(t-s)H(s)ds \in C([0,T];L^2(\Omega)),
$$

that is,

$$
\int_0^t K(t-s)H(s)ds \in C([0,T];\mathcal{D}(A^\gamma)).
$$

Thus the proof of Lemma 8.1 is completed. ■

Now we proceed to the proof of Theorem 8.1. In view of Theorem 6.2 (i), the inclusion $a \in \mathcal{D}(A^\gamma)$ implies

$$
S(t)a \in C([0,T];\mathcal{D}(A^\gamma)). \tag{8.9}
$$

Indeed,

$$\|A^\gamma(S(t)a - S(s)a)\|^2 = \|S(t)(A^\gamma a) - S(s)(A^\gamma a)\|^2$$
$$= \sum_{n=1}^{\infty} |E_{\alpha,1}(-\lambda_n t^\alpha) - E_{\alpha,1}(-\lambda_n s^\alpha)|^2 |(A^\gamma a, \varphi_n)|^2.$$

Applying the Lebesgue convergence theorem and the estimate (see, e.g., Theorem 1.6 (p. 35) in [36])

$$|E_{\alpha,1}(-\lambda_n t^\alpha)| \le \frac{C}{1+\lambda_n t^\alpha} \quad \text{for all } n \in \mathbb{N} \text{ and } t > 0$$

we can verify the inclusion (8.9).

Because of the condition (8.5) and $\mathcal{D}(A^\gamma) \subset H^1(\Omega)$ for $v \in C([0,T];\mathcal{D}(A^\gamma))$, we obtain $F(v) \in C([0,T];L^2(\Omega))$. Now, applying Lemma 8.1, in view of (8.9), we reach the inclusion

$$Lv \in C([0,T];\mathcal{D}(A^\gamma)) \quad \text{for } v \in C([0,T];\mathcal{D}(A^\gamma)). \tag{8.10}$$

For the further proof, we need the following properties that are valid for a sufficiently small $T > 0$:
(i) $LV \subset V$, V being the set defined by (8.8).
(ii) There exists a constant $\sigma \in (0,1)$ such that for any $u_1, u_2 \in V$, the norm estimate

$$\|Lu_1 - Lu_2\|_{C([0,T];\mathcal{D}(A^\gamma))} \le \sigma \|u_1 - u_2\|_{C([0,T];\mathcal{D}(A^\gamma))}$$

holds true.

Proof of (i). Let $u \in V$. Then, the inclusion (8.10) implicates $Lu \in C([0,T];\mathcal{D}(A^\gamma))$. Now we consider the expression

$$A^\gamma(Lu(t) - S(t)a) = \int_0^t A^\gamma K(t-s)F(u(s))ds, \quad 0 < t < T. \tag{8.11}$$

For any $u \in V$, using the norm estimates

$$\|a\|_{\mathcal{D}(A^\gamma)} = \|A^\gamma a\| \le \rho, \quad \|u - S(\cdot)a\|_{C([0,T];\mathcal{D}(A^\gamma))} \le \rho,$$

we obtain

$$\|u(t)\|_{\mathcal{D}(A^\gamma)} \le \rho + \|A^\gamma S(t)a\| = \rho + \|S(t)A^\gamma a\| \le \rho + C_1\rho =: C_2\rho. \tag{8.12}$$

The first condition from (8.5) implies that

$$\|F(u(t))\| \le C_F(C_2\rho) \quad \text{for all } u \in V \text{ and } 0 < t < T. \tag{8.13}$$

Applying (8.13), by means of Theorem 6.2 (i), we obtain the norm estimates

$$\begin{aligned}
&\|Lu(t) - S(t)a\|_{\mathcal{D}(A^\gamma)} \\
=&\left\| \int_0^t A^\gamma K(t-s)F(u(s))ds \right\| \\
\le& C \int_0^t (t-s)^{(1-\gamma)\alpha-1} C_F(C_2\rho)ds \le C_3 \frac{t^{(1-\gamma)\alpha}}{(1-\gamma)\alpha} \le C_3 \frac{T^{(1-\gamma)\alpha}}{(1-\gamma)\alpha}.
\end{aligned}$$

The constant $C_3 > 0$ depends on $\rho > 0$ but is independent on $T > 0$. Therefore, choosing $T > 0$ sufficiently small, we complete the proof of the property (i). ■

Proof of (ii).

Estimate (8.12) yields that $\|u_1(t)\|_{\mathcal{D}(A^\gamma)} \le C_2\rho$ and $\|u_2(t)\|_{\mathcal{D}(A^\gamma)} \le C_2\rho$ for any $u_1, u_2 \in V$. The condition (8.5) leads then to the norm estimate

$$\|F(u_1(s)) - F(u_2(s))\| \le C_F(C_2\rho)\|u_1(s) - u_2(s)\|_{\mathcal{D}(A^\gamma)}, \quad 0 < s < T.$$

Hence, we have the following chain of estimates:

$$\begin{aligned}
&\|Lu_1(t) - Lu_2(t)\|_{\mathcal{D}(A^\gamma)} = \left\| \int_0^t A^\gamma K(t-s)(F(u_1(s)) - F(u_2(s)))ds \right\| \\
\le& C_F(C_2\rho) \int_0^t (t-s)^{\alpha(1-\gamma)-1} \|(u_1 - u_2)(s)\|_{\mathcal{D}(A^\gamma)} ds \\
\le& C_4 T^{\alpha(1-\gamma)} \sup_{0<s<T} \|u_1(s) - u_2(s)\|_{\mathcal{D}(A^\gamma)}.
\end{aligned}$$

In the last inequality, the constant $C_4 > 0$ is independent of T, and thus we can choose a sufficiently small constant $T > 0$ satisfying the inequality

$$\rho := C_4 T^{\alpha(1-\gamma)} < 1.$$

The proof of the property (ii) is completed. ■

Due to the properties (i) and (ii), the contraction theorem can be applied to the equation $u = Lu$. As a result, this equation has a unique solution $u \in V$ for $0 < t < T$. This solution $u \in C([0,T]; \mathcal{D}(A^\gamma))$ satisfies the estimate (8.12) and the equation

$$u(t) = S(t)a + \int_0^t K(t-s)F(u(s))ds, \quad 0 < t < T. \tag{8.14}$$

This proves part (i) of Theorem 8.1.

Next, for the solution u of the equation $u = Lu$, we prove the inclusions (8.6).

In the condition (8.5), we can choose a sufficiently small $\varepsilon > 0$ such that $0 < \varepsilon < \frac{1}{2}$. By the equation (8.14), we obtain

$$Au(t) = A^{1-\gamma}S(t)A^{\gamma}a + \int_0^t A^{1-\varepsilon}K(t-s)A^{\varepsilon}F(u(s))ds, \quad 0 < t < T.$$

Furthermore, by the second condition in (8.5), the inequality (8.12) yields the estimate $\|A^{\varepsilon}F(u(s))\| \le C_F(C_2\rho)$. Thus, we obtain the chain of the inequalities

$$\begin{aligned}\|Au(t)\| &\le Ct^{-\alpha(1-\gamma)}\|A^{\gamma}a\| + C\int_0^t (t-s)^{\alpha\varepsilon-1}C_F(C_2\rho)ds \\ &\le Ct^{-\alpha(1-\gamma)}\|A^{\gamma}a\| + C_F(C_2\rho), \quad 0 < t < T.\end{aligned}$$

For $0 < \alpha < 1$, the inequality $-\alpha(1-\gamma) > -1$ and $t^{-\alpha(1-\gamma)} \in L^1(0,T)$ hold true. Application of the generalized Gronwall inequality yields

$$\begin{aligned}&\|Au(t)\| \le (Ct^{-\alpha(1-\gamma)}\|A^{\gamma}a\| + C_F(C_2\rho)) \\ &+C\int_0^t (t-s)^{\alpha\varepsilon-1}(s^{-\alpha(1-\gamma)}\|A^{\gamma}a\| + C_F(C_2\rho))ds \\ &\le Ct^{-\alpha(1-\gamma)}\|A^{\gamma}a\| + C_F(C_2\rho) + (\|A^{\gamma}a\| + C_F(C_2\rho))t^{\alpha(\varepsilon-(1-\gamma))}, \quad 0 < t < T.\end{aligned}$$

Therefore, noting that $-\alpha(1-\gamma) < \alpha(\varepsilon-(1-\gamma))$, we have the norm estimate

$$\|A_0u(t)\| \le C_5(1+T^{\alpha\varepsilon})(t^{-\alpha(1-\gamma)}+1), \quad 0 < t < T,$$

where $C_5 > 0$ depends on $\|A^{\gamma}a\|$ and the constants C_F, C_2, ρ, α, and ε. For $\frac{1}{2} < \gamma \le 1$, we can directly verify that $-2\alpha(1-\gamma) > -1$, so that $\int_0^T \|Au(t)\|^2dt < \infty$, that is, the inclusion

$$u \in L^2(0,T;H^2(\Omega)) \tag{8.15}$$

holds true.

It remains to prove that $u - a \in H_{\alpha}(0,T;L^2(\Omega))$. The inequality (8.13) implies the inclusion $F(u) \in L^2(0,T;L^2(\Omega))$.

Then we apply (6.6) and obtain the inclusion

$$\int_0^t K(t-s)F(u(s))ds \in H_{\alpha}(0,T;L^2(\Omega)).$$

Here we note that we have the same estimate as (6.6) for A with the zero Neumann boundary condition $\partial_{\nu_A}u = 0$. Applying (6.6) to the equation (8.14), we reach the inclusion $u - a \in H_{\alpha}(0,T;L^2(\Omega))$, which completes the proof of (8.6) from the theorem.

Finally, we have to prove the estimate (8.7). By the construction of the solutions u_a, u_b as the fixed points of the equation $u = Lu$, we have

$$\|u_a\|_{L^2(0,T;H^2(\Omega))} \le C(\rho), \quad \|u_b\|_{L^2(0,T;H^2(\Omega))} \le C(\rho). \tag{8.16}$$

On the other hand,

$$u_a(t) - u_b(t) = S(t)a - S(t)b + \int_0^t K(t-s)(F(u_a(s)) - F(u_b(s)))ds, \quad 0 < t < T.$$

In view of (8.16), we can use the condition (8.5) and apply the generalized Gronwall inequality. Further details of the derivations are similar to the ones employed in the proof of Theorem 1 from [32], [33], and we omit them here. Thus, the proof of Theorem 8.1 is completed. ■

9 Comparison principle and blow-up for semilinear time-fractional diffusion equations

In this section, we assume that the spatial dimension n is $1, 2, 3$. We fix $\gamma > 0$ such that $\frac{3}{4} < \gamma < 1$.

9.1 Comparison principle

We consider an initial boundary value problem (8.4) with A defined by (8.1). For simplicity, we assume (8.2), although we can relax the conditions. In what follows, we suppose that the semilinear terms $f(x, u(x,t))$ depend only on the spatial variable x and a function u, but not on $\nabla u(x,t)$. Moreover, we introduce a class of semilinear terms F via smooth functions from the space $C^1(\overline{\Omega} \times [-\rho, \rho])$. That is, for a function $f \in C^1(\overline{\Omega} \times [-\rho, \rho])$, in terms of $\mathcal{D}(A^\gamma) \subset C(\overline{\Omega})$ by $\frac{3}{4} < \gamma < 1$ and $n = 1, 2, 3$, we can define a mapping $F : \{v \in \mathcal{D}(A^\gamma) : \|v\|_{\mathcal{D}(A^\gamma)} \le \rho\} \longrightarrow L^2(\Omega)$ by

$$F(v) := f(x, v(x)), \quad x \in \Omega, \, 0 < t < T. \tag{9.1}$$

For a fixed constant $M > 0$, we set

$$\mathcal{F}_M := \{f \in C^1(\overline{\Omega} \times [-\rho, \rho]); \, \|f\|_{C^1(\overline{\Omega}\times[-\rho,\rho])} \le M\}. \tag{9.2}$$

Now we are ready to formulate a comparison principle for initial boundary value problems (8.4) for a semilinear time-fractional diffusion equation.

Theorem 9.1 (Luchko and Yamamoto [33]).
For $f_1, f_2 \in \mathcal{F}_M$ and $a_1, a_2 \in \mathcal{D}(A^\gamma)$, we assume that there exist solutions $u(f_k, a_k)$, $k = 1, 2$ to the initial boundary value problem (8.4) with the semilinear terms f_k, $k = 1, 2$ and the initial values a_k, $k = 1, 2$, respectively, which satisfy (8.6) and

$$|u(f_k, a_k)(x,t)| \le \rho, \quad x \in \Omega,\ 0 < t < T,\ k = 1, 2.$$

If $f_1(\cdot,\cdot) \ge f_2(\cdot,\cdot)$ on $\Omega \times (-\rho, \rho)$ and $a_1(\cdot) \ge a_2(\cdot)$ in Ω, then

$$u(f_1, a_1)(x,t) \ge u(f_2, a_2)(x,t) \quad \text{in } \Omega \times (0, T).$$

9.2 Blow-up

The comparison principle provides upper and lower estimates for the solution to (8.4). Upper estimates can be used to guarantee the global existence in time of solutions to (8.4), which means that we can choose arbitrary $T > 0$ in Theorem 8.1, and as for examples for such estimates, we can refer to [33]. We recall that we assume $c(x) < 0$, $x \in \overline{\Omega}$ in (8.2), and so all the eigenvalues λ_k, $k \in \mathbb{N}$ of A are positive.

Same as the semilinear parabolic equations, of course we cannot always prove the global existence of solution. As such an issue, we here consider the blow-up. The study of this topic has just started and we state only one result by Huang, Liu and Yamamoto [19], which treats more general cases.

In (8.4), we consider f which does not depend on x, that is, $F(v) := f(v(x))$, $x \in \Omega$. We assume

$$f \in C^1[0,\infty), \quad f \ge 0 \quad \text{in } [0,\infty), \quad f \text{ is convex}, \tag{9.3}$$

and

$$a \in \mathcal{D}(A), \quad a \ge 0, \quad \not\equiv 0 \quad \text{in } \Omega. \tag{9.4}$$

Then, similarly to Theorem 8.1, for each $a \in \mathcal{D}(A)$, we can find $T_a > 0$ such that there exists a unique solution

$$u \in C([0, T_a]; \mathcal{D}(A)) \quad \text{satisfying } u - a \in H_\alpha(0, T_a; L^2(\Omega)), \tag{9.5}$$

and Theorem 9.1 yields

$$u(x,t) \ge 0, \quad x \in \Omega,\ 0 < t < T_a.$$

Now we are mainly concerned with the non-existence of global solution in time to (8.4) within the class (9.5).

In addition to (9.3) and (9.4), we further assume that

$$\text{there exist constants } c_0 > 0 \text{ and } p > 1$$
$$\text{such that } f(\xi) \ge c_0 \xi^p \text{ for all } \xi \ge 1. \tag{9.6}$$

We set

$$T_a := \sup\{t > 0;\ \|u(\,\cdot\,, t)\|_{L^1(\Omega)} < \infty\} \tag{9.7}$$

and we call $T_a > 0$ the blow-up time if $T_a < \infty$. If $T_a < \infty$, then by the definition we see that

$$\limsup_{t \uparrow T_a} \|u(\,\cdot\,, t)\|_{L^1(\Omega)} = \infty.$$

We recall that $\lambda_1 > 0$ is the minimum eigenvalue of A. It is known that the corresponding eigenfunction does not change the sign. Hence we can choose $\varphi_1(x)$ satisfying

$$A\varphi_1 = \lambda_1 \varphi_1, \quad \varphi_1 > 0 \text{ in } \Omega, \quad \int_\Omega \varphi_1(x)\, dx = 1. \tag{9.8}$$

Now we are ready to state our main result.

Theorem 9.2 ([19])

For initial value $a(x)$ satifying (9.4) and the constants $c_0 > 0$ and $p > 1$ defined in (9.6), we further assume

$$a_0 := \int_\Omega a(x)\varphi_1(x) dx > \left(\frac{\lambda_1}{c_0}\right)^{\frac{1}{p-1}}.$$

Then $T_a < \infty$ and

$$T_a \le T_a^* := \left\{(p-1)\Gamma(2-\alpha)(c_0 a_0^{p-1} - \lambda_1)\right\}^{-\frac{1}{\alpha}}.$$

Theorem 9.2 generalizes the result for the case $\alpha = 1$ (i.e., the classical parabolic equation) which is found for example in Theorem 17.1 (p.104) in Quittner and Souplet [37]. In the case of $\alpha = 1$, concerning the non-existence of global solutions in time, there have been enormous works since Fujita [10], and we can refer to a comprehensive monograph by Quittner and Souplet [37]. See also Fujishima and Ishige [9], Ishige and Yagisita [20].

There are very rapidly increasing interests on nonlinear time-fractional differential equations, and so here we refer to only a few works: Borikhanov, Ruzhansky and Torebek [4], Floridia, Liu and Yamamoto [7], Ghergu, Miyamoto and Suzuki [12], Hnaien, Kellil, and Lassoued [18], Kian and Yamamoto [23], Kirane, Laskri and Tatar [25], Kojima [27], Suzuki [39, 40], Vergara and Zacher [43], Zhang and Sun [49].

The proof of Theorem 9.2 is based on the comparison of solutions to initial value problems for time-fractional ordinary differential equations. Such a method can date back to Kaplan [21] for $\alpha = 1$, and see also Payne [34].

10 Concluding remarks

1. Here we sketch a theory for the forward problem for time-fractional diffusion equation whose order α in time is between 0 and 1. The case $\alpha \in (1, 2)$ is quite important from the physical viewpoint, and here we omit that case. There are many works on the numerical analysis for time-fractional differential equations, but in this article we do not discuss them.
2. We should have many topics to be clarified. For example, the non-homogeneous boundary value problems should be studied and we refer only to Yamamoto [45] and the references therein.
3. Here we do not touch inverse problems for time-fractional differential equations, but explain mainly the forward problem, that is, only the initial boundary value problems. For the forward problem, mostly the properties are similar to the case of $\alpha = 1$. However, many results on inverse problems for the case $0 < \alpha < 1$ are drastically different from $\alpha = 1$. Researches on inverse problems are very rapidly developing and it is difficult to provide updated references and here we refer only to three surveys [26] as of the year 2019.
4. We do not explain the physical backgrounds for the time-fractional diffusion equations. The time-fractional diffusion equation is a model equation for diffusion in heterogeneous media such as soil. Real field data often indicate longstanding stay of the diffusive substance near a source and lower averaging effects of the density in x and t, and data are deviated from simulation results by means of the classical diffusion equation with $\alpha = 1$, in other words, the classical diffusion equations may not be an adequate model equation, and so one should consider several alternative models. The time-fractional diffusion equation is one possible model.

Here, in view of the fundamental solutions, we compare the character of the time-fractional diffusion equation with diffusion process on a fractal. We can understand that the fractal is NOT heterogeneous media and we can expect that the diffusion model on a fractal is quite different from the character of the time-fractional diffusion. For comparison, we consider the behavior near $t = 0$ of the fundamental solutions $G(x, y, t)$, which can be interpreted as the particle density at (x, t) if $G(x, y, 0) = \delta(y - x)$: the Dirac delta function, which describes that the particles concentrate at one point at the initial time $t = 0$.

- For the classical diffusion equation in $\mathbb{R}^n$:

  $$\partial_t u(x,t) = \Delta u(x,t), \quad x \in \mathbb{R}^n,\ t > 0,$$

  we have

  $$G(x,y,t) \sim C_1 t^{-\frac{n}{2}} \exp\left(-\frac{C_2|x-y|^2}{t}\right)$$

  for small $t > 0$. Here $C_1 > 0$ and $C_2 > 0$ are constants. Since $G(x, y, t)$ has no singularity for $t > 0$, we can understand that with the fundamnetal solution, particles concentrating on the point y at $t = 0$ immediately diffuses, which means no long stay of particles near a source.

- For diffusion on the Sierpinski gasket, we have

  $$G(x,y,t) \sim C_3 t^{-\frac{d_s}{2}} \exp\left(-C_4\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

  for small $t > 0$ (Barlow and Perkins [3]), where $C_3 > 0$ and $C_4 > 0$ are constants, and d_s and d_w are positive constants, and especially $d_s = \frac{2\log 3}{\log 5}$ is called the spectral dimensions of the Sierpinsky gasket. The fundamental solution is characterized with different parameters d_s, d_w but the character is still similar to the case of the classical diffusion.

- For $\partial_t^\alpha u = \Delta u$, we have

  $$G(x,y,t) \sim C_5|x-y|^{-\frac{n(1-\alpha)}{2-\alpha}} t^{-\frac{\alpha n}{2(2-\alpha)}} \exp\left(-C_6\left(\frac{|x-y|^{\frac{2}{\alpha}}}{t}\right)^{\frac{\alpha}{2-\alpha}}\right)$$

  for small $t > 0$ (Eidelman and Kochubei [5]). Here the costants $C_5 > 0$ and $C_6 > 0$ are given by only n. In the case $\alpha = 1$, this coincides with the fundamental solution for the classical diffusion equation. However for $0 < \alpha < 1$, the fundamental solution contains a factor $|x-y|^{-\frac{n(1-\alpha)}{2-\alpha}}$ which is singular in $x - y$. By this singular factor, the fundamental solution can explain that the particles stay longer near a source point.

These examples demonstrates that the time-fractional diffusion keeps a quite different character even from a diffusion in a fractal.

11 Appendix

Lemma 11. 1 (generalized Gronwall inequality)
Let $C_0 > 0$ be a constant and $0 < \alpha < 1$. Moreover let $r \in L^1(0,T)$, ≥ 0 in $(0,T)$. We assume that $u \in L^1(0,T)$ satisfies

$$0 \leq u(t) \leq r(t) + C_0 \int_0^t (t-s)^{\alpha-1} u(s)ds, \quad 0 \leq t \leq T.$$

Then

$$u(t) \leq r(t) + C_1 e^{C_2 t} \int_0^t (t-s)^{\alpha-1} r(s)ds, \quad 0 \leq t \leq T. \tag{11.1}$$

Here the constants $C_1 > 0$ and $C_2 > 0$ are dependent on α, C_0, but independent of $T > 0$. We note that if $C_0 > 0$ is independent of $T > 0$, then (11.1) holds for $t > 0$ with $C_1, C_2 > 0$ which are independent of $t > 0$.

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