**Fano bundles on the projective space with fiber bundle structure**
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**Introduction. 0**

All varieties (manifolds) we consider here are smooth complex projective varieties although almost facts hold over the algebrically closed fields in any characteristic. In this report an $m$-dimensional homotopy complex projective space means a smooth projective variety which is homotopy equivalent to an $m$-dimensional complex projective space, sometimes written as $\bar{P}^m$ for simplicity .

(0.0) Let us consider a smooth projective variety with two projective space bundle structures.

Let $X,\ Y_1,\ Y_2$ be smooth projective varieties with $\dim Y_1 = m_1 \geq 1,\ \dim Y_2 = m_2 \geq 1$. Then we say that X has two projective space bundle structures when $X$ has the following conditions: (PS: $(Y_1, P^r, p), (Y_2, P^s, q)$) : $p : X \to Y_1$ is $P^r$-bundle ($r \geq 1$) over $Y_1$, and $q : X \to Y_2$ a $P^s$-bundle ($s \geq 1$) over $Y_2$ in etale topology where the image of the natural morphism $\theta = (p, q) : X \to Y_1 \times Y_2$ is of dimension $> max\{m_1, m_2\}$, Then such $X$ is written simply as $X$ with (PS).

**Assumption 0.1** Here it is supposed that $(1 \leq) r \leq m_2$ and $(1 \leq) s \leq m_1$.

In this report we state two Thereoms which are joint-work with Kachi Yasuyuki.

**Main Theorem (\$.3 $\sim$ \$.6)** Let $X$ be a smooth projective variety with two projective space bundle structures $(PS : (Y_1, P^r, p), (Y_2, P^s, q))$ and $\dim Y_i = m_i (i = 1,\ 2)$ with the Assumption 0.1. Suppose that $Y_1$ is an $m_1$- dimensional "complex projective space". Then $X$ has one of the four types:

(1) $P^{m_1} \times P^{m_2}$ $\quad (m_1 = s,\ m_2 = r)$

(2) $X = P_{P^n}(T_{P^n}(-1))$ where $Y_2$ is a complex projective space $P^m$. $(m_1 = m_2 = r + 1 = s + 1 = n)$

(3) Example 3 in section 2.

$$\begin{array}{ccc} P_{P^n}(\Omega_{P^n}(2)) = X = P_{Gr(n.1)}(U) & \xrightarrow[P^1-bundle]{q} & Gr(n.1) = Y_2 \\ p \big\downarrow P^{n-1}-bundle & & \\ Y_1 = P^n & & \end{array}$$

Here $Gr(n.1)$ is a Grassmann variety parameterizing lines in $P^n$ and $U$ the universal quotient vector bundle of rank 2 on $Gr(n.1)$.

(4) Example 4 in section 2. $n(\geq 3)$ is odd.

$$\begin{array}{ccc} P_{P^n}(N1) = X = P_{Y_2}(U|_{Y_2}) & \xrightarrow[P^1-bundle]{q} & Y_2 \\ p \big\downarrow P^{n-2}-bundle & & \\ Y_1 = P^n & & \end{array}$$

Here $Y_2$ is a smooth hypersection divisor in $Gr(n.1)$ with $Y_2 \in |\mathcal{O}_{Gr(n.1)}(1)|$. $N1 := N \otimes \mathcal{O}_{P^n}(1)$ is a vector bundle of rank n-1 on $P^n$ satisfying the following exact sequence: $0 \to \mathcal{O} \to \Omega_{P^n}(2) \to N1 \to 0$.

**Remark 0.0**

1) For the proof of Main Thereom we first prove Theorem 1 stated below. The proof of Main Theorem is divided to two cases: the one is about that $Y_2$ is a homotopy complex projective space and the other about $Y_2$ is not. The former corresponds to 1) 2) of the conclusions and the latter to 3), 4).

**Theorem 1 (in \$.3 - \$.4)** Let $X$ be a smooth projective variety with two projective space bundle structures $(PS : (Y_1, P^r, p), (Y_2, P^s, q))$ and $\dim Y_i = m_i (i = 1,\ 2)$ with the Assumption 0.1. Suppose that $Y_1,\ Y_2$ are homotopy complex projective spaces. Then we have

1. $(m_1, r) = (s, m_2)$ or $(m_2, s)$.

2) When $(m_1, r) = (s, m_2)$, each $Y_i$ is a projective space and $X$ is a product of $P^{m_1} \times P^{m_2}$.

3) When $(m_1, r) = (m_2, s)$, assume, moreover that $Y_1$ is a projective space. Then $Y_2$ is a complex projective space $P^{m_1}$ and $X = P_{P^{m_1}}(T_{P^{m_1}}(-1))$ with the equality $r = m_1 - 1$.

**Remark 0.2.**

1. When $Y_2$ is not a homotopy complex projective space, $q : X \to Y_2$ is a $P^1$-bundle from Theorem A stated below.

2. Moreover Theorem B implies that the image of each fiber of $q$ via $p$ is a line or a conic on $Y_1 = P^n$ in our situation of algebraic geometry. This means $\bar{a} = 1$ or $2$. See (2.2) (3.3.5) about $\bar{a}$.

Here we introduce two thereoms of algebraic topology due to Iriye below which is useful for the proof of Main Thereom. [Iri90].

Let $X$ be (differentiable) manifold which has two bundle structures:
$CP^r \overset{i_1}{\to} X \overset{p_1}{\to} CP^m, \quad CP^s \overset{i_2}{\to} X \overset{p_2}{\to} Y$ where $r, s, m \geq 1$ and $Y$ is a manifold.

Then the following are shown:

**Thereom A** If $Y$ is homotopy equivalent to a complex projective space, then $Y$ is homotopy equivalent to $CP^r$ or $CP^m$. If $Y$ is not homotopy equivalent to a complex projective space, then the fiber of $X \to Y$ is $CP^1$, namely $s = 1$.

**Thereom B** If $Y$ is not homotopy equivalent to a complex projective space, then $\beta = 0,\ \pm 1 \pm 2$. Here we denote the map $p_1 i_2 : CP^1 \to CP^m$ as an element of $\pi_2(CP^1) \cong H_2(CP^m, Z)$ under a certain isomorphism $H_2(CP^m, Z) \cong Z$, we denote it as $\beta$.

We state the motivation of our problem.
Kollar, Miyaoka and Mori showed in [92KoMiMo] the following Corollary:

Let $f : X \to Y$ be a smooth morphism between smooth projective varieties $X, Y$. Then if X is Fano, then so is Y in any characteristic.

(A) such a morphism in the above Corollary is a fiber bundle over Y particularly under the condition of the dimension of the fiber $\geq 3$.

It is not easy to find a couterexample for it.

Therefore we present the following problem with the additional condition:

(B) Classify smooth projective varieties X with two kinds of fiber bundle structures, moreover under the condition that each fibers are Fano with $\rho(X) = 2$.

Then one can expect that there are fews examples in (B).

Moreover we would like to state another point of view for motivation. Hartshorne conjecture is well-known about vector bundle $E$ on the projective space $P^n$ , namely $E$ of low rank $r$ is decomposable if $n \geq 3r$. Therefore the immediate problem from our point of view is to clarify and classify the structure of Fano bundles $E$ of the low rank where the other extremal contraction $q : P(E) \to Y_2$ different from a canonical projection $P(E) \to P^n$ is of " fiber type" . This research for $P(E)$ with the fiber bundle $q$ coincides with the one for the above (B).

**Convensions.** For a vector bundle E on a variety X $P_X(E)$ means $Proj \oplus_{m\geq 0} S^m(E)$, written simply as $P(E)$ without confusion.

For a line bundle $L$ on a variety X $E(mL)$ means $E \otimes L^{\otimes m}$ for an integer $m$. Moreover in case that $H^2(Y,Z) \cong ZL$ $E(mL)$ is written simply as $E(m)$. For a vector bundle $E$ on a variety when $E$ is 'generated by its global sectons', it is somotimes abbrieviated as GS.

## $.1 Basic Facts

(1.1) Let $Z, M$ be smooth projective varieties over $\mathbf{C}$ and $p : Z \to M$ $P^r$-bundle over $Y$ in etale topology. We give a sufficient condition for the above bundle $p$ to be of $P^r$-bundle in Zarisiki-topology.

**Proposition 1.1.1** Under the notations (1.1) let us assume that
1) $M$ is simply-connected.
2) $H^3(M, \mathbf{Z}) = 0$ (e.g. homotopy complex projective space). Then we have
i) there is the following exact sequence:
$0 \to H^2(M, \mathbf{Z}) \to H^2(Z, \mathbf{Z}) \to H^2(F, \mathbf{Z}) \to 0$ with a fiber $F \cong P^r$ of $p$ and $H^3(Z, \mathbf{Z}) = 0$.

(ii) Moreover assume $H^1(Z, \mathcal{O}_Z) = H^2(Z, \mathcal{O}_Z) = 0$. Then the above exact sequence means $0 \to PicM \to PicZ \to H^2(F, \mathbf{Z}) \to 0$. Furthermore we have an algebraic vector bundle $G$ of the rank $r+1$ over $M$ with $Z \cong P(G)$.

(1.2) In this subsection only here we study a differential manifold $X$ which is complex $P^r$-bundle over a differential manifold $Y$ from the homotopical point of view. Then in algebraic topology the following is known:

For a fiber bundle : $F \to X \to Y$ where $Y$ and $F$ are simply-connected we can consider the Serre spectral sequence for cohomology:
$E_2^{p,q} = H^p(Y, H^q(F, \mathbf{Z})) \to H^*(X, \mathbf{Z})$. See Theorem 5.15 [Allen Hatcher]

**Proposition 1.2.1.** Let $X$ be a complex $P^r$- bundle over a differential manifold $Y$. Assume $Y$ is a homotopy complex projective space. Then we have
0) $H^{odd}(Y, \mathbf{Z}) = 0$ and $H^{odd}(F, \mathbf{Z}) = 0$.
$H^{2i}(Y, \mathbf{Z})$ is generated by $h^i$ over $\mathbf{Z}$ where $h$ is a base of $H^2(Y, \mathbf{Z})$ and
$H^{2j}(F, \mathbf{Z})$ by $k^j$ over $\mathbf{Z}$ where $k$ the one of $H^2(F, \mathbf{Z})$.

1) $E_2^{p,q} = H^p(Y, \mathbf{Z}) \otimes H^q(F, \mathbf{Z})$ and therefore $E_2^{p,q} = 0$ for odd $p$ or for odd $q$.
Thus $E_2^{p,q} = E_3^{p,q} = ... = E_{r'}^{p,q}$ for $2 \leq r' \leq \infty$. Consequeuctly $E_2$ degenerates.

2)$H^n(X,\mathbf{Z}) = \oplus_{p+q=n} H^p(Y,\mathbf{Z}) \otimes H^q(F,\mathbf{Z})$ which is generated by $h^j \otimes k^j$ over $\mathbf{Z}$ for $i+j=n$ since $E_2^{p,q}$ is 0 or $\mathbf{Z}$.

Thus $H^{odd}(X,\mathbf{Z}) = 0$ and $H^{2m}(X,\mathbf{Z}) \cong \oplus_{i+j=m} \mathbf{Z} h^j \otimes k^j$ where they are $(m+1)$ free basis over $Z$. In particular $H^*(X,\mathbf{Z}) = (\oplus_{i\geq 0} \mathbf{Z} h^j) \otimes (\oplus_{j \geq 0} \mathbf{Z} k^j)$.

(1.3) We return to the category of algebraic varieties. The following is an application of (1.1) by using Propositin 1.2.

**Corollary 1.3** Let $Z, M$ be smooth projective varieties over $\mathbf{C}$ and $p: Z \to M$ $P^r$-bundle over $M$ in etale topology. Assume that $M$ is a homotopy complex projective space. Then we have

0) There is Serre exact sequence: $0 \to H^2(M,\mathbf{Z}) \cong \mathbf{Z} \to H^2(Z,\mathbf{Z}) \to H^2(F,\mathbf{Z}) \cong \mathbf{Z} \to 0$ and $H^2(Z,\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$.

1) $Z$ is simply-connected and $H^3(Z,\mathbf{Z}) = 0$.

2) $H^i(M, \mathcal{O}_M) = 0$ for $i \geq 1$ and therefore $H^i(Z, \mathcal{O}_Z) = 0$.

3) There is an algebraic vector bundle $E$ of the rank $r+1$ over $M$ with $Z \cong P(E)$ where $\xi_E$ in $PicP(E)$ induces a base of $H^2(F,\mathbf{Z})$ under the sequence in 0).(See Proposition 1.5.4)

4) Therefore the exact sequence of 0) means the following: $0 \to PicM(\cong \mathbf{Z}) \to PicP(E) \to H^2(F,\mathbf{Z}) \to 0$. Hence $PicZ \cong \mathbf{Z} \oplus \mathbf{Z}$.

5) The Poincare polynomial $P(Z)$ of $Z$ is equal to $P(F) \times P(M)$. Here Poincare polynomial $P(Z)$ of $Z$ denotes $\Sigma_{i \geq 0} b_i t^i$ where $b_i(Z)$ is the i-th Betti number of $Z$. Consequently $P(Z) = (1 + t^2 +, ..., + t^{2m})(1 + t^2 +, ..., + t^{2r})$ with $\dim_{\mathbf{C}} M = m$.

● **Corollary 1.3.1** Let $X$, $Y_1$, $Y_2$ be smooth projective varieties with $\dim Y_1 = m_1 \geq 2$, $\dim Y_2 = m_2 \geq 2$. Let us assume that (PS) X has two projective space bundle structures where $p: X \to Y_1$ is $P^r$-bundle over $Y_1$, $q: X \to Y_2$ a $P^s$-bundle over $Y_2$ in etale topology. Moreover suppose $Y_1$ is a homotopy projective space.

Then we have

1) $Y_2$ is simply-connected,

2) $H^3(Y_2,\mathbf{Z}) = 0$. Therefore $H^2(Y_2,\mathbf{Z}) \cong Z$.

3) $H^i(Y_2, \mathcal{O}) = 0$ for $i = 1, 2$ and $PicY_2 \cong H^2(Y_2,\mathbf{Z})(\cong Z)$

4) $q$ is an $P^s$-bundle is in Zariski topology.

5) Suppose $Y_2$ is a homotopy projective space. Then $(m_2, s) = (m_1, r)$ or $(r, m_1)$.

(1.4) Summarizing the above argument, we state

**Fact 1.4.** Let $X$ be a smooth projective variety with (PS) in (0.0). Assume that $Y_1$ is a homotopy complex projective space. Then we have

0. $H^3(Y_2,\mathbf{Z}) = 0$ and $PicY_2 \cong H^2(Y_2, Z)(\cong Zk)$ ( 2), 3) of Corollary 1.3.1)

1. There are algebraic vector bundles $E$ of the rank $r+1$ over $Y_1$ with $X \cong P_{Y_1}(E)$ and the one $F$ of $s+1$ over $Y_2$ with $X \cong P_{Y_2}(F)$.

2. There are two isomorphisms;

$PicX \cong \mathbf{Z}\xi \oplus \mathbf{Z}p^*h$. $PicX \cong \mathbf{Z}\eta \oplus \mathbf{Z}q^*k$. where $h$, $k$ are ample generators on $Y_1$, $Y_2$ respectively and $\xi$, $\eta$ tautological line bundles of $E$, $F$ respectively.

$.2 Picard group $PicX$ of $X$ with (PS) and their examples.

(2.1) Let us consider $X$ of (PS) where $Y_1$ are homotopy complex projective space. Then we from Fact 1.4 we have a relation:

(2.2) **Fact** $\eta = a\xi + bp^*h \quad q^*k = \bar{a}\xi + \bar{b}p^*h$ with $\bar{a}, \bar{b}, a, b$ are integers. Moreover $a\bar{b} - \bar{a}b = 1$ or $-1$.

Consequently we get

**Fact 2.3** Under the condition (2.1) we have $\bar{a} > 0$ and $a\bar{b} - \bar{a}b = -1$.

**Remark 2.3.2.** In case $\bar{a} = 1$ tensoring a suitable linebundle for the vector bundle $E$, we can take $q^*k = \xi, \; p^*h = \eta$ from the begining, which makes our argument easier.

Here we give simple examples with the structure PS

**EXAMPLE 0.**. $(n > d > e)$

$$\begin{array}{ccc} F(n,d,e) & \xrightarrow{q} & Gr(n,e) \\ {\scriptstyle p}\downarrow & & \\ Gr(n,d) & & \end{array}$$

Here $Gr(n,c)$ is a Grassmann variety parameterizing linear $c$-spaces in $P^n$ and $U(n,d)$ the universal quotient vector bundle of rank $c+1$ and $E(n,d)$ the universal subbundle vector bundle of rank $n-c$ on $Gr(n.c)$ satisfying the exact sequeuce: $0 \to E(n,d) \to \oplus^{n+1}\mathcal{O}_{Gr(n.c)} \to U(n,d) \to 0$. $F(n,d,e)$ is a flag variety $\{(x,y) \in Gr(n,d) \times Gr(n,e) | L_x \supset L_y\}$ where $L_x$ is a linear $d$-space corresponding to an element $x \in Gr(n,d)$ and $L_y$ $e$-space corresponding to an element $y \in Gr(n,e)$.

**EXAMPLE 1.** $X = P^r \times P^s$ (1) of main Thereom

**EXAMPLE 2.** $X = F(n.n-1.0)$ (2) of main Thereom

$$\begin{array}{ccc} P_{Y_1}(T_{P^n}(-1)) = X = P_{Y_2}(T_{P^n}(-1)) & \xrightarrow[P^{n-1}-bundle]{q} & P^n = Y_2 \\ {\scriptstyle p}\downarrow {\scriptstyle P^{n-1}-bundle} & & \\ Y_1 = P^n & & \end{array}$$

**EXAMPLE 3.** $X = F(n.1.0)$ (3) of main Thereom

**EXAMPLE 4.** $n (\geq 3)$ is odd. (4) of main Thereom

**Remark 2.4** 1) Each $Y_1$ of the four examples above is a projective space.

2) When we denote the vector bundle on $Y_1$ stated in each Example $i(= 0, 2, 3, 4)$ by $E$ and the one $Y_2$ stated by $F$, we have the relations: $p^*\mathcal{O}_{Y_1}(1) = \eta_F, \; q^*\mathcal{O}_{Y_2}(1) = \xi_E$. For $i = 1, 2$ $\mathcal{O}_{Y_i}(1)$ is an ample generator of $H^2(Y_i, Z)$ respectively.

3) Neither $Y_1$ and $Y_2$ of two examples below are homotopy complex projective spaces.

**EXAMPLE 5. Watanebe[Wa14]**.

$$\begin{array}{ccc} P_{Y_1}(E) = X = P_{Y_2}(F) & \xrightarrow[P^1-bundle]{q} & K(G_2) = Y_2 \\ {\scriptstyle p}\downarrow {\scriptstyle P^1-bundle} & & \\ Y_1 = Q_5 & & \end{array}$$

**EXAMPLE 6.** [Ot90][Ka19]

$$\begin{array}{ccc} P_{Y_1}(E) = X = P_{Y_2}(F) & \xrightarrow[P^2-bundle]{q} & Q_5 = Y_2 \\ p \downarrow P^2-bundle & & \\ Y_1 = Q_5 & & \end{array}$$

**\$. 3 The proof of Thereom 1 1), 2) and $\bar{a} = 1$ of 3)**

(3.0) Here we treat the cases 1), 2) and 3) of Thereom 1.

(3.0.1) In this and next section it is supposed that $Y_1,\ Y_2$ are homotopy complex projective spaces.

(3.1) First 1) of Thereom 1 is already shown in 5) of Corollary 1.3.1.

To get 2) and 3) we need to prove $\bar{a} = 1$ where $\eta = a\xi + bp^*h \ \ q^*k = \bar{a}\xi + \bar{b}p^*h \ \ a\bar{b} - b\bar{a} = -1$ and $\bar{a} > 0$ with $a, b, \bar{a}, \bar{b}$ integers.

Here we show 1), 2) and $\bar{a} = 1$ (Lemma 3.3.5) in case 3). The proof of 3) is completed in the next section. Note that to get $\bar{a} = 1$ in case 3) the assumption of $Y_1 \cong P^m$ is not needed.

Recall $X \cong P_{Y_1}(E)$ and $X \cong P_{Y_2}(F)$ (Fact 1.4) where $E$ is a vector bundle of rank $r+1$ and $F$ the one of rank $s+1$. Let $h,\ k$ be ample generators of $H^*(Y_1, \mathbf{Z})$ and $H^*(Y_2, \mathbf{Z})$ and $\xi,\ \eta$ the tautological line bundles of $E,\ F$ respectively.

On the cohomology ring $H^*(X, \mathbf{Z})$ vector bundle $E$ induces the following relation in $H^{2(r+1)}(P_{Y_1}(E), Z)$:
(3.1.2.1) $0 = \xi^{r+1} - c_1 \xi^r p^*h^1 +, ..., +(-1)^{r+1} c_r p^*h^{r+1}$ $(c_i \in Z)$ which is set as $f(\xi, p^*h)$ where $f(x, y)$ is a polynomial with the coefficient of integers. Vector bundle $F$ yields the following relation in $H^{2(s+1)}(P_{Y_2}(F), Z)$ :
(3.1.2.2) $0 = \eta^{s+1} - d_1\eta^s q^*k +, ..., +(-1)^{s+1} d_{s+1} q^*k^{s+1}$ set as $g(\eta, q^*k)$ where $g(x, y)$ is a polynomial with the coefficient of integers.
Therefore the isomorphism $\iota : P_{Y_1}(E) \cong P_{Y_2}(F)$ by the assumption of Theorem 1 yields a ring isomorphism $\iota^* : H^*(P_{Y_2}(F), Z) \cong H^*(P_{Y_1}(E), Z)$.
Hence we can consider a ring isomorphism between finite graded rings as preserving the degrees as follows:

(3.1.3) $\iota^* : Z[u, v]/(g(u, v), v^{m_2+1}) \cong Z[w, z]/(f(w, z), z^{m_1+1})$.

**The proof of case 2))** $(m_2, s) = (r, m_1)$
**Lemma 3.2.** Let us consider the case 1 $(m_1, r) = (s, m_2)$. Then we have $\bar{a} = 1$, and therefore $\xi = q^*k \ \ \eta = p^*h$. $Y_1 \cong P^s$ and $Y_2 \cong P^r$. Thus $X = P^s \times P^r$.

**Proof of $\bar{a} = 1$ for case 3)** $m_1 = m_2 =: m,\ r = s$
(3.3) It is supposed in \$ 3 and \$.4 that $1 \leq r \leq m - 1$ from (PS) and (3.2.1.1).

Until the end of this section we focus to show
(3.3.0) $\bar{a} = 1$ (see also Lemma 3.3.5)

(3.3.2) There is a ring isomorphism $j$ between finite graded rings as preserving the degrees $j : Z[w, z]/(f(w, z), z^{m+1}) \cong Z[u, v]/(g(u, v), v^{m+1}) (m \geqq r)$

$f(w,z) = w^{r+1} - c_1 w^r z^1 +, ..., +(-1)^{r+1} c_{r+1} z^{r+1} = \Sigma_{i=0}^{r+1}(-1)^i c_{r-i} w^{r-i} z^i, \quad c_0 = 1$
$g(u,v) = u^{r+1} - d_1 u^r v^1, ..., +(-1)^{r+1} d_{r+1} v^{r+1} = \Sigma_{i=0}^{r+1}(-1)^i d_{r-i} u^{r-i} v^i, \quad d_0 = 1$

Thus from the above ring isomorphism $\iota^*$ we get
(3.3.2.1) $v^{m+1} = Az^{m+1} + f(w,z)\bar{f}(w,z)$ with $\bar{f} \in Z[w,z]$.
(3.3.2.2) $z^{m+1} = Bv^{m+1} + g(u,v)\bar{g}(u,v)$ with $\bar{g} \in Z[u,v]$.
where $A, B$ are integers and
(3.3.2.3) $\bar{f}(w,z),\ \bar{g}(u,v)$ homogenuous polynomials of the degree $m-r$.

**Lemma 3.3.3** We have $f(w,z) = g(u,v)$, or $-g(u,v)$, moreover $A = B = \pm 1$ under the isomorphism $j$.

**Lemma 3.3.5** Let the notations and conditions be as in Thereom 1. Assume that $Y_1$ and $Y_2$ are homotopy complex projective spaces. Then $\bar{a} = 1$ holds.

This Lemma 3.3.5 is shown after (3.3.6).

First $\bar{a} = 1$ yields the relations:
(3.3.5.1) $\eta = p^*h,\ \ \xi = q^*k\ \ (a = \bar{b} = 0,\ b = \bar{a} = 1)$ by twisting line bundles on base spaces $Y_1,\ Y_2$. Thus we see that $F = q_*\eta = q_*p^*h,\ \ E = p_*\xi = p_*q^*k$ and $X \cong P_{Y_1}(E) \cong P_{Y_2}(F)$.

Till the end of this section we have an argument under this relations (3.3.5.1).

**Corollary 3.3.5.2** 1) $\xi$ and $\eta$ are nef line bundles, namely $E$ and $F$ are nef vector bundles. (See [La0/] for the definition of nef vector bundle) Thus $c_i,\ d_j$ are non-negative integers.
2) In case (3.3) there are relations $c_i = d_{r+1-i}$ for any $i$ $(0 \le i \le r)$, in particularly $c_{r+1} = d_0 = 1$ $d_{r+1} = c_0 = 1$ namely, $c_{r+1}(E) = 1h^{r+1},\ c_{r+1}(F) = 1k^{r+1}$.
3) Moreover $-K_X = (r+1)\xi + (v_{Y_1} - c_1)p^*h = (r+1)\eta + (v_{Y_2} - d_1)q^*k$ where $-K_{Y_1} = v_{Y_1}h$ and $-K_{Y_2} = v_{Y_2}k$. Thus $r+1 = v_{Y_1} - c_1 = v_{Y_2} - d_1$.

**(3.3.6) Proof of Lemma 3.3.5**
There we got $A = \pm 1$ where (3.3.2.1) $v^{m+1} = Az^{m+1} + f(w,z)\bar{f}(w,z)$ with $\bar{f} \in Z[w,z]$.

Now we devide Lemma 3.3.5 into two cases:
**(3.3.5.I)** $r = 1$ and $A = 1$
**(3.3.5.II)** otherwise (shown in 3.3.6.1)

**(3.3.6.1) Proof of Lemma 3.3.5 (the case 3.3.5.II)**
First we treat (3.3.5.II) and show $\bar{a} = 1$. Thus it suffices to the following:

**Proposition 3.3.6.1** Let $f(x), \bar{f}(x)$ be polynomials in $\mathbf{Z}[x]$ so that $(\bar{a}x + \bar{b})^{m+1} - A = f(x)\bar{f}(x)$ where $A = 1$ or $-1$ and where $\bar{a}, \bar{b}, m$ are integers with $\bar{a} > 0, m(\ge 2)$,. Assume that $f(x)$ is a monic polynomialof the degree $r+1$.
Then $\bar{a} = 1$, if either of the following holds:
1) $1 < r \le m$ or
2) $r = 1$ and $A = -1$.

Here we need the following result on algebraic number theory

**Proposition 3.3.7** (Proposition 2.2 Sa85) Let $\Phi_n(x)$ be a cyclotonic polynomial of $n-$th root of unity ($n \geq 3$) and $\alpha(\neq 0)$, $\beta$ integers. Assume that $\alpha^{\phi(n)}$ divides $\Phi_n(\alpha x+\beta)$. Then $\alpha = \pm 1$. Here $\phi(n)$ is Euler number of $n$.

The above is obtained by the famous result:
**Theorem** (Thereom 4 [La70] ) Let $\sigma$ be the n-th primitive root of unity and $Q(\sigma)$ the field generated by the rational number field $Q$ and $\sigma$. Let $A$ be the ring of integers of $Q(\sigma)$. Then $A = \oplus_{i=0}^{\phi(n)} \mathbf{Z}\sigma^i$.

We omit the proof of the case (3.3.5.I)
Thus we complete the proof of Lemma 3.3.5.

**\$. 4 The proof of Theorem 1. 3)** $(X = P(T_{P^m}(-1)))$

This section is the continuation of the previous one. We have the condition:
(4.0) 1) $Y_1, Y_2$ are $m$-dimensional homotopy complex projective spaces $\bar{P}^m$ with $m_1 = m_2(= m)$, $r = s$ and
2) $\bar{a} = 1$, shown in \$.3, with the additional assumption that
(4.0.0) $Y_1 = P^m$ $(h = \mathcal{O}_{P^m}(1))$.

Moreover from 3.3.5.1 we have the argument hereafter under the condition
(4.0.1) $\eta = p^*h,\ \xi = q^*k,\ q_*\eta = q_*p^*h = F$ and $c_{r+1}(F) = 1k^{r+1}$ (Corollary 3.3.5.2) obtained from 2) $\bar{a} = 1$. Here $k$ is the ample generator of $PicY_2$.

Then we show $r = m-1$ and $X = P(T_{P^m})$, which implies Thereom 1.

Therefore what we do first as a starting point is to prepare the following fact.
**Proposition 4.1** Under the condition (PS) in (0.0) let us maintain the condition (4.0). Suppose (4.0.0). Then $q_*p^*\mathcal{O}_{P^m}(1)(= F)$ is a rank (s+1)-vector bundle on the $m$-dimensioanl homotopy complex projective space $Y_2$ which is generated by its global sections with the following exact sequence of vector bundles on $Y_2$:
(4.1.1) $0 \to G' \to \mathcal{O}_{Y_2}^{m+1} \to q_*p^*\mathcal{O}_{P^m}(1) \to 0$ $\ rankF = s+1(\geq 2)$.
Therefore there is a morphism $f : Y_2 = \bar{P}^m \to Gr(m, s)$ $(s \geq 1)$

The flow of the proof hereafter is as follows:
**4.1** $\to$ **4.4.1** $+$ **4.4.2** $+$ **4.4.3** $\to$ **4.5** ($<=$ **4.5.0.2**) $\to$ **4.4.0** $\to$ **Theorem 1**

**We change the notations in the orthodox style** as follows in \$.4.
the dimension of $Y_2 = M$ : $\quad m = m_2 \longrightarrow n$
the generator of $PicY_2$ : $\quad k \longrightarrow h$
the dimension of a fiber of $q$: $r = s \longrightarrow d$
the vector bundle $\quad F \longrightarrow E$

Then the main aim is to show the following:
**Proposition 4.4.0** Let $E$ be a vector bundle of rank $d+1$ $(n-1 \geq d \geq 0)$ on an $n$-dimensional smooth complex projective variety $M$ satisifying that $E$ is GS with the exact sequence of vector bundles: $0 \to G \to \mathcal{O}_M^{n+1} \to E \to 0$. Here let $f : M \to Gr(n, d)$ be a morphism induced by the above exact sequence.(4.3.3.0)

Suppose $M = \bar{P}^n$ and $c_{d+1}(E) = h^{d+1}$ in $H^{2d+2}(M, Z) \cong Zh^{d+1}$. (Note that $f$ is not constant from Proposition 4.3.3).

Then $M = P^n$ and $d = 0$ or $n - 1$.

In the first case $d = 0$ the exact sequece turns out to be:

(4.4.0.1) $0 \to \Omega_{P^n}(1)(= G) \to \oplus^{n+1}\mathcal{O}_{P^n} \to \mathcal{O}_{P^n}(1)(= E) \to 0$.

In the second case $d = n - 1$ it does:

(4.4.0.2) $0 \to \mathcal{O}_{P^n}(-1) \to \oplus^{n+1}\mathcal{O}_{P^n} \to T_{P^n}(-1) \to 0$.

Note that if $rankE \geq 2$, $E = T_{P^n}(-1)$ (Thereom 1).

First we consider the following condition:

(4.3.3.0) Let $E$ be a vector bundle of rank $d + 1$ $(n - 1 \geq d \geq 0)$ on an $n$-dimensional smooth complex projective variety $M$ satisifying that $E$ is GS with the exact sequence of vector bundles: $0 \to G \to \mathcal{O}_M^{n+1} \to E \to 0$. Here let $f : M \to Gr(n, d)$ be a morphism induced by the above exact sequence.

**Proposition (4.3.3)** Let $M,\ E$ be as stated in (4.3.3.0). Assume $M$ is a homotopy complex projective space $\bar{P}^n$ where $H^2(M, Z) \cong Zh$ with the ample generator $h$. Then we have

1) if $f$ is a constant map, then each $c_i$ and $d_j$ are zero with $c_i(E) = c_i h^i,\ c_j(E) = d_j h^j$. (Remark 4.3.2.1)

2) if $f$ is not a constant map, then each $c_i$ and $d_j$ are positive.

● **Proposition 4.4.1** Let $M,\ E$ be as stated in (4.3.3.0). Suppose that $M$ is $\bar{P}^n$. Moreover assume that $d = 0$ or $n - 1$ and $d + 1$-th Chern class $c_{d+1}(E) = h^{d+1}$.

Then $M$ is an n-dimensional projective space. Moreover if $d = 0$, then $E = \mathcal{O}_{P^n}(1)$. If $d = n - 1$, then $E = T_{P^n}(-1)$.

Hereafter we have preparations in order to show that there is no non-constant morphism from $\bar{P}^n$ to $G(n, d)$ with $n - 1 > d > 0$ and $c_{d+1}(E) = h^{d+1}$.

■ **Proposition 4.4.2** Let us consider the following three polynomials in $Z[t]$

$E(t) := 1 + c_1 t +, ..., +t^{d+1}$ $(c_{d+1} = 1)$ $(0 \leq d \leq n - 1)$

$G(t) := 1 - d_1 t +, ..., +(-1)^{n-d} d_{n-d} t^{n-d}$

$P(t) := E(t)G(t) = 1 + (-1)^{n-d} d_{n-d} t^{n+1}$

Assume each $c_i$ is a positive integer for $1 \leq i \leq d + 1$ and $d_j$ so for $1 \leq j \leq n - d$.

Then we have

1) Suppose $n - d$ is even $(n > d \geq 0)$.

Then $d = 0$ ($n$ is even), $c_1 = 1$ and each $d_j = 1$. Particularly $d_{n-d} = 1$. Consequently $E(t) = 1 + t, G(t) = 1 - t + t^2 -, ..., +t^n$ and $P(t) = 1 + t^{n+1} = (1 + t)(1 - t + t^2 -, ..., +t^n)$.

2) Suppose $n - d$ is odd $(n > d \geq 0)$.

Then we have

i) if n is even (d odd), then $d = n - 1$, $d_1 (= d_{n-d}) = 1$ and each $c_i = 1$. Consequently $E(t) = 1 + t + t^2 +, ..., +t^n$, $G(t) = 1 - t$ and $P(t) = 1 - t^{n+1} = (1 + t + t^2 +, ..., +t^n)(1 - t)$.

ii) if n is odd (d even), then $d_{n-d} = 1$ and the following two relations holds:

(4.4.2.1) $E(-1) = 1 - c_1 +, ..., -c_{d+1} = 0, \quad G(1) = 1 - d_1 +, ..., -d_{n-d} = 0$

In this case $P(t) = 1 - t^{n+1} = E(t)G(t) = (1 + c_1 t +, ..., +t^{d+1})(1 - d_1 t +, ..., -t^{n-d})$

Moreover we have

(4.4.2.2) If $n-d \geq d+1$, then $1+d_1+,...,+d_{n-d}=n+1$..

If $n-d \leq d+1$, then $1+c_1+,...,+c_{d+1}=n+1$..

(4.4.2.3) $c_i = c_{d+1-i}$ for each i and $d_j = d_{n-d-j}$ for each j.

(4.4.2.4) Note that for all above cases $d_{n-d}=1$ .

**Remark 4.4.3** As for the case 2) ii) that n is odd and d even we cannot completely determine the structure $E, G$ by this argument.

(4.5) Thus for the proof of Proposition 4.4.0, recalling the conclusion $c_{n-d}(G)=(-1)^{n-d}h^{n-d}$ of 4.4.2, we have only to prove

**Proposition 4.5** Let $E$ and $G$ be vector bundles on $\bar{P}^n$ of rank $E=d+1$ and rank $G=n-d$ with $n-1>d>0$ enjoying the exact sequence: $0 \to G \to \mathcal{O}_{\bar{P}^n}^{\oplus n+1} \to E \to 0$. Assume that $c_{d+1}(E)=1h^{d+1}$ and $c_{n-d}(G)=(-1)^{n-d}h^{n-d}$. Then there is no morphisms $f:\bar{P}^n \to Gr(n.d)$ induced from the above exact sequence.

**The proof of Theorem 1** From 4.1, 4.4.1, 4.4.2, 4.4.3. and 4.5 we get 4.4.0. We complete the proof of Theorem 1 by virtue of Proposition 4.5.

The above proposition 4.5 yields an application which will be used in \$.5

**Corollary 4.6.** Let $G$ be a vector bundle of rank $r$ on $P^n$ $(n \geq 2)$ Assume there is an exact sequence of vector bundles : $0 \to N \to T_{P^n}(-1) \to G \to 0$ with $0 \leq r \leq n$. Then $G=T_{P^n}(-1)$ $(r=n)$ or $G=0$ $(r=0)$. When $n$ is odd and $r=1$, $G=\mathcal{O}_{P^n}(1)$ with the exact sequence: $0 \to N \to T_{P^n}(-1) \to 0(1) \to 0$.

### \$.5 The proof of Main Theorem (3), (4) ($q: X \to Y_2$ is a $P^1$-bundle with $\bar{a}=1$)

(5.0.0) In §.5 and \$.6 let us consider the case that $Y_2$ is not a hopotopy complex projective space, namely, $p: X \to Y_1 := P^n$ is a $P^r$-bundle and $q: X \to Y_2 := Y$ a $P^1$-bundle We treat the case $\bar{a}=1$ here and $\bar{a}=2$ in the next section. The former case corresponds to 3), 4) of Main Thereom and the latter is shown not to exist.

Throughout in this section it is assumend from Fact 1.6 and Remark 2.3.2 that

(5.0) $\bar{a}=1$, $\xi=q^*k, \eta=p^*h$ with $p_*\xi=E$, $q_*\eta=F$. $P_{P^n}(E)=X= P_Y(F)$. Here $h$ means $\mathcal{O}_{P^n}(1)$ and $k$ does $\mathcal{O}_Y(1)$ which is the ample generator of $PicY$.

For the proof we organize as follows:

I) For each fiber $L$ of $p$ $q|_L: L \to Y := Y_2$ is a closed embedding (Proposition 5.1)

II) the construction of a morphism $\theta: X=P(E) \to P(\Omega_{P^n}(2))$ over $P^n$ (5.3.9) and a morphism $\theta_2: Y \to Gr(n.1)$.

III) $\theta$ is a linear embedding on each fiber $L$ of $p$. (Proposition 5.3.14 $b=1$)

IV) The proof of 3) and 4) of Main Thereom (5.15).

**Proposition 5.1.** For each fiber $L$ of $P^r$-bundle $p: X \to P^n$ $q|_L: L \to Y := Y_2$ is a closed embedding where $L \cong P^r$ and $\mathcal{O}_Y(1)|_{q(L)} \cong \mathcal{O}_{P^r}(1)$. $(k=\mathcal{O}_Y(1))$

#### (5.3) The constraction of a morphism $\theta: X=P(E) \to P(\Omega_{P^n}(2))$ over $P^n$

(5.3.0) From the conditions we have two smooth morphisms $p: X \to P^n$ and $q: X \to Y$, which yield the following exact sequences on $X$:

(5.3.1) $0 \to T_p \overset{i_1}{\to} T_X \overset{i_2}{\to} p^*T_{P^n} \to 0$.

(5.3.2) $0 \to T_q \overset{j_1}{\to} T_X \overset{j_2}{\to} q^*T_Y \to 0$.

(5.3.2.1) Note $T_q \cong p^*\mathcal{O}_{P^n}(2) \otimes q^*\mathcal{O}_Y(-b))$ with an integer $b$. It follows from $T_q|_{q^{-1}(y)} \cong \mathcal{O}_{P^1}(2)$ $(y \in Y)$ and see-saw lemma.

Then we get

**Lemma 5.3.3.** The induced homomorphism $T_p \oplus T_q \overset{(i_1,j_1)}{\to} T_X$ is injective as a vector bundle with the following exact sequence: $0 \to T_p \oplus T_q \overset{(i_1,j_1)}{\to} T_X \to G \to 0$ where $G$ is the cokernel of $(i_1, j_1)$ which is a vector bundle of rank $n-1$. Moreover $N_{L'/Y}$ is isomorphic to $G|_L$ which is GS with $L' = q(L)$.

**Remark (5.3.9) Grassmann functor** $Grass_{n-1}T_{P^n} = P(\Omega_{P^n})$ .

Taking the dual of the exact sequence (5.3.4): $0 \to T_q \to p^*T_{P^n} \to G \to 0$ on $X$, we get

(5.3.4)': $0 \to G^\vee \to p^*\Omega_{P^n} \to T_q^\vee \to 0$.

Next tensoring $p^*\mathcal{O}_{P^n}(2)$ for the above, we obtain

(5.3.10): $0 \to G^\vee \otimes p\mathcal{O}_{P^n}(2) \to p^*\Omega_{P^n}(2) \to T_q^\vee \otimes p^*\mathcal{O}_{P^n}(2) \to 0$,

which yields the following $P^n$−morphism $\theta : X \to P(\Omega_{P^n}(2)) = Grass_{n-1}(T_{P^n})$ with the diagram ($p = \theta\bar{p}$) (see Remark 1.4 of 1. Grassmannian by Kleiman, Steven [Kl69]).

$$\begin{array}{ccc} X = P(E) & \xrightarrow{\theta} & P(\Omega_{P^n}(2)) \\ p\downarrow & \swarrow \bar{p} & \\ P^n & & \end{array}$$

where $\bar{p} : P(\Omega_{P^n}(2)) \to P^n$ is a canonical projection. Recall $T_q^\vee \otimes p^*\mathcal{O}_{P^n}(2) = q^*\mathcal{O}_Y(b)$ from (5.3.2.1).

By the universal mapping property the above exact sequence (5.3.10) :

(5.3.10.0) $0 \to G^\vee \otimes p^*\mathcal{O}_{P^n}(2) \to p^*\Omega_{P^n}(2) \to \bar{q}^*\mathcal{O}_Y(b) \to 0$

is the pull-back of the following universal exact sequence on $\mathcal{X}$ $(:= P(\Omega_{P^n}(2))$:

(5.3.10.1) $0 \to \bar{q}^*\mathcal{E} \otimes \bar{p}^*\mathcal{O}_{P^n}(1) \to \bar{p}^*\Omega_{P^n}(2) \to \bar{q}^*\mathcal{O}_{Gr(n.1)}(1) \to 0$ via $\theta^*$. (Remark 5.3.10.2 below). Here $\mathcal{E}$ denotes the universal subbundle bundle of rank $n-1$ and $\mathcal{U}$ the universal quotient bundle of rank 2 on $\mathcal{X}\colon = P(\Omega_{P^n}(2)) \cong P_{Gr(n.1)}(\mathcal{U})$ where there is a natural exact sequence: $0 \to \mathcal{E} \to \mathcal{O}_{Gr(n.1)}^{\oplus n+1} \to \mathcal{U} \to 0$.

(5.3.11) Consequently from (5.3.10.1) and (5.3.10.0) we have two relations:

(5.3.11.1) : $\theta^*(\bar{q}^*\mathcal{O}_{Gr(n.1)}(1)) = q^*\mathcal{O}_Y(b)$,

(5.3.11.2) : $\theta^*(\bar{q}^*\mathcal{E} \otimes \bar{p}^*\mathcal{O}_{P^n}(1)) = G^\vee \otimes p^*\mathcal{O}_{P^n}(2)$

Now we claim

$\mathbf{p}^*\mathcal{O}_{\mathbf{P^n}}(\mathbf{1}) = \theta^*\bar{\mathbf{p}}^*\mathcal{O}_{\mathbf{P^n}}(\mathbf{1})$.

Thus $\theta$ gives rise to an isomorphism $\theta_1 : P^n \to P^n$ with the commutative diagram $p\theta_1 = \theta\bar{p}$. Similarly the relation (5.3.11.1) does a morphism $\theta_2 : Y \to Gr(n.1)$ with the commutative diagram $q\theta_2 = \theta\bar{q}$.

(5.3.12)

$$\begin{tikzcd}
Y \arrow[r, "\theta_2"'] & Gr(n,1) \\
P(E) = X = P(F) \arrow[u, "q"] \arrow[r, "\theta"] \arrow[d, "p"] & P(\Omega_{P^n}(2)) = \mathcal{X} = P(\mathcal{U}) \arrow[u, "\bar{q}"] \arrow[d, "\bar{p}"] \\
P^n \arrow[r, "\theta_1"] & P^n
\end{tikzcd}$$

(5.3.13) From now on we study the property of the morphism $\theta : X(:= P(E)) \to \mathcal{X}(\cong P(\Omega_{P^n}(1))$ over $P^n$ by means of Chow rings. The Picard groups and the cohomology rings are important ingredients for the proof.

**Fact (5.3.13.2)**

1) $\xi = \mathcal{O}_{P(E)}(1) = q^*\mathcal{O}_Y(1)$, $\mathcal{O}_{P(\Omega_{P^n}(1))}(1) = \bar{q}^*\mathcal{O}_{Gr}(1) \otimes \bar{p}^*\mathcal{O}_{P^n}(-1)$.
Recall that $\mathcal{O}_{P(\Omega_{P^n}(2))}(1) = \bar{q}^*\mathcal{O}_{Gr}(1)$.
2) $\theta^*\bar{p}^*\mathcal{O}_{P^n}(1) = p^*\mathcal{O}_{P^n}(1)$ from (5.3.11.4)
3) $\theta^*\mathcal{O}_{P(\Omega_{P^n}(1))}(1) = b\xi - p^*\mathcal{O}_{P^n}(1)$ for integer $b > 0$.

**Proposition 5.3.14** Let $E$ be a vector bundle of rank $r+1$ on $P^n$ and $\theta : X(:= P_{P^n}(E)) \to \mathcal{X}(:= P(\Omega_{P^n}(1))$ a nonconstant morphism over $P^n$. Let the notations be as in Fact in (5.3.13.1). If $r \geq 1$, then $b = 1$.

Main Theorem 3) 4) is obtained by Corollary 4.6.

**6. The non-existence of the case that $q : X \to Y_2$ is a $P^1$-bundle with $\bar{a} = 2$**

Let us consider the case that $p : X \to Y_1 := P^n$ is a $P^r$-bundle and $q : X \to Y_2 := Y$ a $P^1$-bundle where $Y_2$ is not a homotopy complex projective space with $\bar{a} = 2$ as in (5.0.0). The aim is to show that the above case does not exist.

(6.0.0) Thus the condition $\bar{a} = 2$ is assumed in this section. Note that $p^*h = 2\eta - aq^*k$ from Fact 2.3. Recall $h = \mathcal{O}_{P^n}(1)$ and $k = \mathcal{O}_Y(1)$ (5.0). Namely for each point $y \in Y$ $p^*h|_{q^{-1}(y)} \cong \mathcal{O}_{P^1}(2)$. Hereafter $Y_2$ is written as $Y$ and $q^{-1}(y) \cong P^1$ as $X_y$

The arguements are made as follows:

(6.0) Suppose the existence of $P^1$-bundle $q : X \to Y_2$ with $\bar{a} = 2$. After five steps below we get a contradiction.

**Step 6.1.** The coherent sheaf $q_*p^*h$ is a rank 3 vector bundle over $Y =: Y_2$, written as $G$. Moreover there is an exact sequence of vector bundles on X: $0 \to J \to q^*G \to p^*h \to 0$ where $J$ is a vector bundle of rank 2 on $X$.

**Step 6.2.** The above exact sequence induces a natural morphism $g : X \to P(G)$ over $Y$ which is a closed embedding and $g^*\lambda_G = p^*h$. Here $\lambda_G$ denotes the tauological line bundle of $G$ and $\theta : P(G) \to Y$ is a canonical projection.

Moreover the restriction of $g$ to each fiber $X_y$ yields $g^*\mathcal{O}_{P^2}(1) = \mathcal{O}_{X_y}(2) = \mathcal{O}_{P^1}(2)$ from $g\theta = q$. Thus $X$ is a conic bundle as a divisor in $P^2$-bundle $Z := P(G)$ over $Y$.

$$\begin{tikzcd}
X = P(E) \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, "q"] & P(G) \arrow[d, "\theta"] \arrow[dl, "\bar{p}"'] \\
P^n & Y
\end{tikzcd}$$

**Step 6.3.** For each point $y \in Y$ $p(X_y)$ is a smooth conic.

**Step 6.4.** $G$ is GS, namely $\lambda_G$ is base -point-free. The morphism $p : X \to P^n$ is extended to a morphism $\bar{p} : P(G) \to P^n$ induced by the line bundle $\lambda_G = \bar{p}^* h$ with $\bar{p}|_X = p$.

**Step 6.5.** $X$ is an ample divisor in $P(G)$, which yields a contradiction.

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