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RESEARCH ARTICLE

Local newforms for generic representations of unramified even unitary groups I: Even conductor case

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Abstract
In this paper, we define compact open subgroups of quasi-split even unitary groups for each even non-negative integer and establish the theory of local newforms for irreducible tempered generic representations with a certain condition on the central characters. To do this, we use the local Gan–Gross–Prasad conjecture, the local Rankin–Selberg integrals and the local theta correspondence.

Contents

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1. Introduction

In the 1970s, Atkin–Lehner [1] and Li [17] introduced the notion of *newforms* for elliptic modular forms and showed the multiplicity one theorem. Together with their results, Casselman's theory of *local newforms* [5] is a bridge between modular forms and automorphic representations of $\mathrm{GL}_2/\mathbb{Q}$. Since then, the theory of local newforms was developed for several groups. For example, for low rank cases, Roberts–Schmidt [22] and Lansky–Raghuram [16] established this theory for GSp_4 and $\mathrm{U}(1,1)$, respectively. Casselman's result was extended to GL_n by Jacquet–Piatetski-Shapiro–Shalika [13] (see also [12]) and by Atobe–Kondo–Yasuda [2]. For other general rank cases,

- Tsai [23] studied the local newforms of generic supercuspidal representations of SO_{2n+1}; and
- the author together with Oi and Yasuda [3] treated the case for unramified U_{2n+1}.

In this paper, for a bridge to hermitian modular forms, we try to establish the theory of local newforms for $\mathrm{U}(n,n)$.

Let us describe our results. Let E/F be an unramified quadratic extension of non-archimedean local fields of characteristic 0 and of residue characteristic $p > 2$. Fix a nontrivial additive character ψ of F such that $\psi|_{\mathfrak{o}_F} = \mathbf{1}$ but $\psi|_{\mathfrak{p}_F^{-1}} \neq \mathbf{1}$, and set $\psi_E(x) = \psi(\frac{x+\overline{x}}{2})$ for $x \in E$. Consider a quasi-split unitary group of $2n$ variables given by

$$\mathrm{U}_{2n} = \left\{ g \in \mathrm{GL}_{2n}(E) \,\middle|\, {}^t\overline{g}\begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix} \right\}$$

with

$$w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(E).$$

We denote by $W = E^{2n}$ the vector space where U_{2n} acts. The center of U_{2n} is identified with $E^1 = \{x \in E^\times \mid N_{E/F}(x) = 1\}$. Define a compact subgroup K_{2m}^W of U_{2n} by $K_0^W = \mathrm{U}_{2n} \cap \mathrm{GL}_{2n}(\mathfrak{o}_E)$, and by

$$K_{2m}^W = \begin{array}{c} \\ 1 \\ 2n-2 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2n-2 & 1 \end{array} \\ \begin{pmatrix} 1+\mathfrak{p}_E^m & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^m & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^{2m} & \mathfrak{p}_E^m & 1+\mathfrak{p}_E^m \end{pmatrix} \end{array} \cap \mathrm{U}_{2n}$$

for $2m > 0$. For an irreducible smooth representation π of U_{2n}, we denote by π_ψ the maximal quotient of π on which the subgroup

$$Z = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & \mathbf{1}_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}_{2n} \,\middle|\, z \in F \right\} \cong F$$

acts by ψ. This is a local analogue of the Fourier–Jacobi expansions of hermitian modular forms and is called the *Fourier–Jacobi module* of π. We write $\pi_\psi^{K_{2m}^W}$ for the image of the subspace $\pi^{K_{2m}^W}$ consisting of K_{2m}^W-fixed vectors via the canonical surjection $\pi \twoheadrightarrow \pi_\psi$.

The main theorem is stated as follows. For other notations, in particular for the notion of ψ_E-generic, see Section 2 below.

Theorem 1.1 (Theorem 2.2). *Let π be an irreducible tempered representation of U_{2n} with the L-parameter ϕ_π and the central character ω_π. We denote by $c(\phi_\pi)$ the conductor of ϕ_π.*

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(1) *If π is not ψ_E-generic, then $\pi_\psi^{K_{2m}^W} = 0$ for any $2m \geq 0$. Conversely, if π is ψ_E-generic, then there exists $2m \geq 0$ such that $\pi_\psi^{K_{2m}^W} \neq 0$.*
(2) *Suppose that π is ψ_E-generic. If $2m < c(\phi_\pi)$, then $\pi_\psi^{K_{2m}^W} = 0$. If $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi)+1$, then*

$$\dim_{\mathbb{C}}(\pi_\psi^{K_{2m}^W}) \leq 1.$$

(3) *Set $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi) + 1$. Suppose that π is ψ_E-generic and that ω_π is trivial on $E^1 \cap (1 + \mathfrak{p}_E^m)$. Then $\pi_\psi^{K_{2m}^W} \neq 0$.*

If $2m = c(\phi_\pi)$ and if ω_π is trivial on $E^1 \cap (1 + \mathfrak{p}_E^m)$, we shall call an element in $\pi^{K_{2m}^W}$ whose image in π_ψ is nonzero a *local newform* of π.

Remark 1.2.

(1) If $\pi^{K_{2m}^W} \neq 0$, then ω_π is trivial on $E^1 \cap (1 + \mathfrak{p}_E^m)$ since $E^1 \cap (1 + \mathfrak{p}_E^m) \subset K_{2m}^W$.
(2) Even if $2m = c(\phi_\pi)$ or $c(\phi_\pi) + 1$, the dimension of $\pi^{K_{2m}^W}$ can be greater than 1. A counterexample already appears in the case where $n = 1$, which was treated by Lansky and Raghuram. See [16, Theorem 4.2.1].
(3) As well as in [3], one might expect the existence of K_m^W for all integers $m \geq 0$ such that Theorem 1.1 holds. Unfortunately, we do not know how to define K_m^W for odd integers $m > 0$ at this moment.

We expect that Theorem 1.1 has several applications such as a higher level generalization of a result of Chenevier–Renard [7]. We will try it as a next project.

A usual method to establish the theory of local newforms is to apply the *Rankin–Selberg integrals*, which are based on the multiplicity one theorem for several *Gan–Gross–Prasad (GGP) pairs*. For example, Tsai [23] and Cheng [8] used the pairs $(\mathrm{SO}_{2n+1}(F), \mathrm{SO}_{2n}(F))$ and $(\mathrm{U}_{2n+1}, \mathrm{U}_{2n})$ to obtain knowledge about newforms. In this paper, we will also use this method as well. However, in our case, one needs the GGP pair $(\mathrm{U}_{2n}, \mathrm{U}_{2n-2})$, which is not a 'basic' case. More precisely, we have to consider the restrictions of irreducible representations of U_{2n} to the *Jacobi group*. Since the Jacobi group is not reductive, several arguments in [23] would not work.

For example, to prove an analogue of Theorem 1.1 (1) in [23], Tsai used a lemma of Moy–Prasad ([23, Lemma 3.4.1]). We do not know whether this lemma can be extended to our case. Instead of this lemma, we use the local period integrals for the refined GGP conjecture. Using the absolutely convergence of these integrals, the argument of Gan–Savin [11, Lemma 12.5] can show Theorem 1.1 (1). See Section 3.2 below. This is the same idea as in the previous paper [3, Theorem 4.5].

The proof of Theorem 1.1 (2) is the same as usual. Namely, it is an application of the Rankin–Selberg integrals for $\mathrm{U}_{2n} \times \mathrm{GL}_{n-1}(E)$. This theory in this case was established by Ben-Artzi–Soudry [4] and Morimoto [21], and is recalled in Theorem 4.2. Especially, the multiplicativity of the gamma factors is included in [21, Theorem 3.1]. Using the Rankin–Selberg integrals, we will define certain formal power series. Lemma 4.4 is a key computation to give lower bounds of the degrees. Using the functional equations of the Rankin–Selberg integrals, we would obtain an upper bound of the dimension of $\pi_\psi^{K_{2m}^W}$. However, since the Rankin–Selberg integrals for $\mathrm{U}_{2n} \times \mathrm{GL}_{n-1}(E)$ factors through $\pi \twoheadrightarrow \pi_\psi$, we cannot estimate the dimension of $\pi^{K_{2m}^W}$ itself.

For the proof of Theorem 1.1 (3), the fact that we have to deal with the Jacobi group complicates the situation. Indeed, the arguments in [23, Chapter 8] and in the previous paper [3, Theorem 4.3] might not work. In this paper, we give a new, or rather old, idea.

Recall that the theory of newforms was initiated by Atkin–Lehner [1] and Li [17] for elliptic modular forms of integral weights. Kohnen [14] established a similar theory to the half-integral weights case. Moreover, he proved that the newforms of integral weights and the ones of half-integral weights are related to each other by the *Shimura correspondence*. Since the *theta correspondence* is a generalization

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of the Shimura correspondence, the local newforms will be compatible with the local theta correspondence in the future. Instead, the local theta correspondence would be useful to show the existence of the local newforms. This is our idea.

In fact, if we let $\sigma = \theta_\psi(\pi)$ be the theta lift of π to U_{2n+1}, then σ is nonzero irreducible tempered and generic, and its conductor and central character are the same as the ones of π. By the definition of the theta lifting, we have a surjective $\mathrm{U}_{2n+1} \times \mathrm{U}_{2n}$-equivariant map

$$\omega_\psi \longrightarrow \sigma \boxtimes \pi,$$

where ω_ψ is the Weil representation of $\mathrm{U}_{2n+1} \times \mathrm{U}_{2n}$. Let K_{2m}^V be a conjugate of the compact subgroup of U_{2n+1} defined in [3], where $V = E^{2n+1}$ is the vector space on which U_{2n+1} acts. Set J_{2m}^V to be the subgroup of U_{2n+1} generated by K_{2m}^V and the central subgroup $E^1 \cap (1+\mathfrak{p}_E^m)$. Then by using a lattice model and Waldspurger's result (Proposition 5.3), one can show that $\omega_\psi^{J_{2m}^V}$ is generated by $\omega_\psi^{J_{2m}^V \times K_{2m}^W}$ as a representation of U_{2n}. Hence, if $2m \geq c(\phi_\pi)$ and $\omega_\pi|_{E^1 \cap (1+\mathfrak{p}_E^m)} = \mathbf{1}$, then $\pi^{K_{2m}^W} \neq 0$ since $\sigma^{J_{2m}^V} \neq 0$. See Proposition 5.6 for the details.

However, it is much harder to show $\pi_\psi^{K_{2m}^W} \neq 0$ when $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi)+1$. Let $l_\sigma : \sigma \to \mathbb{C}$ be a nonzero Whittaker functional. Then the composition

$$\omega_\psi \longrightarrow \sigma \boxtimes \pi \xrightarrow{l_\sigma \otimes \mathrm{id}} \pi$$

factors through a twisted Jacquet module of ω_ψ along a maximal unipotent subgroup of U_{2n+1}. By the same argument as Mao–Rallis [18, Proposition 2.3], this twisted Jacquet module is isomorphic to the compact induction $\mathrm{ind}_{N'_{2n}}^{\mathrm{U}_{2n}}(\mu)$, where N'_{2n} is a maximal unipotent subgroup of U_{2n} and μ is a generic character of N'_{2n}. By Cheng's result [8, Theorem 1.4, Lemma 7.5], l_σ is nonzero on the one-dimensional subspace $\sigma^{J_{2m}^V}$ if l_σ is suitably chosen. Hence, there is $\phi \in \omega_\psi^{J_{2m}^V \times K_{2m}^W}$ such that it is nonzero under the all maps in the following diagram:

$$\begin{array}{ccccc} \omega_\psi & \longrightarrow & \sigma \boxtimes \pi & \xrightarrow{l_\sigma \otimes \mathrm{id}} & \pi \\ \downarrow & & & \nearrow & \\ \mathrm{ind}_{N'_{2n}}^{\mathrm{U}_{2n}}(\mu) & & & & \end{array}$$

Lemma 5.7 asserts that the support of the image of ϕ in $\mathrm{ind}_{N'_{2n}}^{\mathrm{U}_{2n}}(\mu)$ is small enough. It implies that $\pi_\psi^{K_{2m}^W} \neq 0$ immediately. See Section 5.5 for the details. Finally, to prove Lemma 5.7, we need to change models of the Weil representation and review the argument of Mao–Rallis [18, Proposition 2.3].

This paper is organized as follows. In Section 2, we introduce several notations and state our main theorem. Using the local Fourier–Jacobi periods, we show Theorem 1.1 (1) in Section 3. Theorem 1.1 (2) is obtained as an application of the Rankin–Selberg integrals in Section 4. Finally, we study theta liftings to prove Theorem 1.1 (3) in Section 5.

Notation

Let E/F be an unramified quadratic extension of non-archimedean local fields of characteristic 0 and of residue characteristic $p > 2$. The nontrivial element in $\mathrm{Gal}(E/F)$ is denoted by $x \mapsto \overline{x}$. Set $\mathfrak{o}_E$ (resp. $\mathfrak{o}_F$) to be the ring of integers of E (resp. F), and $\mathfrak{p}_E$ (resp. $\mathfrak{p}_F$) to be its maximal ideal. Let $E^1 = \{x \in E^\times \mid x\overline{x} = 1\}$ denote the kernel of the norm map $N_{E/F} : E^\times \to F^\times$. Fix a uniformizer ϖ of F, which is also a uniformizer of E. When $x \in E^\times$ can be written as $x = u\varpi^l$ for some $u \in \mathfrak{o}_E^\times$, we write

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$\mathrm{ord}(x) = l$. Set $q = |\mathfrak{o}_F/\mathfrak{p}_F|$ so that $q^2 = |\mathfrak{o}_E/\mathfrak{p}_E|$. Let $|\cdot|_E$ be the normalized absolute value of E so that $|x|_E = q^{-2\mathrm{ord}(x)}$ for $x \in E^\times$.

We fix $\delta \in \mathfrak{o}_E^\times$ such that $\overline{\delta} = -\delta$, and a nontrivial additive character $\psi\colon F \longrightarrow \mathbb{C}^\times$ such that $\psi|_{\mathfrak{o}_F} = \mathbf{1}$ but $\psi|_{\mathfrak{p}_F^{-1}} \neq \mathbf{1}$. Set $\psi_E(x) = \psi(\frac{1}{2}\mathrm{tr}_{E/F}(x)) = \psi(\frac{x+\overline{x}}{2})$ and $\psi_E^\delta(x) = \psi_E(x/\delta)$. Then ψ_E and ψ_E^δ are nontrivial additive characters of E such that $\psi_E|_F = \psi$ and $\psi_E^\delta|_F = \mathbf{1}$. The unique nontrivial quadratic unramified character of $E^\times$ is denoted by χ. Namely, $\chi|_{\mathfrak{o}_E^\times} = \mathbf{1}$ and $\chi(\varpi) = -1$. In particular, if we write $\chi = |\cdot|_E^{s_0}$, we have $q^{-2s_0} = -1$.

A representation π of a p-adic group G means a smooth representation over a complex vector space. When K is a compact open subgroup of G, we write π^K for the subspace of π consisting of K-fixed vectors. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible representations of G, and $\mathrm{Irr}_{\mathrm{temp}}(G)$ be its subset consisting of tempered representations.

2. Statement of the main theorem

In this section, we define families of compact open subgroups of unitary groups, and we state our main theorem.

2.1. Unitary groups

Let $V = V_{2n+1}$ (resp. $W = W_{2n}$) be a hermitian (resp. skew-hermitian) space over E of dimension $2n+1$ (resp. $2n$) equipped with a nondegenerate hermitian form $\langle \cdot,\cdot \rangle_V$ (resp. skew-hermitian form $\langle \cdot,\cdot \rangle_W$). Assume that there are bases $\{e_n, \dots, e_1, e_0, e_{-1}, \dots, e_{-n}\}$ of V and $\{f_n, \dots, f_1, f_{-1}, \dots, f_{-n}\}$ of W, respectively, such that

$$\left\langle e_i, e_j \right\rangle_V = \left\langle f_i, f_j \right\rangle_W = 0$$

unless $j = -i$, and

$$\langle e_0, e_0 \rangle_V = \langle e_i, e_{-i} \rangle_V = \langle f_i, f_{-i} \rangle_W = 1$$

for $1 \leq i \leq n$.

Using these bases, we often identify the associated unitary groups $\mathrm{U}(V)$ and $\mathrm{U}(W)$ with

$$\mathrm{U}_{2n+1} = \left\{ h \in \mathrm{GL}_{2n+1}(E) \,\middle|\, {}^t\overline{h} w_{2n+1} h = w_{2n+1} \right\},$$
$$\mathrm{U}_{2n} = \left\{ g \in \mathrm{GL}_{2n}(E) \,\middle|\, {}^t\overline{g} J_{2n} g = J_{2n} \right\},$$

respectively, where we set

$$w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(E), \quad J_{2n} = \begin{pmatrix} 0 & w_n \\ -w_n & 0 \end{pmatrix} \in \mathrm{GL}_{2n}(E).$$

2.2. Representations of unitary groups

Let N_{2n+1} (resp. N_{2n}) be the upper triangular unipotent subgroup of U_{2n+1} (resp. U_{2n}). We define generic characters of N_{2n+1} and N_{2n} by the same formula

$$u \mapsto \psi_E\left(\sum_{k=1}^{n} u_{k,k+1} \right).$$

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By abuse of notation, we denote these characters by ψ_E. We say that an irreducible representation σ of U_{2n+1} (resp. π of U_{2n}) is *generic* (resp. ψ_E*-generic*) if $\mathrm{Hom}_{N_{2n+1}}(\sigma, \psi_E) \neq 0$ (resp. $\mathrm{Hom}_{N_{2n}}(\pi, \psi_E) \neq 0$).

For an irreducible representation π of U_{2n}, we denote by $\pi^\vee$ the contragredient representation of π. By a result in [19, Chapter 4. II. 1], we know $\pi^\vee \cong \pi^\theta$, where $\pi^\theta(g) = \pi(\theta(g))$ with

$$\theta\colon \mathrm{U}_{2n} \to \mathrm{U}_{2n},\ g \mapsto \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix} \overline{g} \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}^{-1}.$$

In particular, π is ψ_E-generic if and only if $\pi^\vee$ is ψ_E^{-1}-generic.

By the local Langlands correspondence established by Mok [20], to an irreducible representation σ of U_{2n+1} (resp. π of U_{2n}), one can attach a conjugate self-dual representation ϕ_σ (resp. ϕ_π) of $W_E \times \mathrm{SL}_2(\mathbb{C})$ of dimension $2n+1$ (resp. $2n$), where W_E is the Weil group of E. We call ϕ_σ (resp. ϕ_π) the *L-parameter* for σ (resp. π). Then we define the *conductor* $c(\phi_\sigma)$ of ϕ_σ by the non-negative integer satisfying

$$\varepsilon(s, \phi_\sigma, \psi_E) = \varepsilon(0, \phi_\sigma, \psi_E) q^{-2c(\phi_\sigma)s}.$$

Similarly, the *conductor* $c(\phi_\pi)$ of ϕ_π is defined.

The center of U_{2n+1} (resp. U_{2n}) is U_1 which is identified with E^1. For an irreducible representation σ (resp. π) of U_{2n+1} (resp. U_{2n}), we denote its central character by ω_σ (resp. ω_π). If σ (resp. π) corresponds to ϕ_σ (resp. ϕ_π), then the L-parameter of ω_σ (resp. ω_π) is given by $\det(\phi_\sigma)$ (resp. $\det(\phi_\pi)$).

2.3. Jacobi group

Set

$$\mathbf{v}(x, y; z) = \begin{pmatrix} 1 & x & y & z + \frac{1}{2}(x w_{n-1} {}^t\overline{y} - y w_{n-1} {}^t\overline{x}) \\ 0 & \mathbf{1}_{n-1} & 0 & w_{n-1} {}^t\overline{y} \\ 0 & 0 & \mathbf{1}_{n-1} & -w_{n-1} {}^t\overline{x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}_{2n}$$

for $x, y \in E^{n-1}$ and $z \in F$. Here, E^{n-1} is the space of row vectors. Let $H_{n-1} = \{\mathbf{v}(x, y; z) \mid x, y \in E^{n-1}, z \in F\} \cong E^{2n-2} \oplus F$ be a Heisenberg group in $4n-3$ variables over F with the multiplication law

$$\mathbf{v}(x, y; z)\mathbf{v}(x', y'; z') = \mathbf{v}\left(x + x', y + y'; z + z' + \frac{1}{2}\mathrm{tr}_{E/F}(x w_{n-1} {}^t\overline{y} - y w_{n-1} {}^t\overline{x})\right).$$

We write

$$\begin{aligned} X_{n-1} &= \{\mathbf{v}(x, 0; 0) \mid x \in E^{n-1}\}, \\ Y_{n-1} &= \{\mathbf{v}(0, y; 0) \mid y \in E^{n-1}\}, \\ Z &= \{\mathbf{v}(0, 0; z) \mid z \in F\}. \end{aligned}$$

By abuse of notation, we denote the character $Z \ni \mathbf{v}(0, 0; z) \mapsto \psi(z)$ by ψ.

We identify U_{2n-2} as a subgroup of U_{2n} by the inclusion

$$\mathrm{U}_{2n-2} \ni g' \mapsto \begin{pmatrix} 1 & & \\ & g' & \\ & & 1 \end{pmatrix} \in \mathrm{U}_{2n}.$$

Then U_{2n-2} normalizes H_{n-1}. We call $J_{n-1} = H_{n-1} \rtimes \mathrm{U}_{2n-2}$ the *Jacobi group*. Note that Z is the center of J_{n-1}.

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For an irreducible representation π of U_{2n}, we denote by π_ψ the maximal quotient of π on which Z acts by ψ. We call π_ψ the *Fourier–Jacobi module* of π. For a compact open subgroup K of U_{2n}, we denote by π_ψ^K the image of π^K via the canonical surjection $\pi \twoheadrightarrow \pi_\psi$. Note that π_ψ is a smooth representation of J_{n-1} so that K does not act on π_ψ itself.

For $t \in E^\times$, if we put $\psi'(x) = \psi(N_{E/F}(t)x)$ and

$$K' = \begin{pmatrix} t & & \\ & \mathbf{1}_{2n-2} & \\ & & \bar{t}^{-1} \end{pmatrix}^{-1} K \begin{pmatrix} t & & \\ & \mathbf{1}_{2n-2} & \\ & & \bar{t}^{-1} \end{pmatrix},$$

then $\pi(\mathrm{diag}(t, \mathbf{1}_{2n-2}, \bar{t}^{-1}))$ induces isomorphisms

$$\pi^{K'} \xrightarrow{\sim} \pi^K, \quad \pi_{\psi'} \xrightarrow{\sim} \pi_\psi.$$

Hence, we have $\pi_{\psi'}^{K'} \cong \pi_\psi^K$.

2.4. Compact subgroups

For each non-negative even integer $2m \geq 0$, we define compact subgroups $K_{2m}^V \subset \mathrm{U}(V) \cong \mathrm{U}_{2n+1}$ and $K_{2m}^W \subset \mathrm{U}(W) \cong \mathrm{U}_{2n}$ as follows. When $2m = 0$, we set $K_0^V = \mathrm{U}_{2n+1} \cap \mathrm{GL}_{2n+1}(\mathfrak{o}_E)$ and $K_0^W = \mathrm{U}_{2n} \cap \mathrm{GL}_{2n}(\mathfrak{o}_E)$. If $2m > 0$, we set

$$K_{2m}^V = \begin{matrix} n \\ 1 \\ n \end{matrix} \overset{\begin{matrix} n & 1 & n \end{matrix}}{\begin{pmatrix} \mathfrak{o}_E & \mathfrak{p}_E^m & \mathfrak{o}_E \\ \mathfrak{p}_E^m & 1+\mathfrak{p}_E^{2m} & \mathfrak{p}_E^m \\ \mathfrak{o}_E & \mathfrak{p}_E^m & \mathfrak{o}_E \end{pmatrix}} \cap \mathrm{U}_{2n+1},$$

$$K_{2m}^W = \begin{matrix} 1 \\ 2n-2 \\ 1 \end{matrix} \overset{\begin{matrix} 1 & 2n-2 & 1 \end{matrix}}{\begin{pmatrix} 1+\mathfrak{p}_E^m & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^m & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^{2m} & \mathfrak{p}_E^m & 1+\mathfrak{p}_E^m \end{pmatrix}} \cap \mathrm{U}_{2n}.$$

Note that

$$\begin{pmatrix} \varpi^{-m}\cdot\mathbf{1}_n & & \\ & 1 & \\ & & \varpi^m\cdot\mathbf{1}_n \end{pmatrix} K_{2m}^V \begin{pmatrix} \varpi^{-m}\cdot\mathbf{1}_n & & \\ & 1 & \\ & & \varpi^m\cdot\mathbf{1}_n \end{pmatrix}^{-1}$$

$$= \begin{matrix} n \\ 1 \\ n \end{matrix} \overset{\begin{matrix} n & 1 & n \end{matrix}}{\begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-2m} \\ \mathfrak{p}_E^{2m} & 1+\mathfrak{p}_E^{2m} & \mathfrak{o}_E \\ \mathfrak{p}_E^{2m} & \mathfrak{p}_E^{2m} & \mathfrak{o}_E \end{pmatrix}} \cap \mathrm{U}_{2n+1},$$

which is denoted by $\mathbb{K}_{2m,\mathrm{U}(V)}$ in [3], and by $K_{n,2m}$ in [8]. If we set ${}^tK_{2m}^W = \{{}^tk \mid k \in K_{2m}^W\}$ to be the transpose of K_{2m}^W, then

$$K_{2m}^W = \begin{pmatrix} \varpi^{-m} & & \\ & \mathbf{1}_{2n-2} & \\ & & \varpi^m \end{pmatrix} {}^tK_{2m}^W \begin{pmatrix} \varpi^{-m} & & \\ & \mathbf{1}_{2n-2} & \\ & & \varpi^m \end{pmatrix}^{-1}.$$

The theory of local newforms for U_{2n+1} is established by the author together with Oi and Yasuda [3, Theorem 1.1] and by Cheng [8, Theorem 1.2] as follows.

Theorem 2.1. *Let σ be an irreducible tempered representation of U_{2n+1} with the L-parameter ϕ_σ.*

(1) *If σ is not generic, then $\sigma^{K_{2m}^V} = 0$ for any $2m \geq 0$.*
(2) *If σ is generic, then*

$$\dim_{\mathbb{C}}(\sigma^{K_{2m}^V}) = \begin{cases} 0 & \text{if } 2m < c(\phi_\sigma), \\ 1 & \text{if } 2m = c(\phi_\sigma) \text{ or } c(\phi_\sigma)+1. \end{cases}$$

Moreover, if $2m > c(\phi_\sigma)$, then $\sigma^{K_{2m}^V} \neq 0$.

In this paper, we will prove an analogue of this theorem for U_{2n} as follows.

Theorem 2.2. *Let π be an irreducible tempered representation of U_{2n} with the L-parameter ϕ_π and the central character ω_π.*

(1) *If π is not ψ_E-generic, then $\pi_\psi^{K_{2m}^W} = 0$ for any $2m \geq 0$. Conversely, if π is ψ_E-generic, then there exists $2m \geq 0$ such that $\pi_\psi^{K_{2m}^W} \neq 0$.*
(2) *Suppose that π is ψ_E-generic. If $2m < c(\phi_\pi)$, then $\pi_\psi^{K_{2m}^W} = 0$. If $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi)+1$, then*

$$\dim_{\mathbb{C}}(\pi_\psi^{K_{2m}^W}) \leq 1.$$

(3) *Set $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi)+1$. Suppose that π is ψ_E-generic and that ω_π is trivial on $E^1 \cap (1+\mathfrak{p}_E^m)$. Then $\pi_\psi^{K_{2m}^W} \neq 0$.*

When $2m = c(\phi_\pi)$, we shall call an element in $\pi^{K_{2m}^W}$ whose image in π_ψ is nonzero a *local newform* of π.

3. Local Fourier–Jacobi periods

In this section, we will prove Theorem 2.2 (1). To do this, we use the local Gan–Gross–Prasad conjecture for $(\mathrm{U}_{2n}, \mathrm{U}_{2n-2})$.

3.1. Weil representation

Let W_0 be the subspace of W generated by $\{f_{n-1}, \dots, f_1, f_{-1}, \dots, f_{-n+1}\}$. We write $G_n = \mathrm{U}(W)$ and $G_{n-1} = \mathrm{U}(W_0)$ in this section. Hence, the Jacobi group J_{n-1} is written as $J_{n-1} = H_{n-1} \rtimes G_{n-1}$.

Recall that we have a compact subgroup K_{2m}^W of $G_n = \mathrm{U}(W)$. Note that the intersections

$$K^J = K_{2m}^W \cap J_{n-1}, \quad K^H = K_{2m}^W \cap H_{n-1}, \quad K^{W_0} = K_{2m}^W \cap \mathrm{U}(W_0)$$

are independent of $2m$. Moreover, K^{W_0} is a hyperspecial maximal compact subgroup of $G_{n-1} = \mathrm{U}(W_0)$.

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We consider the Weil representation ω_ψ of J_{n-1} associated to ψ and χ. It is realized on the Schwartz space $\mathcal{S}(E^{n-1})$ as follows. For $\phi \in \mathcal{S}(E^{n-1})$ and $\xi \in E^{n-1}$,

$$\begin{aligned}
&\omega_\psi(\mathbf{v}(x,0;0))\phi(\xi) = \phi(\xi + x), \quad x \in E^{n-1},\\
&\omega_\psi(\mathbf{v}(0,y;0))\phi(\xi) = \psi_E(2\xi w_{n-1}{}^t\overline{y})\phi(\xi), \quad y \in E^{n-1},\\
&\omega_\psi(\mathbf{v}(0,0;z))\phi(\xi) = \psi(z)\phi(\xi), \quad z \in F,\\
&\omega_\psi(\mathbf{m}(a))\phi(\xi) = \chi(\det(a))|\det(a)|^{\frac{1}{2}}\phi(\xi a), \quad a \in \mathrm{GL}_{n-1}(E),\\
&\omega_\psi(\mathbf{n}(b))\phi(\xi) = \psi_E\left(\overline{\xi b} w_{n-1}{}^t\xi\right)\phi(\xi), \quad b \in \mathrm{M}_{n-1}(E), {}^t(w_{n-1}\overline{b}) = w_{n-1}b,\\
&\omega_\psi(J_{2n-2})\phi(\xi) = \int_{E^{n-1}} \phi(x)\psi_E(2\overline{x}\cdot {}^t\xi)dx,
\end{aligned}$$

where we set

$$\mathbf{m}(a) = \begin{pmatrix} a & 0 \\ 0 & w_{n-1}{}^t\overline{a}^{-1}w_{n-1}^{-1}\end{pmatrix}, \quad \mathbf{n}(b) = \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \in G_{n-1},$$

and the measure dx on E^{n-1} is the self-dual Haar measure with respect to ψ_E. The Weil representation ω_ψ is unitary with respect to the pairing

$$(\phi_1, \phi_2) = \int_{E^{n-1}} \phi_1(\xi)\overline{\phi_2(\xi)}d\xi.$$

Set $\phi_0 \in \mathcal{S}(E^{n-1})$ to be the characteristic function on $\mathfrak{o}_E^{n-1}$. Note that ϕ_0 is fixed by $\omega_\psi(K^J)$. Moreover, the subspace $\omega_\psi^{K^H}$ is one-dimensional spanned by ϕ_0.

3.2. Proof of Theorem 2.2 (1)

Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G_n)$ and $\pi' \in \mathrm{Irr}_{\mathrm{temp}}(G_{n-1})$. Fix a nonzero G_n-invariant (resp. G_{n-1}-invariant) bilinear pairing $(\cdot,\cdot)_\pi : \pi \times \pi^\vee \to \mathbb{C}$ (resp. $(\cdot,\cdot)_{\pi'} : \pi' \times \pi'^\vee \to \mathbb{C}$). For $\varphi \in \pi, \varphi^\vee \in \pi^\vee, \varphi' \in \pi', \varphi'^\vee \in \pi'^\vee$ and $\phi, \phi^\vee \in \mathcal{S}(E^{n-1})$, we define the *local Fourier–Jacobi period* by

$$\begin{aligned}
&\alpha(\varphi, \varphi^\vee, \varphi', \varphi'^\vee, \phi, \phi^\vee)\\
&= \int_{G_{n-1}}\int_{H_{n-1}} (\pi(hg)\varphi, \varphi^\vee)_\pi(\pi'(g)\varphi', \varphi'^\vee)_{\pi'}\overline{(\omega_\psi(hg)\phi, \phi^\vee)}dhdg.
\end{aligned}$$

Proposition 3.1. *The integral $\alpha(\varphi, \varphi^\vee, \varphi', \varphi'^\vee, \phi, \phi^\vee)$ is absolutely convergent.*

Proof. This is exactly the same as the symplectic-metaplectic case ([25, Proposition 2.2.1]). We omit the details. □

Since the central character of ω_ψ is ψ, if $\alpha(\varphi, \varphi^\vee, \varphi', \varphi'^\vee, \phi, \phi^\vee) \neq 0$, then

$$\int_F (\pi(hg\cdot \mathbf{v}(0,0;z))\varphi, \varphi^\vee)_\pi \overline{\psi(z)}dz \neq 0$$

for some $h \in H_{n-1}$ and $g \in G_{n-1}$. This means that the image of φ in π_ψ is nonzero. The converse holds in the following sense.

Lemma 3.2. *Let $\varphi \in \pi$. Assume that the image of φ in π_ψ is nonzero. Then there exists $\varphi^\vee \in \pi^\vee$ such that*

$$\int_F (\pi(\mathbf{v}(0,0;z))\varphi, \varphi^\vee)_\pi \overline{\psi(z)}dz \neq 0.$$

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Proof. Note that by Proposition 3.1, the integral

$$\int_F (\pi(\mathbf{v}(0,0;z))\varphi, \varphi^\vee)_\pi \overline{\psi(z)} dz$$

converges absolutely. Suppose that this integral is equal to zero for all $\varphi^\vee \in \pi^\vee$. We will show that the image of φ in π_ψ is zero.

For an integer $j > 0$, set

$$T_j = \left\{ t(1+a) = \begin{pmatrix} 1+a & & \\ & \mathbf{1}_{2n-2} & \\ & & (1+a)^{-1} \end{pmatrix} \middle| a \in \mathfrak{p}_F^j \right\} \subset \mathrm{U}_{2n}.$$

Recall that $\psi|_{\mathfrak{o}_F} = \mathbf{1}$ but $\psi|_{\mathfrak{p}_F^{-1}} \neq \mathbf{1}$. Hence, for fixed $z \in F$ with $-k = \mathrm{ord}(z)$, the map

$$T_j \ni t(1+a) \mapsto \frac{\psi((1+a)^2 z)}{\psi(z)} \in \mathbb{C}^\times$$

is a character if $k \leq 2j$. Moreover, it is trivial if $k \leq j$. Hence,

$$q^{-j} \int_{\mathfrak{p}_F^j} \psi((1+a)^2 z) da = \begin{cases} \psi(z) & \text{if } k \leq j, \\ 0 & \text{if } j < k \leq 2j. \end{cases}$$

Since π is smooth, there is an integer $j > 0$ such that φ is T_j-fixed. To show that the image of φ in π_ψ is zero, it suffices to prove that

$$\int_{\mathfrak{p}_F^{-j}} \pi(\mathbf{v}(0,0;z)\varphi) \overline{\psi(z)} dz = 0.$$

This is equivalent to saying that

$$\int_{\mathfrak{p}_F^{-j}} (\pi(\mathbf{v}(0,0;z))\varphi, \varphi^\vee)_\pi \overline{\psi(z)} dz = 0 \tag{$\dagger$}$$

for all $\varphi^\vee \in \pi^\vee$. We claim that we may assume that $\varphi^\vee$ is T_j-fixed. Indeed, if $z \in \mathfrak{p}_F^{-j}$, since $k = -\mathrm{ord}(z) \leq j$, we have

$$\begin{aligned} \int_{\mathfrak{p}_F^{-j}} (\pi(\mathbf{v}(0,0;z))\varphi, \varphi^\vee)_\pi \overline{\psi(z)} dz &= \int_{\mathfrak{p}_F^{-j}} (\pi(\mathbf{v}(0,0;z))\varphi, \varphi^\vee)_\pi \left(q^{-j} \int_{\mathfrak{p}_F^j} \overline{\psi((1+a)^2 z)} da \right) dz \\ &= q^{-j} \int_{\mathfrak{p}_F^{-j}} \int_{\mathfrak{p}_F^j} (\pi(t(1+a)^{-1}\mathbf{v}(0,0;z)t(1+a))\varphi, \varphi^\vee)_\pi \overline{\psi(z)} da dz \\ &= q^{-j} \int_{\mathfrak{p}_F^{-j}} \int_{\mathfrak{p}_F^j} (\pi(\mathbf{v}(0,0;z))\varphi, \pi^\vee(t(1+a))\varphi^\vee)_\pi \overline{\psi(z)} da dz. \end{aligned}$$

Hence, ($\dagger$) holds for $\varphi^\vee$ if it holds for

$$q^{-j} \int_{\mathfrak{p}_F^j} \pi^\vee(t(1+a))\varphi^\vee da$$

which is T_j-fixed.

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Now assume that $\varphi^\vee$ is T_j-fixed. Then we claim that

$$\int_{\mathfrak{p}_F^{-j}} (\pi(\mathbf{v}(0,0;z))\varphi,\varphi^\vee)_\pi \overline{\psi(z)}dz = \int_F (\pi(\mathbf{v}(0,0;z))\varphi,\varphi^\vee)_\pi \overline{\psi(z)}dz,$$

and hence, the left-hand side is zero by assumption. Indeed, for $k > j > 0$, since $k \geq 2$ so that $k-1 < k \leq 2(k-1)$, we have

$$\begin{aligned}
&\int_{\mathfrak{p}_F^{-k}\backslash\mathfrak{p}_F^{-k+1}} (\pi(\mathbf{v}(0,0;z))\varphi,\varphi^\vee)_\pi \overline{\psi(z)}dz \\
&= q^{-k+1}\int_{\mathfrak{p}_F^{-k}\backslash\mathfrak{p}_F^{-k+1}}\int_{\mathfrak{p}_F^{k-1}} (\pi(t(1+a)^{-1}\mathbf{v}(0,0;z)t(1+a))\varphi,\varphi^\vee)_\pi \overline{\psi(z)}dadz \\
&= \int_{\mathfrak{p}_F^{-k}\backslash\mathfrak{p}_F^{-k+1}} (\pi(\mathbf{v}(0,0;z))\varphi,\varphi^\vee)_\pi \left(q^{-k+1}\int_{\mathfrak{p}_F^{k-1}} \overline{\psi((1+a)^2z)}da\right)dz \\
&= 0.
\end{aligned}$$

This completes the proof of the lemma. □

Now, we prove Theorem 2.2 (1).

Proof of Theorem 2.2 (1). Let π be an irreducible tempered representation of $G_n = \mathrm{U}_{2n}$. Suppose that $\pi_\psi^{K_{2m}^W} \neq 0$ for some $2m \geq 0$. We will show that π must be ψ_E-generic.

Fix $\varphi \in \pi^{K_{2m}^W}$ such that the image of π_ψ is nonzero. By Lemma 3.2, one can find $\varphi^\vee \in \pi^\vee$ such that

$$\int_F (\pi(\mathbf{v}(0,0;z))\varphi,\varphi^\vee)_\pi \overline{\psi(z)}dz \neq 0.$$

Since Z is the center of H_{n-1}, we may assume that $\varphi^\vee$ is fixed by K^H. Hence, the matrix coefficient $H_{n-1} \ni h \mapsto (\pi(h)\varphi,\varphi^\vee)_\pi$ is bi-K^H-invariant. Since ω_ψ is the unique irreducible representation of H_{n-1} whose central character is ψ, there are $\phi,\phi^\vee \in \mathcal{S}(E^{n-1})$ such that

$$\int_{H_{n-1}} (\pi(h)\varphi,\varphi^\vee)_\pi \overline{(\omega_\psi(h)\phi,\phi^\vee)}dh \neq 0.$$

We may also assume that both ϕ and $\phi^\vee$ are fixed by K^H. Since $\omega_\psi^{K^H} = \mathbb{C}\phi_0$, we can take $\phi = \phi^\vee = \phi_0$. Hence,

$$\int_{H_{n-1}} (\pi(h)\varphi,\varphi^\vee)_\pi \overline{(\omega_\psi(h)\phi_0,\phi_0)}dh \neq 0.$$

Now by applying the same argument as [11, Lemma 12.5] to the integral on G_{n-1}, one can find $\pi' \in \mathrm{Irr}_{\mathrm{temp}}(G_{n-1})$ and $(\varphi',\varphi'^\vee) \in \pi' \times \pi'^\vee$ such that

$$\alpha(\varphi,\varphi^\vee,\varphi',\varphi'^\vee,\phi_0,\phi_0) \neq 0.$$

We may assume that φ' is fixed by K^{W_0} since so are φ and ϕ_0. This means that π' is unramified. By the local Gan–Gross–Prasad conjecture ([9, Conjecture 17.3, Theorem 19.1]), whose basic case is proven by Gan–Ichino [10, Theorem 1.3], we can deduce that π is ψ_E-generic.

Conversely, if π is ψ_E-generic, by the local Gan–Gross–Prasad conjecture, one can find an irreducible tempered unramified representation π' of G_{n-1} such that $\mathrm{Hom}_{J_{n-1}}(\pi\otimes\pi'\otimes\overline{\omega_\psi},\mathbb{C}) \neq 0$. Since π' and ω_ψ are irreducible as representations of G_{n-1} and H_{n-1}, respectively, for any nonzero unramified vector $\varphi_0' \in \pi'$ and for any nonzero element $\mathcal{L} \in \mathrm{Hom}_{J_{n-1}}(\pi\otimes\pi'\otimes\overline{\omega_\psi},\mathbb{C})$, one can take $\varphi \in \pi$ such that

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$\mathcal{L}(\varphi \otimes \varphi_0' \otimes \overline{\phi_0}) \neq 0$. We may assume that φ is fixed by K^J. Since π is smooth, φ is fixed by K_{2m}^W for $2m \gg 0$. In this case, φ gives a nonzero element in $\pi_\psi^{K_{2m}^W}$.

This completes the proof of Theorem 2.2 (1). □

Recall in [9, Corollary 16.3] that for $\pi \in \mathrm{Irr}(G_n)$ and $\pi' \in \mathrm{Irr}(G_{n-1})$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{J_{n-1}}(\pi \otimes \pi' \otimes \overline{\omega_\psi}, \mathbb{C}) \leq 1.$$

It is worth to state the following result which was obtained by the above argument.

Proposition 3.3. *Let π be an irreducible tempered representation of G_n. Suppose that there is $\varphi \in \pi^{K_{2m}^W}$ whose image in π_ψ is nonzero for some $2m \geq 0$. Then there exists an irreducible tempered unramified representation π' of G_{n-1} together with an unramified vector $\varphi_0' \in \pi'$ such that $\mathcal{L}(\varphi \otimes \varphi_0' \otimes \overline{\phi_0}) \neq 0$ for any nonzero $\mathcal{L} \in \mathrm{Hom}_{J_{n-1}}(\pi \otimes \pi' \otimes \overline{\omega_\psi}, \mathbb{C})$.*

4. Uniqueness

In this section, we will prove Theorem 2.2 (2). As usual, this is an application of Rankin–Selberg integrals.

4.1. Rankin–Selberg integrals

Let τ be an irreducible generic representation of $\mathrm{GL}_{n-1}(E)$ which is realized on the Whittaker space $\mathcal{W}(\tau, \psi_E^{-1})$ with respect to the inverse of ψ_E. For $s \in \mathbb{C}$, we consider the normalized parabolically induced representation

$$\mathrm{Ind}_{Q_{n-1}}^{G_{n-1}}\left(\tau |\det|^{s-\frac{1}{2}}\right)$$

of G_{n-1}, where $Q_{n-1} = M_{n-1}U_{n-1}$ denotes the standard Siegel parabolic subgroup so that

$$M_{n-1} = \{\mathbf{m}(a) \mid a \in \mathrm{GL}_{n-1}(E)\},$$
$$U_{n-1} = \{\mathbf{n}(b) \mid b \in \mathrm{M}_{n-1}(E), {}^t(w_{n-1}\overline{b}) = w_{n-1}b\}.$$

We realize it on the space $V_{Q_{n-1}}^{G_{n-1}}(\mathcal{W}(\tau, \psi_E^{-1}), s)$ of smooth functions $f_s \colon G_{n-1} \times \mathrm{GL}_{n-1}(E) \to \mathbb{C}$ such that

- $f_s(\mathbf{n}(b)\mathbf{m}(a)g, a') = |\det a|_E^{s+\frac{n}{2}-1} f_s(g, a'a)$ for $g \in G_{n-1}$, $a, a' \in \mathrm{GL}_{n-1}(E)$ and $\mathbf{n}(b) \in U_{n-1}$;
- the function $a \mapsto f_s(g, a)$ belongs to $\mathcal{W}(\tau, \psi_E^{-1})$ for any $g \in G_{n-1}$.

Define a new representation τ^* by $\tau^*(a) = \tau(a^*)$, where $a^* = w_{n-1}{}^t\overline{a}^{-1}w_{n-1}^{-1}$. Note that $\tau^* \cong \overline{\tau}^\vee$, where $\overline{\tau}(a) = \tau(\overline{a})$. As in [21, Section 2.3], one can define a normalized intertwining operator

$$M^*(\tau, s) \colon V_{Q_{n-1}}^{G_{n-1}}(\mathcal{W}(\tau, \psi_E^{-1}), s) \to V_{Q_{n-1}}^{G_{n-1}}(\mathcal{W}(\tau^*, \psi_E^{-1}), 1-s).$$

Let π be an irreducible ψ_E-generic representation of G_n realized on the Whittaker space $\mathcal{W}(\pi, \psi_E)$. For $W \in \mathcal{W}(\pi, \psi_E)$, $f_s \in V_{Q_{n-1}}^{G_{n-1}}(\mathcal{W}(\tau, \psi_E^{-1}), s)$ and $\phi \in \mathcal{S}(E^{n-1})$, we define the *Rankin–Selberg integral* $\mathcal{L}(W, f_s, \overline{\phi})$ by

$$\int_{N_{n-1}\backslash G_{n-1}} \int_{E^{n-1}} W(w_{1,n-1}\mathbf{v}(x, 0; 0)g) f_s(g, \mathbf{1}_{n-1}) \overline{\omega_\psi(g)\phi(x)} dx dg,$$

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where we set

$$w_{1,n-1} = \left(\begin{array}{cc|cc} & \mathbf{1}_{n-1} & & \\ 1 & & & \\ \hline & & & 1 \\ & & \mathbf{1}_{n-1} & \end{array}\right) \in G_n.$$

Remark 4.1. Note that

$$W(w_{1,n-1}\mathbf{v}(x,0;0)g \cdot \mathbf{v}(0,0;z)) = \psi(z)W(w_{1,n-1}\mathbf{v}(x,0;0)g)$$

for $W \in \mathcal{W}(\pi, \psi_E)$. Hence, the restriction map $W \mapsto W(w_{1,n-1}\mathbf{v}(x,0;0)g)$ factors through $\pi \twoheadrightarrow \pi_\psi$. In particular, if π is ψ_E-generic, then π_ψ is nonzero.

Theorem 4.2. *Keep the notations.*

(1) *The integral $\mathcal{L}(W, f_s, \overline{\phi})$ converges absolutely for* $\mathrm{Re}(s) \gg 0$. *It is a rational function in q^{-s} so that it admits a meromorphic continuation to the whole s-plane.*

(2) *Let $I(\pi \times \tau \times \chi)$ be the fractional ideal of $\mathbb{C}[q^{-s}, q^s]$ generated by $\mathcal{L}(W, f_s, \overline{\phi})$ for $W \in \mathcal{W}(\pi, \psi_E)$, $f_s \in V_{Q_{n-1}}^{G_{n-1}}(\mathcal{W}(\tau, \psi_E^{-1}), s)$ and $\phi \in \mathcal{S}(E^{n-1})$. Then there is a unique polynomial $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$ such that $I(\pi \times \tau \times \chi) = (P(q^{-s})^{-1})$. We define the L-function attached to $\pi \times \tau$ and χ by*

$$L(s, \pi \times \tau, \chi) = P(q^{-s})^{-1}.$$

(3) *There is a meromorphic function $\Gamma(s, \pi \times \tau, \psi)$ such that*

$$\mathcal{L}(W, M^*(\tau, s)f_s, \overline{\phi}) = \omega_\pi(-1)^{n-1}\omega_\tau(-1)^n\Gamma(s, \pi \times \tau, \chi, \psi)\mathcal{L}(W, f_s, \overline{\phi})$$

for any $W \in \mathcal{W}(\pi, \psi_E)$, $f_s \in V_{Q_{n-1}}^{G_{n-1}}(\mathcal{W}(\tau, \psi_E^{-1}), s)$ and $\phi \in \mathcal{S}(E^{n-1})$. We call $\Gamma(s, \pi \times \tau, \chi, \psi)$ the gamma factor attached to $\pi \times \tau$, χ and ψ.

(4) *The gamma factor $\Gamma(s, \pi \times \tau, \chi, \psi)$ satisfies several properties (including the multiplicativity), which determine $\Gamma(s, \pi \times \tau, \chi, \psi)$ uniquely.*

(5) *Define the ε-factor attached to $\pi \times \tau$, χ and ψ by*

$$\varepsilon(s, \pi \times \tau, \chi, \psi) = \Gamma(s, \pi \times \tau, \chi, \psi)\frac{L(s, \pi \times \tau, \chi)}{L(1-s, \pi^\vee \times \tau^\vee, \chi)}.$$

Then it satisfies that

$$\varepsilon(1-s, \pi \times \tau^*, \chi, \psi)\varepsilon(s, \pi \times \tau, \chi, \psi) = 1.$$

In particular, $\varepsilon(s, \pi \times \tau, \chi, \psi) \in \mathbb{C}^\times (q^{-s})^{\mathbb{Z}}$.

Proof. (1) is [4, Proposition 6.4]. By [4, Proposition 6.5], we see that $1 \in I(\pi \times \tau \times \chi)$, which implies (2). The assertion (3) follows from the multiplicity one theorem proven in [9, Corollary 16.3]. (4) is proven by Morimoto [21, Theorem 3.1]. Since $M^*(\tau^*, 1-s) \circ M^*(\tau, s) = \mathrm{id}$, using $\omega_{\tau^*}(-1) = \omega_\tau(-1)$, we have

$$\begin{aligned}\mathcal{L}(W, f_s, \overline{\phi}) &= \mathcal{L}(W, M^*(\tau^*, 1-s) \circ M^*(\tau, s)f_s, \overline{\phi}) \\ &= \omega_\pi(-1)^{n-1}\omega_{\tau^*}(-1)^n\Gamma(1-s, \pi \times \tau^*, \chi, \psi)\mathcal{L}(W, M^*(\tau, s)f_s, \overline{\phi}) \\ &= \Gamma(1-s, \pi \times \tau^*, \chi, \psi)\Gamma(s, \pi \times \tau, \chi, \psi)\mathcal{L}(W, f_s, \overline{\phi})\end{aligned}$$

for any W, f_s and ϕ. It means that

$$\Gamma(1-s, \pi \times \tau^*, \chi, \psi)\Gamma(s, \pi \times \tau, \chi, \psi) = 1,$$

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which is equivalent to saying that

$$\varepsilon(1-s,\pi\times\tau^*,\chi,\psi)\varepsilon(s,\pi\times\tau,\chi,\psi)=1.$$

Hence, $\varepsilon(s,\pi\times\tau,\chi,\psi)\in\mathbb{C}[q^{-s},q^{s}]^\times=\mathbb{C}^\times(q^{-s})^{\mathbb{Z}}$. □

4.2. Unramified representations

In this subsection, we consider the Rankin–Selberg integrals when τ varies over irreducible unramified representations of $\mathrm{GL}_{n-1}(E)$.

Recall that $K^{W_0}=K_0^W\cap G_{n-1}$. It is a hyperspecial maximal compact subgroup of G_{n-1}, and the Iwasawa decomposition $G_{n-1}=Q_{n-1}K^{W_0}$ holds.

Irreducible unramified representations of $\mathrm{GL}_{n-1}(E)$ are parametrized by the *Satake parameters* $\underline{x}=(x_1,\dots,x_{n-1})\in(\mathbb{C}^\times)^{n-1}/S_{n-1}$. We write the unramified representation associated to $\underline{x}$ by $\tau_{\underline{x}}$. Then for almost all $\underline{x}$, since $\tau_{\underline{x}}$ is generic, there exists a unique function $f_s(\underline{x})\in V_{Q_{n-1}}^{G_{n-1}}(\mathcal{W}(\tau_{\underline{x}},\psi_E^{-1}),s)$ such that

- $f_s(gk,a;\underline{x})=f_s(g,a;\underline{x})$ for any $g\in G_{n-1}$, $k\in K^{W_0}$ and $a\in\mathrm{GL}_{n-1}(E)$; and
- the function $W(a;\underline{x})=f_s(\mathbf{1}_{2(n-1)},a;\underline{x})$ is right $\mathrm{GL}_{n-1}(\mathfrak{o}_E)$-invariant with $W(\mathbf{1}_{n-1};\underline{x})=1$.

Lemma 4.3. *For $\underline{x}=(x_1,\dots,x_{n-1})$, we write $\underline{x}^{-1}=(x_1^{-1},\dots,x_{n-1}^{-1})$. Then we have*

$$\frac{M^*(\tau_{\underline{x}},s)f_s(\underline{x})}{\prod_{i=1}^{n-1}(1-q^{-s}x_i)\prod_{1\le i<j\le n-1}(1-q^{-2s}x_ix_j)}
=\frac{f_{1-s}(\underline{x}^{-1})}{\prod_{i=1}^{n-1}(1-q^{-(1-s)}x_i^{-1})\prod_{1\le i<j\le n-1}(1-q^{-2(1-s)}x_i^{-1}x_j^{-1})}.$$

Proof. The assertion follows from [4, Theorem 8.1] and [21, Theorem 3.1]. □

Let π be an irreducible ψ_E-generic tempered representation of G_n with L-parameter ϕ_π. Then by the uniqueness of the gamma factor (Theorem 4.2 (4)), we have

$$\Gamma(s,\pi\times\tau_{\underline{x}},\chi,\psi)=\prod_{i=1}^{n-1}\varepsilon(s+s_i+s_0,\phi_\pi,\psi_E)\frac{L(1-s-s_i-s_0,\phi_\pi^\vee)}{L(s+s_i+s_0,\phi_\pi)}$$

for almost all $\underline{x}=(x_1,\dots,x_{n-1})$, where $s_0,s_1,\dots,s_{n-1}\in\mathbb{C}$ are such that $q^{-2s_0}=-1$ and $x_i=q^{-2s_i}$ for $1\le i\le n-1$. Since ϕ_π is tempered, two meromorphic functions $\prod_{i=1}^{n-1}L(1-s-s_i-s_0,\phi_\pi^\vee)$ and $\prod_{i=1}^{n-1}L(s+s_i+s_0,\phi_\pi)$ have no common pole for almost all $\underline{x}$. In particular, in this case, we have

$$L(s,\pi\times\tau_{\underline{x}},\chi)=\prod_{i=1}^{n-1}L(s+s_i+s_0,\phi_\pi),$$
$$\varepsilon(s,\pi\times\tau_{\underline{x}},\chi,\psi)=\prod_{i=1}^{n-1}\varepsilon(s+s_i+s_0,\phi_\pi,\psi_E).$$

If we write $L(s,\phi_\pi)=P_\pi(q^{-2s})$ and $\varepsilon(s,\phi_\pi,\psi_E)=\varepsilon q^{c(\phi_\pi)(1-2s)}$, then

$$L(s,\pi\times\tau_{\underline{x}},\chi)=\prod_{i=1}^{n-1}P_\pi(-x_iq^{-2s}),$$
$$\varepsilon(s,\pi\times\tau_{\underline{x}},\chi,\psi)=\varepsilon^{n-1}(-q^{1-2s})^{c(\phi_\pi)(n-1)}\prod_{i=1}^{n-1}x_i^{c(\phi_\pi)}.$$

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4.3. Proof of Theorem 2.2 (2)

The symmetric group S_{n-1} acts on $\mathbb{C}[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}]$ canonically. Set

$$\mathcal{T} = \mathbb{C}[X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}]^{S_{n-1}}.$$

Note that

$$\mathcal{T} = \mathbb{C}[T_1, \ldots, T_{n-2}, T_{n-1}, T_{n-1}^{-1}]$$

with

$$T_i = \sum_{\sigma \in S_{n-1}} X_{\sigma(1)} \cdots X_{\sigma(i)}.$$

The degree with respect to T_{n-1} gives a $\mathbb{Z}$-grading on $\mathcal{T}$; that is, $\mathcal{T} = \oplus_{d \in \mathbb{Z}} \mathcal{T}_d$ with

$$\mathcal{T}_d = \mathbb{C}[T_1, \ldots, T_{n-2}] T_{n-1}^d.$$

Write $\underline{X} = (X_1, \ldots, X_{n-1})$ and $q^{1-2s}\underline{X} = (q^{1-2s}X_1, \ldots, q^{1-2s}X_{n-1})$. There is a function

$$W(\underline{X}) \colon \mathrm{GL}_{n-1}(E) \to \mathcal{T}$$

such that $W(\underline{X})|_{\underline{X}=\underline{x}} = W(\underline{x})$ for almost all $\underline{x} \in (\mathbb{C}^\times)^{n-1}$. Similarly, we consider the function $f_s(\underline{X}) \colon G_{n-1} \times \mathrm{GL}_{n-1}(E) \to \mathcal{T}$ so that $f_s(\underline{X})|_{\underline{X}=\underline{x}} = f_s(\underline{x})$ for almost all $\underline{x} \in (\mathbb{C}^\times)^{n-1}$. In particular, $f_s(\mathbf{1}_{2(n-1)}, a; \underline{X}) = W(a; q^{1-2s}\underline{X})$.

We regard $\mathcal{L}(W, f_{1/2}(\underline{X}), \overline{\phi_0})$ as a formal power series of $X_1^{\pm 1}, \ldots, X_{n-1}^{\pm 1}$, or an element of $\mathbb{C}[T_1, \ldots, T_{n-2}][[T_{n-1}^{\pm 1}]]$. For $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{Z}^{n-1}$, we set $|\lambda| = \lambda_1 + \cdots + \lambda_{n-1}$. The following is a key lemma.

Lemma 4.4. *Let $W \in \mathcal{W}(\pi, \psi_E)^{K_{2m}^W}$. Write*

$$\mathcal{L}(W, f_{1/2}(\underline{X}), \overline{\phi_0}) = \sum_{\lambda \in \mathbb{Z}^{n-1}} a_\lambda(W) X_1^{\lambda_1} \cdots X_{n-1}^{\lambda_{n-1}} = \sum_{d \in \mathbb{Z}} \mathcal{L}_d(W) T_{n-1}^d$$

with $a_\lambda(W) \in \mathbb{C}$ and $\mathcal{L}_d(W) \in \mathbb{C}[T_1, \ldots, T_{n-2}]$. Then

- *$a_\lambda(W) = 0$ unless $|\lambda| \geq -(n-1)m$; and*
- *$\mathcal{L}_d(W) = 0$ unless $d \geq -m$.*

Proof. For row vectors $x, u \in E^{n-1}$ and $a \in \mathrm{GL}_{n-1}(E)$, we put $k(x, a, u)$ to be the matrix

$$(w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a))^{-1} \left(\begin{array}{cc|cc} \mathbf{1}_{n-1} & {}^t u & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -\overline{u} w_{n-1} \\ 0 & 0 & 0 & \mathbf{1}_{n-1} \end{array}\right) w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a).$$

By an easy calculation, $k(x, a, u)$ is equal to

$$\left(\begin{array}{cc|cc} 1 - x{}^t u & -x{}^t uxa & 0 & 0 \\ a^{-1}{}^t u & \mathbf{1}_{n-1} + a^{-1}{}^t uxa & 0 & 0 \\ \hline 0 & 0 & \mathbf{1}_{n-1} - w_{n-1}{}^t\overline{a}{}^t\overline{xu}{}^t\overline{a}^{-1} w_{n-1} & w_{n-1}{}^t\overline{a}{}^t\overline{xu}{}^t\overline{x} \\ 0 & 0 & -\overline{u}{}^t\overline{a}^{-1} w_{n-1} & 1 + \overline{u}{}^t\overline{x} \end{array}\right).$$

In particular, if $xa \in \mathfrak{o}_E^{n-1}$ and $u{}^t a^{-1} \in (\mathfrak{p}_E^m)^{n-1}$, then $x{}^t u \in \mathfrak{p}_E^m$ so that $k(x, a, u) \in K_{2m}^W$.

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As functions on $g \in G_{n-1}$, all of $W(w_{1,n-1}\mathbf{v}(x,0;0)g)$, $f_s(g, \mathbf{1}_{n-1}; \underline{X})$ and $\overline{\omega_\psi(g)\phi_0}$ are right K^{W_0}-invariant. Hence, by the integral formula with respect to the Iwasawa decomposition, we can write $\mathcal{L}(W, f_s(\underline{X}), \overline{\phi_0})$ as

$$\int_{T_{n-1}} \int_{E^{n-1}} W(w_{1,n-1}\mathbf{v}(x,0;0)t) f_s(t, \mathbf{1}_{n-1}; \underline{X}) \overline{\omega_\psi(t)\phi_0(x)} \delta_{B_{n-1}}^{-1}(t)\, dx dt,$$

where $B_{n-1} = T_{n-1}N_{n-1}$ is the upper triangular Borel subgroup of G_{n-1} with the diagonal torus T_{n-1}. Write $t = \mathbf{m}(a)$ with $a = \mathrm{diag}(a_1, \dots, a_{n-1})$ being a diagonal matrix in $\mathrm{GL}_{n-1}(E)$. Then $\omega_\psi(\mathbf{m}(a))\phi_0(x) \neq 0 \iff xa \in \mathfrak{o}_E^{n-1}$. In this case, if $W(w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a)) \neq 0$, then for $u = (u_1, \dots, u_{n-1}) \in E^{n-1}$ such that $u\,{}^t a^{-1} \in (\mathfrak{p}_E^m)^{n-1}$, we have

$$\begin{aligned} 0 &\neq W(w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a)) \\ &= W(w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a) \cdot k(x,a,u)) \\ &= W\left(\left(\begin{array}{cc|cc} \mathbf{1}_{n-1} & {}^t u & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -\overline{u} w_{n-1} \\ 0 & 0 & 0 & \mathbf{1}_{n-1} \end{array} \right) w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a) \right) \\ &= \psi_E(u_{n-1}) W(w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a)). \end{aligned}$$

This shows that

$$u_{n-1} \in \mathfrak{p}_E^{\mathrm{ord}(a_{n-1})+m} \implies \psi_E(u_{n-1}) = 1.$$

This means that $\mathrm{ord}(a_{n-1}) + m \geq 0$.

Recall that $f_s(\mathbf{m}(a), \mathbf{1}_{n-1}; \underline{X}) = \delta_{Q_{n-1}}^{\frac{1}{2}}(\mathbf{m}(a)) W(a; q^{1-2s}\underline{X})$. By a similar (and well-known) argument, if $W(a; q^{1-2s}\underline{X}) \neq 0$, then $\mathrm{ord}(a_1) \geq \cdots \geq \mathrm{ord}(a_{n-1})$. Hence, we conclude that if

$$W(w_{1,n-1}\mathbf{v}(x,0;0)\mathbf{m}(a)) W(a; \underline{X}) \overline{\omega_\psi(\mathbf{m}(a))\phi_0(x)} \neq 0,$$

then

$$\mathrm{ord}(a_1) \geq \cdots \geq \mathrm{ord}(a_{n-1}) \geq -m$$

so that

$$\mathrm{ord}(\det(a)) = \sum_{i=1}^{n-1} \mathrm{ord}(a_i) \geq -(n-1)m.$$

Since the Casselman–Shalika formula [6] tells us that

$$W(a; \underline{X}) \in \left(\bigoplus_{\substack{\lambda \in \mathbb{Z}^{n-1} \\ |\lambda| = \mathrm{ord}(\det(a))}} \mathbb{C} X_1^{\lambda_1} \cdots X_{n-1}^{\lambda_{n-1}} \right) \cap \mathbb{C}[T_1, \dots, T_{n-2}] T_{n-1}^{\mathrm{ord}(a_{n-1})},$$

we obtain the assertions. □

For $W \in \mathcal{W}(\pi, \psi_E)^{K_{2m}^W}$, we define $\Psi(W; \underline{X})$ by

$$\Psi(W; \underline{X}) = \frac{\prod_{i=1}^{n-1} P_\pi(-q^{-1}X_i) \mathcal{L}(W, f_{1/2}(\underline{X}), \phi_0)}{\prod_{i=1}^{n-1}(1 - q^{-1}X_i) \prod_{1 \leq i < j \leq n-1}(1 - q^{-2}X_iX_j)}.$$

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Proposition 4.5. *If* $2m < c(\phi_\pi)$*, then* $\Psi(W;\underline{X}) = 0$ *for* $W \in \mathcal{W}(\pi,\psi_E)^{K^W_{2m}}$*. If* $2m = c(\phi_\pi)$ *or* $2m = c(\phi_\pi)+1$*, then*

$$\dim_{\mathbb{C}}\left\{\Psi(W;\underline{X}) \,\middle|\, W \in \mathcal{W}(\pi,\psi_E)^{K^W_{2m}}\right\} \le 1.$$

Proof. Since $P_\pi(X)$ is a polynomial of X with $P_\pi(0) = 1$, and since $(1-q^{-1}X_i)^{-1} = \sum_{k=0}^{\infty}(q^{-1}X_i)^k$ and $(1-q^{-2}X_iX_j)^{-1} = \sum_{k=0}^{\infty}(q^{-2}X_iX_j)^k$, if we write

$$\Psi(W;\underline{X}) = \sum_{\lambda\in\mathbb{Z}^{n-1}} \alpha_\lambda(W)X_1^{\lambda_1}\cdots X_{n-1}^{\lambda_{n-1}} = \sum_{d\in\mathbb{Z}} \Psi_d(W;\underline{X})T_{n-1}^d$$

with $\alpha_\lambda(W) \in \mathbb{C}$ and $\Psi_d(W;\underline{X}) \in \mathbb{C}[T_1,\dots,T_{n-2}]$, by Lemma 4.4, we see that

- $\alpha_\lambda(W) = 0$ unless $|\lambda| \ge -(n-1)m$; and
- $\Psi_d(W;\underline{X}) = 0$ unless $d \ge -m$.

Write $\underline{X}^{-1} = (X_1^{-1},\dots,X_{n-1}^{-1})$. By the functional equation (Theorem 4.2 (3), (5)) together with Lemma 4.3, we see that

$$T_{n-1}^{-c(\phi_\pi)}\Psi(W;\underline{X}^{-1}) = \varepsilon_0\Psi(W;\underline{X}) \tag{$*$}$$

with

$$\varepsilon_0 = ((-1)^{c(\phi_\pi)}\varepsilon\cdot\omega_\pi(-1))^{n-1}.$$

The left-hand side and the right-hand side of $(*)$ belong to

$$\bigoplus_{d\le m-c(\phi_\pi)} \mathbb{C}[T_1,\dots,T_{n-2}]T_{n-1}^d, \qquad \bigoplus_{d\ge -m} \mathbb{C}[T_1,\dots,T_{n-2}]T_{n-1}^d,$$

respectively. Hence, if $\Psi_d(W;\underline{X}) \ne 0$, then $-m \le d \le m - c(\phi_\pi)$ so that $2m \ge c(\phi_\pi)$. A similar argument shows that if $\alpha_\lambda(W) \ne 0$, then

$$-(n-1)m \le |\lambda| \le (n-1)(m-c(\phi_\pi)).$$

Now we assume that $2m = c(\phi_\pi)$. Then $\Psi_d(W;\underline{X}) = 0$ unless $d = -m$. Hence,

$$T_{n-1}^m\Psi(W;\underline{X}) \in \mathbb{C}[T_1,\dots,T_{n-2}] \subset \mathbb{C}[X_1,\dots,X_{n-1}].$$

This implies that $\alpha_\lambda(W) = 0$ unless $\lambda_i \ge -m$ for any $1 \le i \le n-1$. However, since $\alpha_\lambda(W) = 0$ unless $|\lambda| = -(n-1)m$, we see that $\alpha_\lambda(W) = 0$ unless $\lambda_1 = \cdots = \lambda_{n-1} = -m$. This means that

$$\Psi(W;\underline{X}) \in \mathbb{C}T_{n-1}^{-m}$$

so that

$$\dim_{\mathbb{C}}\left\{\Psi(W;\underline{X}) \,\middle|\, W \in \mathcal{W}(\pi,\psi_E)^{K^W_{c(\phi_\pi)}}\right\} \le 1.$$

Next, we assume that $2m = c(\phi_\pi)+1$. Then $\Psi_d(W;\underline{X}) = 0$ unless $d = -m, -m+1$, and $\alpha_\lambda(W) = 0$ unless $|\lambda| = -(n-1)m, -(n-1)(m-1)$. In particular, $\Psi_{-m+1}(W;\underline{X})$ is a scalar so that

$$\Psi_{-m+1}(W;\underline{X}^{-1}) = \Psi_{-m+1}(W;\underline{X}).$$

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By the functional equation $(*)$, we have

$$\Psi_{-m+1}(W;\underline{X}^{-1}) = \varepsilon_0\Psi_{-m}(W;\underline{X}),$$
$$\Psi_{-m}(W;\underline{X}^{-1}) = \varepsilon_0\Psi_{-m+1}(W;\underline{X}).$$

Hence, $\Psi_{-m}(W;\underline{X})$ is also a scalar. Therefore,

$$\Psi(W;\underline{X}) \in \mathbb{C}(T_{n-1}^{-m} + \varepsilon_0 T_{n-1}^{-m+1})$$

so that

$$\dim_{\mathbb{C}}\left\{\Psi(W;\underline{X}) \,\middle|\, W \in \mathcal{W}(\pi,\psi_E)^{K^W_{c(\phi_\pi)+1}}\right\} \leq 1.$$

This completes the proof. □

By Proposition 3.3, we see that $\mathcal{W}(\pi,\psi_E)^{K^W_{2m}} \ni W \mapsto \Psi(W;\underline{X})$ gives an injective linear map

$$\Psi\colon \pi_\psi^{K^W_{2m}} \hookrightarrow \mathcal{T}.$$

Hence, by Proposition 4.5, we have

- $\pi_\psi^{K^W_{2m}} = 0$ if $2m < c(\phi_\pi)$; and
- $\dim_{\mathbb{C}}(\pi_\psi^{K^W_{2m}}) \leq 1$ if $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi)+1$.

This completes the proof of Theorem 2.2 (2).

5. Existence

In this section, we will prove Theorem 2.2 (3). To do this, we will use the theta correspondence for $(\mathrm{U}(V),\mathrm{U}(W))$.

5.1. *Theta correspondence*

Recall that $V = V_{2n+1}$ (resp. $W = W_{2n}$) is a hermitian (resp. skew-hermitian) space over E of dimension $2n+1$ (resp. $2n$). Then $\mathbb{W} = V \otimes_E W$ forms a symplectic space of dimension $4n(2n+1)$ equipped with the symplectic form

$$\langle v\otimes w, v'\otimes w'\rangle = \mathrm{tr}_{E/F}(\langle v,v'\rangle_V \cdot \langle w,w'\rangle_W).$$

Here, $\mathrm{U}(V)$, $\mathrm{U}(W)$ and $\mathrm{Sp}(\mathbb{W})$ act on V, W and $\mathbb{W}$, respectively, all from the left. We have a canonical map $\mathrm{U}(V)\times\mathrm{U}(W) \to \mathrm{Sp}(\mathbb{W})$.

Recall that χ is the unique nontrivial quadratic unramified character of $E^\times$. Note that $\chi|_{F^\times}$ is equal to the quadratic character corresponding to E/F. Let $\widetilde{\mathrm{Sp}}(\mathbb{W})$ be the metaplectic $\mathbb{C}^\times$-cover. Using the pair $(\chi_V,\chi_W) = (\chi^{2n+1},\chi^{2n})$, we have Kudla's splitting [15]

$$\mathrm{U}(V)\times\mathrm{U}(W) \to \widetilde{\mathrm{Sp}}(\mathbb{W}).$$

Let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ associated to the additive character ψ. By the pullback, we obtain the Weil representation $\omega_{\psi,V,W}$ of $\mathrm{U}(V)\times\mathrm{U}(W)$. For an irreducible representation π of $\mathrm{U}(W)$, it is known that the maximal π-isotypic quotient of $\omega_{\psi,V,W}$ is of the form

$$\Theta_\psi(\pi)\boxtimes\pi$$

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for a smooth representation $\Theta_\psi(\pi)$ of $\mathrm{U}(V)$ of finite length. The Howe duality conjecture, proven by Waldspurger [24], asserts that if $\Theta_\psi(\pi)$ is nonzero, then it has a unique irreducible quotient $\theta_\psi(\pi)$. We call $\theta_\psi(\pi)$ the *theta lift* of π.

The following is a special case of Prasad's conjecture, which was proven by Gan–Ichino [10]. See also Theorem 4.4 in that paper.

Theorem 5.1. *Let π be an irreducible ψ_E-generic representation of* $\mathrm{U}(W)$ *with L-parameter ϕ_π. Then $\Theta_\psi(\pi)$ is always nonzero. Moreover, $\sigma = \theta_\psi(\pi)$ is generic, and its L-parameter is given by*

$$\phi_\sigma = \phi_\pi\chi \oplus \mathbf{1},$$

where $\phi_\pi\chi = \phi_\pi \otimes \chi$.

In particular, if $\sigma = \theta_\psi(\pi)$, then we have $c(\phi_\sigma) = c(\phi_\pi)$ and $\omega_\sigma = \omega_\pi$. Moreover, if π is tempered, then so is σ, so that we have $\sigma^{K_{2m}^V} \neq 0$ for $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi) + 1$ by Theorem 2.1.

5.2. Lattice model

First, we will show that $\pi^{K_{2m}^W} \neq 0$. To do this, we use a lattice model $\mathcal{S} = \mathcal{S}(A)$ of the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(\mathbb{W})$. In this subsection, we recall this model.

Let $\mathbb{W}$ be a symplectic space over F of dimension $2N$ equipped with a symplectic form $\langle\cdot,\cdot\rangle$. The group law of the Heisenberg group $H(\mathbb{W}) = \mathbb{W} \oplus F$ is given by

$$(w_1, t_1) \cdot (w_2, t_2) = \left(w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2\rangle\right),$$

whose center is $\{0\} \oplus F \cong F$. By the Stone–von Neumann theorem, there is a unique (up to isomorphism) irreducible admissible representation $(\rho_\psi, \mathcal{S})$ of $H(\mathbb{W})$ whose central character is ψ. The symplectic group $\mathrm{Sp}(\mathbb{W})$ acts on $H(\mathbb{W})$ by $g \cdot (w, t) = (gw, t)$. By the uniqueness, for $g \in \mathrm{Sp}(\mathbb{W})$, we have $M_g \in \mathrm{Aut}(\mathcal{S})$ such that

$$M_g \circ \rho_\psi(h) \circ M_g^{-1} = \rho_\psi(gh) \quad \text{for } h \in H(\mathbb{W}). \tag{$\star$}$$

By Schur's lemma, such M_g is determined uniquely up to a nonzero scalar. Define the metaplectic $\mathbb{C}^\times$-cover $\widetilde{\mathrm{Sp}}(\mathbb{W})$ of $\mathrm{Sp}(\mathbb{W})$ by

$$\widetilde{\mathrm{Sp}}(\mathbb{W}) = \{(g, M_g) \in \mathrm{Sp}(\mathbb{W}) \times \mathrm{Aut}(\mathcal{S}) \mid M_g \text{ satisfies } (\star)\}.$$

We have an exact sequence

$$1 \longrightarrow \mathbb{C}^\times \xrightarrow{\alpha} \widetilde{\mathrm{Sp}}(\mathbb{W}) \xrightarrow{\beta} \mathrm{Sp}(\mathbb{W}) \longrightarrow 1$$

given by $\alpha(z) = (\mathbf{1}_\mathbb{W}, z \cdot \mathrm{id}_\mathcal{S})$ and $\beta(g, M_g) = g$. The *Weil representation* ω_ψ of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ on the space $\mathcal{S}$ is defined by

$$\omega_\psi(g, M_g) = M_g.$$

Now we shall give a realization of the space $\mathcal{S}$. Let A be a lattice of $\mathbb{W}$ (i.e., a free $\mathfrak{o}_F$-submodule of rank $2N$). The *dual lattice* A^* is defined by

$$A^* = \{w \in \mathbb{W} \mid \langle w, a\rangle \in \mathfrak{o}_F \text{for any} a \in A\}.$$

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Suppose that A is self-dual (i.e., $A^* = A$). Let $\mathcal{S}(A)$ be the space of locally constant, compactly supported functions $\phi\colon H(\mathbb{W}) \to \mathbb{C}$ such that

$$\phi((a,t)\cdot h) = \psi(t)\phi(h)$$

for $(a,t) \in A \oplus F$ and $h \in H(\mathbb{W})$. The group $H(\mathbb{W})$ acts on $\mathcal{S}(A)$ by the right translation ρ_ψ. It is known that the representation $(\rho_\psi, \mathcal{S}(A))$ of $H(\mathbb{W})$ is irreducible with the central character ψ. This gives a realization $(\omega_\psi, \mathcal{S}(A))$ of the Weil representation which is called a *lattice model*. Since $(a,0)\cdot(w,0) = (a+w, \frac{1}{2}\langle a,w\rangle)$, by the restriction to $\mathbb{W} \oplus \{0\}$, we can identify $\mathcal{S}(A)$ with the space of locally constant, compactly supported functions $\phi\colon \mathbb{W} \to \mathbb{C}$ such that

$$\phi(a+w) = \psi\left(-\frac{1}{2}\langle a,w\rangle\right)\phi(w)$$

for $a \in A$ and $w \in \mathbb{W}$.

For $g \in \mathrm{Sp}(\mathbb{W})$, we define $M[g] \in \mathrm{Aut}(\mathcal{S}(A))$ by

$$(M[g]\phi)(w) = \int_A \psi\left(\frac{1}{2}\langle a,w\rangle\right)\phi(g^{-1}\cdot(a+w))da$$

for $\phi \in \mathcal{S}(A)$ and $w \in \mathbb{W}$. Here, da is the Haar measure on A normalized so that $\mathrm{vol}(A) = 1$. It is easy to check that $(g, M[g]) \in \widetilde{\mathrm{Sp}}(\mathbb{W})$.

Let K_A be the stabilizer of A in $\mathrm{Sp}(\mathbb{W})$. Then we have

$$(M[k]\phi)(w) = \phi(k^{-1}\cdot w)$$

for $k \in K_A$, $\phi \in \mathcal{S}(A)$, and $w \in \mathbb{W}$. The map $k \mapsto (k, M[k])$ gives a splitting $K_A \to \widetilde{\mathrm{Sp}}(\mathbb{W})$. If we identify K_A with the image, the restriction of the Weil representation $(\omega_\psi, \mathcal{S}(A))$ to K_A is given by $\omega_\psi(k)\phi(w) = \phi(k^{-1}\cdot w)$.

5.3. *Families of lattices*

Take bases $\{e_n, \dots, e_1, e_0, e_{-1}, \dots, e_{-n}\}$ of V and $\{f_n, \dots, f_1, f_{-1}, \dots, f_{-n}\}$ of W, respectively, as in §2.1. Set

$$\Gamma_V = \left(\bigoplus_{i=1}^n \mathfrak{o}_E e_i\right) \oplus \mathfrak{o}_E e_0 \oplus \left(\bigoplus_{i=1}^n \mathfrak{o}_E e_{-i}\right),$$
$$\Gamma_W = \left(\bigoplus_{i=1}^n \mathfrak{o}_E f_i\right) \oplus \left(\bigoplus_{i=1}^n \mathfrak{o}_E f_{-i}\right).$$

Then Γ_V and Γ_W are self-dual lattices (i.e., $\Gamma_V^* = \Gamma_V$ and $\Gamma_W^* = \Gamma_W$).

In this subsection, for two $\mathfrak{o}_E$-modules Γ_1 and Γ_2, we denote by $\Gamma_1 \otimes \Gamma_2$ the tensor product of $\mathfrak{o}_E$-modules. We put

$$A = \Gamma_V \otimes \Gamma_W.$$

This is a self-dual lattice of $\mathbb{W} = V \otimes_F W$, (i.e., $A^* = A$). We will consider the lattice model $(\omega_\psi, \mathcal{S}(A))$ of the Weil representation of $\widetilde{\mathrm{Sp}}(\mathbb{W})$.

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Fix a non-negative even integer $2m \geq 0$. We consider lattices

$$M_{2m} = \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E e_i \right) \oplus \mathfrak{p}_E^m e_0 \oplus \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E e_{-i} \right),$$
$$N_{2m} = \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E f_i \right) \oplus \left(\bigoplus_{i=1}^{n-1} \mathfrak{o}_E f_{-i} \right) \oplus \mathfrak{p}_E^m f_{-n}$$

of V and W, respectively. Then $M_{2m} \subset \Gamma_V$ and $N_{2m} \subset \Gamma_W$. Moreover, the dual lattices are given by

$$M_{2m}^* = \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E e_i \right) \oplus \mathfrak{p}_E^{-m} e_0 \oplus \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E e_{-i} \right),$$
$$N_{2m}^* = \mathfrak{p}_E^{-m} f_n \oplus \left(\bigoplus_{i=1}^{n-1} \mathfrak{o}_E f_i \right) \oplus \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E f_{-i} \right).$$

Recall that in Section 2.4, we defined compact subgroups K_{2m}^V and K_{2m}^W of $\mathrm{U}(V)$ and $\mathrm{U}(W)$, respectively. The following lemma is easy to check.

Lemma 5.2. *We have*

$$K_{2m}^V = \{h \in \mathrm{U}(V) \mid (h-1) \cdot M_{2m}^* \subset M_{2m}\},$$
$$K_{2m}^W = \{g \in \mathrm{U}(W) \mid (g-1) \cdot N_{2m}^* \subset N_{2m}\}.$$

In particular, $K_{2m}^V \times K_{2m}^W$ *is contained in* K_A *under the canonical map* $\mathrm{U}(V) \times \mathrm{U}(W) \to \mathrm{Sp}(\mathbb{W})$.

Let $\mathcal{S}(A)_{M_{2m}}$ be the subspace of $\mathcal{S}(A)$ consisting of functions $\phi\colon \mathbb{W} \to \mathbb{C}$ such that $\mathrm{Supp}(\phi) \subset M_{2m}^* \otimes \Gamma_W$. We will use the following result proven by Waldspurger.

Proposition 5.3 [24, Corollary III.2]. *Let* J_{2m}^V *be a compact subgroup of* $\mathrm{U}(V)$. *Suppose that*

- $J_{2m}^V \supset K_{2m}^V$;
- $\mathcal{S}(A)_{M_{2m}}$ *is stable by* J_{2m}^V;
- $(\mathcal{S}(A)_{M_{2m}})^{J_m^V} \neq \{0\}$.

Then $\mathcal{S}(A)^{J_{2m}^V}$ *is generated by* $(\mathcal{S}(A)_{M_{2m}})^{J_{2m}^V}$ *as a representation of* $\mathrm{U}(W)$.

We will apply this proposition to the compact subgroup J_{2m}^V generated by K_{2m}^V and $E^1 \cap (1 + \mathfrak{p}_E^m)$, where the latter is regarded as a subgroup of the center of $\mathrm{U}(V)$. Namely, $J_0^V = K_0^V$, and

$$J_{2m}^V = \begin{matrix} & n & 1 & n \\ n & \mathfrak{o}_E & \mathfrak{p}_E^m & \mathfrak{o}_E \\ 1 & \mathfrak{p}_E^m & 1+\mathfrak{p}_E^m & \mathfrak{p}_E^m \\ n & \mathfrak{o}_E & \mathfrak{p}_E^m & \mathfrak{o}_E \end{matrix} \cap \mathrm{U}_{2n+1}$$

for $2m > 0$. It is clear that $J_{2m}^V \supset K_{2m}^V$.

We check the second and third conditions in Proposition 5.3.

Lemma 5.4. *The space* $\mathcal{S}(A)_{M_{2m}}$ *is stable by* J_{2m}^V *and fixed by* K_{2m}^V. *Moreover,* $(\mathcal{S}(A)_{M_{2m}})^{J_{2m}^V} \neq \{0\}$.

Proof. For $t \in \mathfrak{p}_E^{-m}$ and $w \in \Gamma_W$, define $\phi_{t,w} \in \mathcal{S}(A)$ so that $\mathrm{Supp}(\phi_{t,w}) = A + te_0 \otimes w$ and $\phi_{t,w}(te_0 \otimes w) = 1$. Then $\mathcal{S}(A)_{M_{2m}}$ is equal to the $\mathbb{C}$-span of

$$\left\{ \phi_{t,w} \;\middle|\; t \in \mathfrak{p}_E^{-m},\ w \in \Gamma_W \right\}.$$

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For $k \in J^V_{2m}$, write $ke_0 = \sum_{i=-n}^{n} k_i e_i$. Then

$$k_i \in \begin{cases} \mathfrak{p}_E^m & \text{if } i \neq 0, \\ 1 + \mathfrak{p}_E^m & \text{if } i = 0. \end{cases}$$

In particular, we see that $(k-1)te_0 \otimes w \in A$. Hence,

$$\begin{aligned} \mathrm{Supp}(\omega_\psi(k)\phi_{t,w}) &= k(A + te_0 \otimes w) \\ &= A + (k-1)te_0 \otimes w + te_0 \otimes w \\ &= A + te_0 \otimes w = \mathrm{Supp}(\phi_{t,w}). \end{aligned}$$

Moreover,

$$\begin{aligned} \omega_\psi(k)\phi_{t,w}(te_0 \otimes w) &= \omega_\psi(k)\phi_{t,w}(kte_0 \otimes w - (k-1)te_0 \otimes w) \\ &= \psi\left(\frac{1}{2}\langle (k-1)te_0 \otimes w, kte_0 \otimes w\rangle\right)\omega_\psi(k)\phi_{t,w}(kte_0 \otimes w) \\ &= \psi_E\left(\langle (k-1)te_0, kte_0\rangle_V \cdot \langle w, w\rangle_W\right)\phi_{t,w}(te_0 \otimes w) \\ &= \psi_E\left(N_{E/F}(t)(\langle ke_0, ke_0\rangle_V - \langle e_0, ke_0\rangle_V) \cdot \langle w, w\rangle_W\right) \\ &= \psi_E\left(N_{E/F}(t)(1 - \overline{k_0})\langle w, w\rangle_W\right). \end{aligned}$$

Hence, for $t \in \mathfrak{p}_E^{-m}$, $w \in \Gamma_W$ and $k \in J^V_{2m}$, there exists $c \in \mathbb{C}^\times$ such that $\omega_\psi(k)\phi_{t,w} = c\phi_{t,w}$. This shows that $\mathcal{S}(A)_{M_{2m}}$ is stable by J^V_{2m}. Moreover, if $k_0 \in \mathfrak{p}_E^{2m}$ or $\langle w, w\rangle_W = 0$, then $c = 1$. Hence, we have $(\mathcal{S}(A)_{M_{2m}})^{K^V_{2m}} = \mathcal{S}(A)_{M_{2m}}$ and $(\mathcal{S}(A)_{M_{2m}})^{J^V_{2m}} \neq \{0\}$. □

Therefore, by Proposition 5.3, we see that $\mathcal{S}(A)^{J^V_{2m}}$ is generated by $(\mathcal{S}(A)_{M_{2m}})^{J^V_{2m}}$ as a representation of $\mathrm{U}(W)$. If $2m > 0$, then $\mathcal{S}(A)_{M_{2m}} \supset \mathcal{S}(A)_{M_{2m-2}}$. Let $\mathcal{S}(A)_{M_{2m} \setminus M_{2m-2}}$ be the subspace spanned by

$$\left\{\phi_{t,w} \;\middle|\; \mathrm{ord}(t) = -m,\ w \in \Gamma_W \setminus \varpi\Gamma_W\right\}.$$

Then we have

$$\mathcal{S}(A)_{M_{2m}} = \mathcal{S}(A)_{M_{2m-2}} \oplus \mathcal{S}(A)_{M_{2m} \setminus M_{2m-2}}.$$

Lemma 5.5. *Suppose that* $2m > 0$. *The image* $(\mathcal{S}(A)_{M_{2m}})^{J^V_{2m}}$ *under the projection* $\mathcal{S}(A)_{M_{2m}} \twoheadrightarrow \mathcal{S}(A)_{M_{2m} \setminus M_{2m-2}}$ *is equal to the one of the subspace spanned by*

$$\left\{\omega_\psi(k')\phi_{t,f_n} \;\middle|\; \mathrm{ord}(t) = -m,\ k' \in K_0^W\right\}.$$

Moreover, ϕ_{t,f_n} *is fixed by* K^W_{2m}, *and* $\phi_{t,f_{-n}}$ *is fixed by* ${}^tK^W_{2m}$.

Proof. As we have seen in the proof of Lemma 5.4, $k \in J^V_{2m}$ acts on $\phi_{t,w}$ by the character

$$\begin{array}{ccccc} J^V_{2m} & \twoheadrightarrow & 1 + \mathfrak{p}_E^m & \longrightarrow & \mathbb{C}^\times, \\ k & \longmapsto & k_0 & \longmapsto & \psi_E\left(N_{E/F}(t)(1 - \overline{k_0})\langle w, w\rangle_W\right). \end{array}$$

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Hence, the image in question is equal to the one of the subspace spanned by $\phi_{t,w}$ with $\mathrm{ord}(t) = -m$ and $w \in \Gamma_W \setminus \varpi\Gamma_W$ such that $\langle w, w\rangle_W \in \mathfrak{p}_E^m$. It means that

$$\langle w, w\rangle_W \equiv \langle f_n, f_n\rangle_W \bmod \mathfrak{p}_E^m.$$

Note that

$$K_0^W = \{g \in \mathrm{U}(V) \mid g\Gamma_W = \Gamma_W\}$$

is a hyperspecial maximal compact subgroup of $\mathrm{U}(W)$. Hence, there exists $k' \in K_0^W$ such that $w \equiv k' \cdot f_n \bmod \varpi^m\Gamma_W$. In particular, we have

$$te_0 \otimes w - te_0 \otimes k' \cdot f_n \in A.$$

Hence, we can find $c \in \mathbb{C}^\times$ such that $\phi_{t,w} = c\phi_{t,k'\cdot f_n} = c \cdot \omega_\psi(k')\phi_{t,f_n}$. This shows the first assertion.

Fix $k' \in K_{2m}^W$. Since $(k'-1)f_n \in \varpi^m\Gamma_W$, we have $(k'-1)(te_0 \otimes f_n) \in A$ for $t \in \mathfrak{p}_E^{-m}$. Hence, $\mathrm{Supp}(\omega_\psi(k')\phi_{t,f_n}) = \mathrm{Supp}(\phi_{t,f_n})$. Moreover,

$$\begin{aligned}\omega_\psi(k')\phi_{t,f_n}(te_0 \otimes f_n) &= \omega_\psi(k')\phi_{t,f_n}(k'(te_0 \otimes f_n) - (k'-1)(te_0 \otimes f_n))\\ &= \psi\left(\frac{1}{2}\langle (k'-1)(te_0 \otimes f_n), k'(te_0 \otimes f_n)\rangle\right)\omega_\psi(k')\phi_{t,f_n}(k'(te_0 \otimes f_n))\\ &= \psi_E\big(N_{E/F}(t)\langle (k'-1)f_n, k'f_n\rangle_W\big).\end{aligned}$$

Since $\langle (k'-1)f_n, k'f_n\rangle_W = \langle -f_n, k'f_n\rangle_W \in \mathfrak{p}_E^{2m}$, we have $\omega_\psi(k')\phi_{t,f_n}(te_0 \otimes f_n) = 1$. Therefore, we conclude that $\omega_\psi(k')\phi_{t,f_n} = \phi_{t,f_n}$ for $k' \in K_{2m}^W$. By a similar calculation, one can prove that $\omega_\psi(k')\phi_{t,f_{-n}} = \phi_{t,f_{-n}}$ for $k' \in {}^tK_{2m}^W$. This completes the proof. □

5.4. *Existence of K_{2m}^W-fixed vectors*

Let π be an irreducible ψ_E-generic tempered representation of $\mathrm{U}(W)$ with the L-parameter ϕ_π and the central character ω_π. Consider its theta lift $\sigma = \theta_\psi(\pi)$. It is an irreducible generic tempered representation of $\mathrm{U}(V)$ with L-parameter $\phi_\sigma = \phi_\pi\chi \oplus \mathbf{1}$. In particular, $c(\phi_\sigma) = c(\phi_\pi)$ so that $\sigma^{K_{2m}^V} \neq 0$ for $2m \geq c(\phi_\pi)$ by Theorem 2.1. Since $\omega_\sigma = \omega_\pi$, we see that $\sigma^{J_{2m}^V} \neq 0$ if $2m \geq c(\phi_\pi)$ and $\omega_\pi|_{1+\mathfrak{p}_E^m} = \mathbf{1}$.

Set $\omega_\psi = \omega_{\psi,V,W}$. By the definition of theta lifts, we have a $\mathrm{U}(V)\times\mathrm{U}(W)$-equivariant surjective map

$$\Phi\colon \omega_\psi \twoheadrightarrow \sigma \boxtimes \pi.$$

Proposition 5.6. *Set $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi) + 1$. Suppose that $2m > 0$ and that ω_π is trivial on $1 + \mathfrak{p}_E^m$. For any sign $\epsilon \in \{\pm 1\}$, there exists $t \in \mathfrak{p}_E^{-m}$ such that $\Phi(\phi_{t,f_{\epsilon n}}) \neq 0$. In particular, $\pi^{K_{2m}^W} \neq 0$.*

Proof. We realize ω_ψ on the lattice model $\mathcal{S}(A)$. Since $\Sigma \mapsto \Sigma^{J_{2m}^V}$ is an exact functor on the category of smooth representations Σ of $\mathrm{U}(V)$, we obtain a $\mathrm{U}(W)$-equivariant surjective map

$$\Phi\colon \mathcal{S}(A)^{J_{2m}^V} \twoheadrightarrow \sigma^{J_{2m}^V} \boxtimes \pi.$$

By Proposition 5.3 together with Lemma 5.4, its restriction to $(\mathcal{S}(A)_{M_{2m}})^{J_{2m}^V}$ is still nonzero. Since $\sigma^{K_{2m-2}^V} = 0$, this map factors through the restriction of the projection $\mathcal{S}(A)_{M_{2m}} \twoheadrightarrow \mathcal{S}(A)_{M_{2m}\setminus M_{2m-2}}$. Hence, by Lemma 5.5, there exists $t \in E^\times$ with $\mathrm{ord}(t) = -m$ such that $\Phi(\phi_{t,f_n}) \neq 0$. Since ϕ_{t,f_n} is fixed by $J_{2m}^V \times K_{2m}^W$, we have $\Phi(\phi_{t,f_n}) \in \sigma^{J_{2m}^V} \boxtimes \pi^{K_{2m}^W}$ so that $\pi^{K_{2m}^W} \neq 0$. By the same argument, one can show that $\Phi(\phi_{t,f_{-n}}) \neq 0$ for some $t \in \mathfrak{p}_E^{-m}$. □

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5.5. *Proof of Theorem 2.2 (3)*

The goal of the rest of this section is to show that $\pi_\psi^{K_{2m}^W} \neq 0$ if $2m = c(\phi_\pi)$ or $2m = c(\phi_\pi)+1$ and if ω_π is trivial on $1+\mathfrak{p}_E^m$. If $2m = c(\phi_\pi) = 0$, then π is unramified (with respect to the hyperspecial maximal compact subgroup K_0^W), and the Casselman–Shalika formula [6] shows that $\pi_\psi^{K_0^W} \neq 0$. See Remark 4.1. Hence, we may assume that $c(\phi_\pi) > 0$ so that $2m > 0$.

We need further notations. Set

$$X = \bigoplus_{i=1}^n Ee_i, \quad V_0 = Ee_0, \quad X^* = \bigoplus_{i=1}^n Ee_{-i}.$$

Hence, $V = X \oplus V_0 \oplus X^*$. For $a \in \mathrm{GL}(X)$, $b \in \mathrm{Hom}(V_0, X)$ and $c \in \mathrm{Hom}(X^*, X)$, we define $a^* \in \mathrm{GL}(X^*)$, $b^* \in \mathrm{Hom}(X^*, V_0)$ and $c^* \in \mathrm{Hom}(X^*, X)$ so that

$$\langle ax, x'\rangle_V = \langle x, a^*x'\rangle_V, \quad \langle be_0, x'\rangle_V = \langle e_0, b^*x'\rangle_V, \quad \langle cx', x''\rangle_V = \langle x', c^*x''\rangle_V$$

for $x \in X$ and $x', x'' \in X^*$. For $a \in \mathrm{GL}(X)$, $b \in \mathrm{Hom}(V_0, X)$ and

$$c \in \mathrm{Herm}(X^*, X) = \{c \in \mathrm{Hom}(X^*, X) \mid c^* = -c\},$$

we put

$$\mathbf{m}_X(a) = \begin{pmatrix} a & & \\ & \mathbf{1}_{V_0} & \\ & & (a^*)^{-1} \end{pmatrix},$$
$$\mathbf{n}_1(b) = \begin{pmatrix} \mathbf{1}_X & b & -\frac{1}{2}bb^* \\ & \mathbf{1}_{V_0} & b^* \\ & & \mathbf{1}_{X^*} \end{pmatrix},$$
$$\mathbf{n}_2(c) = \begin{pmatrix} \mathbf{1}_X & & c \\ & \mathbf{1}_{V_0} & \\ & & \mathbf{1}_{X^*} \end{pmatrix}.$$

These are elements in $\mathrm{U}(V)$.

Similarly, set

$$Y = \bigoplus_{i=1}^n Ef_i, \quad Y^* = \bigoplus_{i=1}^n Ef_{-i}$$

so that $W = Y \oplus Y^*$. For $a \in \mathrm{GL}(Y)$ and $c \in \mathrm{Hom}(Y^*, Y)$, we define $a^* \in \mathrm{GL}(X^*)$ and $c^* \in \mathrm{Hom}(Y^*, Y)$ so that

$$\langle ay, y'\rangle_W = \langle y, a^*y'\rangle_W, \quad \langle cy', y''\rangle_W = \langle y', c^*y''\rangle_W$$

for $y \in Y$ and $y', y'' \in Y^*$. For $a \in \mathrm{GL}(Y)$ and

$$c \in \mathrm{Herm}(Y^*, Y) = \{c \in \mathrm{Hom}(Y^*, Y) \mid c^* = -c\},$$

we put

$$\mathbf{m}_Y(a) = \begin{pmatrix} a & \\ & (a^*)^{-1} \end{pmatrix}, \quad \mathbf{n}(c) = \begin{pmatrix} \mathbf{1}_Y & c \\ & \mathbf{1}_{Y^*} \end{pmatrix}.$$

These are elements in $\mathrm{U}(W)$.

Define $a_\delta \in \mathrm{GL}(X)$ by

$$a_\delta \colon e_i \mapsto \delta^{-i} e_i$$

for $-n \le i \le n$. If we fix a nonzero Whittaker functional $l_\sigma \in \mathrm{Hom}_{N_{2n+1}}(\sigma, \psi_E)$ for σ, then $l'_\sigma = l_\sigma \circ \sigma(\mathbf{m}_X(a_\delta))$ is a nonzero Whittaker functional with respect to the character $\psi_E^\delta \colon N_{2n+1} \to \mathbb{C}^\times$ given by

$$\psi_E^\delta(u) = \psi_E\left(\delta^{-1} \sum_{i=1}^{n} \langle u e_{i-1}, e_{-i}\rangle_V\right).$$

This is the generic character considered in [8].

Now we fix $t \in E^\times$ with $\mathrm{ord}(t) = -m$ such that $\Phi(\phi_{t,f_{-n}}) \neq 0$. This belongs to $\sigma^{J_{2m}^V} \boxtimes \pi$. Note that $\mathbf{m}_X(t \cdot \mathbf{1}_X) K_{2m}^V \mathbf{m}_X(t \cdot \mathbf{1}_X)^{-1}$ is the compact group $K_{n,2m}$ considered in [8]. In particular, the Whittaker functional

$$l'_{\sigma,t} = l'_\sigma \circ \sigma(\mathbf{m}_X(t \cdot \mathbf{1}_X)) = l_\sigma \circ \sigma(\mathbf{m}_X(a_\delta t))$$

with respect to

$$\psi_{E,t}^\delta \colon N_{2n+1} \ni u \mapsto \psi_E^\delta(\mathbf{m}_X(t \cdot \mathbf{1}_X) \cdot u \cdot \mathbf{m}_X(t \cdot \mathbf{1}_X)^{-1}) \in \mathbb{C}^\times$$

is nonzero on $\sigma^{K_{2m}^V}$ by [8, Theorem 1.4, Lemma 7.5]. Therefore, the image $\phi_{t,f_{-n}}$ under the composition of $N_{2n+1} \times \mathrm{U}(W)$-equivariant maps

$$\omega_\psi \xrightarrow{\Phi} \sigma \boxtimes \pi \xrightarrow{l'_{\sigma,t} \otimes \mathrm{id}} \psi_{E,t}^\delta \boxtimes \pi$$

is nonzero.

By the same argument as the proof of [18, Proposition 2.3], one can prove that the maximal quotient of ω_ψ on which N_{2n+1} acts by $\psi_{E,t}^\delta$ is isomorphic to the compact induction $\mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu)$, where N'_{2n} is the unipotent radical of the Borel subgroup of $\mathrm{U}(W)$ stabilizing the flag

$$Ef_1 \subset Ef_1 \oplus Ef_2 \subset \cdots \subset Ef_1 \oplus \cdots \oplus Ef_n = Y,$$

and μ is a character of N'_{2n} given by

$$\mu(u) = \psi_E\left(\sum_{i=1}^{n} \langle u f_{i+1}, f_{-i}\rangle + N_{E/F}(t)\langle u f_{-n}, f_{-n}\rangle\right).$$

Here, we note that N'_{2n} differs from N_{2n} defined in Section 2.2.

Hence, the map

$$\omega_\psi \xrightarrow{\Phi} \sigma \boxtimes \pi \xrightarrow{l'_{\sigma,t} \otimes \mathrm{id}} \psi_{E,t}^\delta \boxtimes \pi$$

factors through $\omega_\psi \to \mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu)$. Namely, we have a nonzero $\mathrm{U}(W)$-equivariant map

$$\mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu) \to \pi.$$

The following is a key lemma, which will be proven in Section 5.7 below.

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Lemma 5.7. *Let* $\widetilde{F}_{t,f_{-n}} \in \mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu)$ *be the image of* $\phi_{t,f_{-n}} \in \mathcal{S}(A)$*. Then* $\widetilde{F}_{t,f_{-n}}$ *is right* ${}^tK_{2m}^W$*-invariant and*

$$\mathrm{Supp}(\widetilde{F}_{t,f_{-n}}) = N'_{2n} \cdot {}^tK_{2m}^W.$$

Note that having a $\mathrm{U}(W)$-equivariant map

$$\mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu) \to \pi$$

is equivalent to giving a $\mathrm{U}(W)$-equivariant map

$$\pi^\vee \to \mathrm{Ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu^{-1}).$$

These are related as follows. Suppose that $\mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu) \ni \widetilde{F} \mapsto v \in \pi$ corresponds to $\pi^\vee \ni v' \mapsto W \in \mathrm{Ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu^{-1})$. Then

$$(v, v')_\pi = \int_{N'_{2n}\backslash \mathrm{U}(W)} \widetilde{F}(g)W(g)dg.$$

By Lemma 5.7, there exists $\widetilde{F} \in \mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu)$ such that

- its image v in π is nonzero;
- $\widetilde{F}$ is right ${}^tK_{2m}^W$-invariant;
- $\mathrm{Supp}(\widetilde{F}) = N'_{2n} \cdot {}^tK_{2m}^W$.

Hence, $v \in \pi^{{}^tK_{2m}^W}$. One can take $v' \in (\pi^\vee)^{{}^tK_{2m}^W}$ such that $(v, v')_\pi \neq 0$. Let $W \in \mathrm{Ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu^{-1})$ be the image of v'. Then W is right ${}^tK_{2m}^W$-invariant, and

$$0 \neq (v, v')_\pi = \int_{N'_{2n}\backslash \mathrm{U}(W)} \widetilde{F}(g)W(g)dg = c\widetilde{F}(\mathbf{1})W(\mathbf{1})$$

for some constant $c > 0$. Hence, $W(\mathbf{1}) \neq 0$. Moreover, since $\mathbf{v}(0,0;z) \in N'_{2n}$, we have

$$W(\mathbf{v}(0,0;z)) = \mu^{-1}(\mathbf{v}(0,0;z))W(\mathbf{1}) = \psi^{-1}(N_{E/F}(t)z)W(\mathbf{1})$$

for $z \in F$. Therefore, via $v' \mapsto W \mapsto W(\mathbf{1})$, we conclude that

$$(\pi^\vee)_{\psi'^{-1}}^{{}^tK_{2m}^W} \neq 0,$$

where we put $\psi'(z) = \psi(N_{E/F}(t)z)$. Since

$${}^tK_{2m}^W = \begin{pmatrix} t & & \\ & \mathbf{1}_{2n-2} & \\ & & \bar{t}^{-1} \end{pmatrix}^{-1} K_{2m}^W \begin{pmatrix} t & & \\ & \mathbf{1}_{2n-2} & \\ & & \bar{t}^{-1} \end{pmatrix},$$

as in Section 2.3, we have

$$(\pi^\vee)_{\psi^{-1}}^{K_{2m}^W} \cong (\pi^\vee)_{\psi'^{-1}}^{{}^tK_{2m}^W} \neq 0.$$

Since π is ψ_E-generic if and only if $\pi^\vee$ is ψ_E^{-1}-generic, by replacing π and ψ with $\pi^\vee$ and ψ^{-1}, respectively, we conclude that

$$\pi_\psi^{K_{2m}^W} \neq 0.$$

Therefore, Theorem 2.2 (3) is reduced to proving Lemma 5.7.

5.6. *Mixed model*

To show Lemma 5.7, we review the argument in the proof of [18, Proposition 2.3]. For this, we use another model of the Weil representation $\omega_\psi = \omega_{\psi,V,W}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$. It is known that the Weil representation ω_ψ can be realized on the space $\mathcal{S}(X^* \otimes W) \otimes \mathcal{S}(V_0 \otimes Y^*)$, which is called a *mixed model*. See, for example, [10, Section 7.4]. Let us recall some formulas for the action of $\mathrm{U}(V) \times \mathrm{U}(W)$ on this space.

For $\varphi_1 \otimes \varphi_2 \in \mathcal{S}(X^* \otimes W) \otimes \mathcal{S}(V_0 \otimes Y^*)$ and $(x, y) \in (X^* \otimes W) \times (V_0 \otimes Y^*)$,

$$\begin{aligned}
&\omega_\psi(g)(\varphi_1 \otimes \varphi_2)(x, y) = \varphi_1(g^{-1}x) \cdot \omega_\psi^0(g)\varphi_2(y), \quad g \in \mathrm{U}(W),\\
&\omega_\psi(h_0)(\varphi_1 \otimes \varphi_2)(x, y) = \varphi_1(x) \cdot \omega_\psi^0(h_0)\varphi_2(y), \quad h_0 \in \mathrm{U}(V_0),\\
&\omega_\psi(\mathbf{m}_X(a))(\varphi_1 \otimes \varphi_2)(x, y) = \chi_W(\det a)|\det a|^n \varphi_1(a^*x) \cdot \varphi_2(y), \quad a \in \mathrm{GL}(X),\\
&\omega_\psi(\mathbf{n}_1(b))(\varphi_1 \otimes \varphi_2)(x, y) = \varphi_1(x) \cdot \rho_\psi^0(b^*x, 0)\varphi_2(y), \quad b \in \mathrm{Hom}(V_0, X),\\
&\omega_\psi(\mathbf{n}_2(c))(\varphi_1 \otimes \varphi_2)(x, y) = \psi\left(\frac{1}{2}\langle cx, x\rangle\right)\varphi_1(x) \cdot \varphi_2(y), \quad c \in \mathrm{Herm}(X^*, X).
\end{aligned}$$

Here, $\mathcal{S}(V_0 \otimes Y^*)$ is regarded as the Schrödinger model of

- the irreducible representation ρ_ψ^0 of the Heisenberg group $H(V_0 \otimes W)$ on $\mathcal{S}(V_0 \otimes Y^*)$ with the central character ψ; and
- the Weil representation ω_ψ^0 of $\mathrm{U}(V_0) \times \mathrm{U}(W)$.

Hence, for $\varphi_2 \in \mathcal{S}(V_0 \otimes Y^*)$ and $y \in V_0 \otimes Y^*$, we have

$$\rho_\psi^0((y_+ + y_-, t))\varphi_2(y) = \psi\left(t + \langle y, y_+\rangle + \frac{1}{2}\langle y_-, y_+\rangle\right)\varphi_2(y + y_-)$$

for $y_+ \in V_0 \otimes Y$ and $y_- \in V_0 \otimes Y^*$, and

$$\begin{aligned}
&\omega_\psi^0(\mathbf{m}_Y(a))\varphi_2(y) = \chi(\det a)|\det a|^{\frac{1}{2}}\varphi(a^*y), \quad a \in \mathrm{GL}(Y),\\
&\omega_\psi^0(\mathbf{n}(c))\varphi_2(y) = \psi\left(\frac{1}{2}\langle cy, y\rangle\right)\varphi_2(y) \quad c \in \mathrm{Herm}(Y^*, Y).
\end{aligned}$$

Moreover, $\omega_\psi^0(J_{2n})\varphi_2$ is given by a Fourier transform of φ_2. For more precision, see [10, Section 7.4].

For $\varphi_1 \otimes \varphi_2 \in \mathcal{S}(X^* \otimes W) \otimes \mathcal{S}(V_0 \otimes Y^*)$, define

$$F_{\varphi_1 \otimes \varphi_2}(g) = \varphi_1(g^{-1}x_0) \cdot \omega_\psi^0(g)\varphi_2(y_0),$$

where we set

$$x_0 = \sum_{i=1}^{n} \frac{1}{2\delta} e_{-i} \otimes f_{n+1-i}, \quad y_0 = te_0 \otimes f_{-n}.$$

Let $Q_{2n} = M_{2n,S}N_{2n,S}$ be the Siegel parabolic subgroup of $\mathrm{U}(W)$ stabilizing Y, where $M_{2n,S} = \{\mathbf{m}_Y(a) \mid a \in \mathrm{GL}(Y)\}$ is its Levi subgroup, and $N_{2n,S}$ is its unipotent radical. Note that $N_{2n,S} \subset N'_{2n}$.

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We regard μ as a character of $N_{2n,S}$ by the restriction. For $u \in N_{2n,S}$, since $u^{-1}x_0 = x_0$ and

$$\begin{aligned}\psi\left(\frac{1}{2}\langle uy_0, y_0\rangle\right) &= \psi_E\left(\langle te_0, te_0\rangle_V \langle uf_{-n}, f_{-n}\rangle_W\right)\\ &= \psi_E\left(N_{E/F}(t)\langle uf_{-n}, f_{-n}\rangle_W\right) = \mu(u),\end{aligned}$$

we see that $F_{\varphi_1\otimes\varphi_2}(g) \in \mathrm{ind}_{N_{2n,S}}^{\mathrm{U}(W)}(\mu)$. Note that $\mathbf{n}_2(c)$ acts trivially on $F_{\varphi_1\otimes\varphi_2}$ for $c \in \mathrm{Herm}(X^*, X)$ since Y is totally isotropic. However, for $b \in \mathrm{Hom}(V_0, X)$, since $\mathbf{n}_1(b)$ commutes with $g \in \mathrm{U}(W)$, we see that

$$\begin{aligned}F_{\omega_\psi(\mathbf{n}_1(b))(\varphi_1\otimes\varphi_2)}(g) &= \omega_\psi(g) \circ \omega_\psi(\mathbf{n}_1(b))(\varphi_1 \otimes \varphi_2)(x_0, y_0)\\ &= \omega_\psi(\mathbf{n}_1(b)) \circ \omega_\psi(g)(\varphi_1 \otimes \varphi_2)(x_0, y_0)\\ &= \rho_\psi^0(b^*x_0, 0) \circ \omega_\psi(g)(\varphi_1 \otimes \varphi_2)(x_0, y_0)\\ &= \psi(\langle y_0, b^*x_0\rangle)\omega_\psi(g)(\varphi_1 \otimes \varphi_2)(x_0, y_0).\end{aligned}$$

Since $\overline{\delta} = -\delta$ and $\langle f_{-n}, f_n\rangle_W = -1$, we have

$$\begin{aligned}\psi(\langle y_0, b^*x_0\rangle) &= \psi_E\left(\sum_{i=1}^{n}\left\langle te_0, b^*\delta^{-1}e_{-i}\right\rangle_V \langle f_{-n}, f_{n+i-1}\rangle_W\right)\\ &= \psi_E(\delta^{-1}t\langle be_0, e_{-1}\rangle_V) = \psi_{E,t}^{\delta}(\mathbf{n}_1(b)).\end{aligned}$$

Hence, $\mathbf{n}_1(b)$ acts on $F_{\varphi_1\otimes\varphi_2}$ by $\psi_{E,t}^{\delta}$.

Define a map

$$\mathrm{ind}_{N_{2n,S}}^{\mathrm{U}(W)}(\mu) \longrightarrow \mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu)$$

by

$$F \mapsto \widetilde{F}(g) = \int_{N_{2n,S}\backslash N'_{2n}} F(ug)\mu(u)^{-1}du.$$

Then by the same argument as in [18, Proposition 2.3], one can prove that the map $\varphi_1 \otimes \varphi_2 \mapsto \widetilde{F}_{\varphi_1\otimes\varphi_2}$ realizes an isomorphism between the maximal quotient of ω_ψ on which N_{2n+1} acts by $\psi_{E,t}^{\delta}$ and $\mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu)$.

5.7. *Proof of Lemma 5.7*

In this subsection, we prove Lemma 5.7. To do this, we relate two models of the Weil representation.

Let $\varphi_1^0 \in \mathcal{S}(X^* \otimes W)$ and $\varphi_2^0 \in \mathcal{S}(V_0 \otimes Y^*)$ be the characteristic functions of

$$\left(\bigoplus_{i=1}^{n} \mathfrak{o}_E e_{-i}\right) \otimes \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E f_i \oplus \bigoplus_{i=1}^{n} \mathfrak{o}_E f_{-i}\right), \quad \mathfrak{o}_E e_0 \otimes \left(\bigoplus_{i=1}^{n} \mathfrak{o}_E f_{-i}\right),$$

respectively. Then the action $\rho = \rho_\psi$ of the Heisenberg group $H(\mathbb{W})$ on $\varphi_1^0 \otimes \varphi_2^0$ satisfies that

$$\rho(a,t)(\varphi_1^0 \otimes \varphi_2^0) = \psi(t) \cdot \varphi_1^0 \otimes \varphi_2^0$$

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for $(a,t) \in A \oplus F$. Moreover, the lattice model $\mathcal{S}(A)$ and the mixed model $\mathcal{S}(X^* \otimes W) \otimes \mathcal{S}(V_0 \otimes Y^*)$ are related by the isomorphism

$$\mathcal{S}(A) \xrightarrow{\sim} \mathcal{S}(X^* \otimes W) \otimes \mathcal{S}(V_0 \otimes Y^*),$$
$$\phi \mapsto \int_{(A\oplus F)\backslash H(\mathbb{W})} \phi(h)\rho(h)^{-1}(\varphi_1^0 \otimes \varphi_2^0)(x,y)dh.$$

In particular, $\phi_{t,f_{-n}} \in \mathcal{S}(A)$ corresponds to

$$\rho(te_0 \otimes f_{-n}, 0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)(x,y) = \varphi_1^0(x) \cdot \rho_\psi^0(te_0 \otimes f_{-n}, 0)^{-1}\varphi_2^0(y)$$

in $\mathcal{S}(X^* \otimes W) \otimes \mathcal{S}(V_0 \otimes Y^*)$ since $\mathrm{Supp}(\phi_{t,f_{-n}}) = (A + te_0 \otimes f_{-n}) \oplus F$. Therefore, under the map

$$\mathcal{S}(A) \longrightarrow \mathrm{ind}_{N'_{2n}}^{\mathrm{U}(W)}(\mu)$$

obtained above, the image of $\phi_{t,f_{-n}}$ is $\widetilde{F}_{\rho(te_0\otimes f_{-n},0)^{-1}(\varphi_1^0\otimes\varphi_2^0)}$.

Now we prove Lemma 5.7.

Proof of Lemma 5.7. First, we consider $F_{\rho(te_0\otimes f_{-n},0)^{-1}(\varphi_1^0\otimes\varphi_2^0)}$. Note that it is left $N_{2n,S}$-invariant and right ${}^tK_{2m}^W$-invariant. We claim that if

$$F_{\rho(te_0\otimes f_{-n},0)^{-1}(\varphi_1^0\otimes\varphi_2^0)}(g) \neq 0,$$

then

$$g \in N_{2n,S} \cdot \mathbf{m}_Y(a) \cdot {}^tK_{2m}^W$$

for some $a \in \mathrm{GL}(Y) \cong \mathrm{GL}_n(E)$ such that $a^{-1} \in \mathrm{M}_n(\mathfrak{o}_E)$ and

$$a^* f_{-n} - f_{-n} \in \bigoplus_{i=1}^n \mathfrak{p}_E^m f_{-i}.$$

By the Iwasawa decomposition, we have $\mathrm{U}(W) = Q_{2n}K_0^W$. Let K_S and K_M be subgroups of K_0^W defined by

$$K_S = \begin{matrix} & \begin{matrix} n & n \end{matrix} \\ \begin{matrix} n \\ n \end{matrix} & \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \end{matrix} \cap \mathrm{U}_{2n}, \quad K_M = \begin{matrix} & \begin{matrix} 1 & 2n-2 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 2n-2 \\ 1 \end{matrix} & \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \end{matrix} \cap \mathrm{U}_{2n}.$$

By the Bruhat decomposition for a finite unitary group over $\mathfrak{o}_F/\mathfrak{p}_F$, we have

$$\begin{aligned} K_0^W &= K_S K_M \cup K_S J_{2n}^{-1} K_M \\ &= (K_S \cap Q_{2n})K_M \cup (K_S \cap Q_{2n})J_{2n}^{-1}K_M. \end{aligned}$$

Since $J_{2n} \in K_0^W$ and $J_{2n}^{-1}K_M J_{2n} = {}^tK_M$, by the multiplication of J_{2n} from the right, we have

$$K_0^W = (K_S \cap Q_{2n})J_{2n}{}^tK_M \cup (K_S \cap Q_{2n}){}^tK_M.$$

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Hence,

$$\mathrm{U}(W) = Q_{2n} J_{2n} {}^t K_M \cup Q_{2n} {}^t K_M = N_{2n,S} M_{2n,S} J_{2n} {}^t K_M \cup N_{2n,S} M_{2n,S} {}^t K_M.$$

Therefore, we may assume that $g = \mathbf{m}_Y(a) J_{2n} k$ or $g = \mathbf{m}_Y(a) k$ for some $a \in \mathrm{GL}(Y)$ and $k \in {}^t K_M$.

Assume that $g = \mathbf{m}_Y(a) J_{2n} k$ is in the former case. Since φ_1^0 and φ_2^0 are fixed by K_0^W, and since $\omega_\psi^0(g) \circ \rho_\psi^0(h) \circ \omega_\psi^0(g)^{-1} = \rho_\psi^0(gh)$, we have

$$\begin{aligned} F_{\rho(te_0 \otimes f_{-n},0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}(g) &= \omega_\psi(g) \circ \rho(te_0 \otimes f_{-n}, 0)^{-1} (\varphi_1^0 \otimes \varphi_2^0)(x_0, y_0) \\ &= \varphi_1(g^{-1} x_0) \cdot \omega_\psi^0(g) \rho_\psi^0(te_0 \otimes f_{-n}, 0)^{-1} \varphi_2^0(y_0) \\ &= \varphi_1(a^{-1} x_0) \cdot \omega_\psi^0(\mathbf{m}_Y(a)) \rho_\psi^0(te_0 \otimes J_{2n} k f_{-n}, 0)^{-1} \varphi_2^0(y_0). \end{aligned}$$

Note that $\varphi_1(a^{-1} x_0) \neq 0$ if and only if $a^{-1} \in \mathrm{M}_n(\mathfrak{o}_E)$. However, since $k \in {}^t K_M$, if we write $J_{2n} k f_{-n} = y + y^*$ with $y \in Y$ and $y^* \in Y^*$, then $y^* \in \oplus_{i=1}^n \mathfrak{p}_E f_{-i}$. Up to a nonzero constant, $\omega_\psi^0(\mathbf{m}_Y(a)) \rho_\psi^0(te_0 \otimes J_{2n} k f_{-n}, 0)^{-1} \varphi_2^0(y_0)$ is equal to

$$\varphi_2^0(te_0 \otimes (a^* f_{-n} - y^*)).$$

If this is nonzero, then we must have $t(a^* f_{-n} - y^*) \in \oplus_{i=1}^n \mathfrak{o}_E f_{-i}$. When $a^{-1} \in \mathrm{M}_n(\mathfrak{o}_E)$, this implies that

$$f_{-n} \in (a^*)^{-1} y^* + \bigoplus_{i=1}^n \mathfrak{p}_E^m f_{-i} \subset \bigoplus_{i=1}^n \mathfrak{p}_E f_{-i}.$$

This is impossible. Hence, we have $F_{\rho(te_0 \otimes f_{-n},0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}(g) = 0$.

Next, we assume that $g = \mathbf{m}_Y(a) k$ is in the latter case. By the Iwahori decomposition, we may further assume that $k f_{-n} = f_{-n}$. Then

$$\begin{aligned} F_{\rho(te_0 \otimes f_{-n},0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}(g) &= \omega_\psi(g) \circ \rho(te_0 \otimes f_{-n}, 0)^{-1} (\varphi_1^0 \otimes \varphi_2^0)(x_0, y_0) \\ &= \varphi_1(g^{-1} x_0) \cdot \omega_\psi^0(g) \rho_\psi^0(te_0 \otimes f_{-n})^{-1} \varphi_2^0(y_0) \\ &= \varphi_1(a^{-1} x_0) \cdot \omega_\psi^0(\mathbf{m}_Y(a)) \rho_\psi^0(te_0 \otimes k f_{-n})^{-1} \varphi_2^0(y_0) \\ &= \varphi_1(a^{-1} x_0) \cdot \omega_\psi^0(\mathbf{m}_Y(a)) \rho_\psi^0(te_0 \otimes f_{-n})^{-1} \varphi_2^0(y_0). \end{aligned}$$

Up to a nonzero constant, it is equal to

$$\varphi_1(a^{-1} x_0) \cdot \varphi_2^0(te_0 \otimes (a^* f_{-n} - f_{-n})).$$

If this is nonzero, then $a^{-1} \in \mathrm{M}_n(\mathfrak{o}_E)$ and

$$a^* f_{-n} - f_{-n} \in \bigoplus_{i=1}^n \mathfrak{p}_E^m f_{-i}.$$

This proves the claim.

Now we consider $\widetilde{F}_{\rho(te_0 \otimes f_{-n},0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}$. Note that it is left N'_{2n}-invariant and right ${}^t K_{2m}^W$-invariant. Suppose that $\widetilde{F}_{\rho(te_0 \otimes f_{-n},0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}(g) \neq 0$. By the claim, we may assume that $g = \mathbf{m}_Y(a)$ with $a \in \mathrm{GL}(Y)$ satisfying the conditions in the claim. By the Iwasawa decomposition, we may further assume that $a = a_d a_0$ such that

- $\langle a_d f_i, f_{-j} \rangle_W = \varpi^{\lambda_i} \delta_{i,j}$ for some $\lambda_i \in \mathbb{Z}$;
- $a_0 \in \mathrm{GL}_n(\mathfrak{o}_E)$ via $\mathrm{GL}(Y) \cong \mathrm{GL}_n(E)$.

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Since $a^{-1} \in \mathrm{M}_n(\mathfrak{o}_E)$, we have $\lambda_i \leq 0$ for $1 \leq i \leq n$. Note that

$$a^* f_{-n} - f_{-n} = a_0^* a_d^* f_{-n} - f_{-n} = a_0^* \varpi^{\lambda_n} f_{-n} - f_{-n}.$$

Since this is in $\oplus_{i=1}^n \mathfrak{p}_E^m f_{-i}$, we have $\lambda_n = 0$ and $\mathbf{m}_Y(a_0) \in {}^t K_{2m}^W$. Hence, we may assume that $a_0 = \mathbf{1}_X$ (i.e., $g = \mathbf{m}_Y(a_d)$). For $2 \leq i \leq n$ and $x \in \mathfrak{o}_E$, define $u_i \in N'_{2n}$ so that

$$u_i f_j - f_j = \begin{cases} x \cdot f_{i-1} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Then $u_i \in {}^t K_{2m}^W$. Hence,

$$\begin{aligned} 0 &\neq \widetilde{F}_{\rho(te_0 \otimes f_{-n}, 0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}(\mathbf{m}_Y(a_d)) \\ &= \widetilde{F}_{\rho(te_0 \otimes f_{-n}, 0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}(\mathbf{m}_Y(a_d) u_i) \\ &= \mu(\mathbf{m}_Y(a_d) u_i \mathbf{m}_Y(a_d)^{-1}) \widetilde{F}_{\rho(te_0 \otimes f_{-n}, 0)^{-1}(\varphi_1^0 \otimes \varphi_2^0)}(\mathbf{m}_Y(a_d)) \end{aligned}$$

so that $\mu(\mathbf{m}_Y(a_d) u_i \mathbf{m}_Y(a_d)^{-1}) = 1$ for any $x \in \mathfrak{o}_E$. Note that

$$\begin{aligned} \mu(\mathbf{m}_Y(a_d) u_i \mathbf{m}_Y(a_d)^{-1}) &= \psi_E(\langle \mathbf{m}_Y(a_d) u_i \mathbf{m}_Y(a_d)^{-1} f_i, f_{-i+1} \rangle) \\ &= \psi_E(\varpi^{\lambda_{i-1} - \lambda_i} x). \end{aligned}$$

Hence, $\psi_E(\varpi^{\lambda_{i-1} - \lambda_i} x) = 1$ for any $x \in \mathfrak{o}_E$. This implies that $\lambda_{i-1} \geq \lambda_i$. In conclusion, we have

$$0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$$

so that $\lambda_1 = \cdots = \lambda_n = 0$. This means that $a_d = \mathbf{1}_X$. This completes the proof of Lemma 5.7. □

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