On a 3-dimensional K-moduli space of Fano 3-folds

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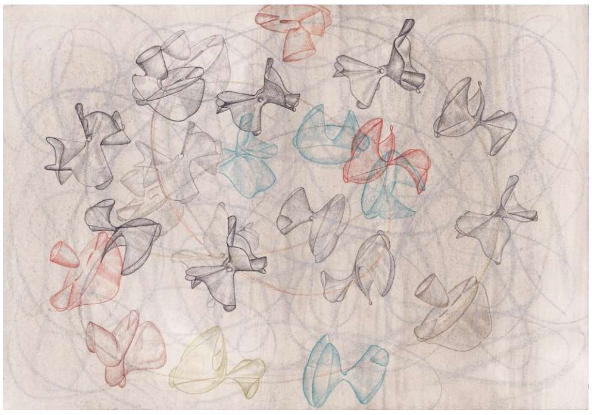

Goal of the talk: K-moduli of pure states of four qubits

- Joint work with I. Cheltsov, M. Fedorchuk and K. Fujita
- Describe component of K-moduli space of Fano 3-folds of anticanonical degree 24 associated to family MM_{4-1}
- An element X in family MM_{4-1} is $(1,1,1,1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Theorem (CFFK'24)

The component of $\mathrm{M}^{\mathrm{Kps}}_{3,24}$ that parametrises K-polystable limits of Fano 3-folds in family MM_{4-1} is the blow up of $(2:2:0:0) \in \mathbb{P}(1,3,4,6)$ with weights $(1,2,3)$.

1. Introduction - K-moduli spaces of Fano 3-folds
2. Geometry of divisors $(1,1,1,1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ - first GIT
3. Another family of degenerations
4. K-moduli and the *right* GIT

Kähler Einstein manifolds and the Calabi problem

Calabi problem

Which compact complex manifolds admit a Kähler-Einstein metric?

- Canonical structure on manifold: Ricci class proportional to Kähler class Hence canonical class of the manifold has a definite sign
- Yau, Aubin/Yau: positive solution when canonical class ample or trivial
- Fano case more subtle: some Fano manifolds are KE, others aren't (Matsushima obstruction)

Yau-Tian-Donaldson conjecture/ Chen-Donaldson-Sun theorem/ Liu-Xu-Zhuang

Let X be a Fano manifold (resp. (X, Δ) a log Fano pair),then:

$$\begin{cases} X \text{ has a KE metric} \\ (X,\Delta) \text{ has a weak KE metric} \end{cases} \iff \begin{cases} X \\ (X,\Delta) \end{cases} \text{is K-polystable}$$

K-polystability

K-polystability introduced to characterise KE Fano manifolds

- original definition in terms of Donaldson-Futaki invariants
- recent valuative formulations of K-polystability

More surprisingly:

K-moduli stacks and spaces [Xu and collaborators]

There is a projective good moduli space $\mathrm{M}^{\mathrm{Kps}}_{n,V}$ whose points parametrise K-polystable $\mathbb{Q}$-Fano varieties of dimension n and volume V.

The $\mathbb{C}$-points of $\mathrm{M}^{\mathrm{Kps}}_{n,V}$ parametrise Kähler-Einstein $\mathbb{Q}$-Fano varieties.

The Calabi problem for Fano surfaces and 3-folds

In any dimension, there are finitely many families of smooth Fano manifolds.

- 10 deformation families of smooth del Pezzo surfaces
- 105 deformation families of smooth Fano 3-folds

For each deformation family of smooth Fano manifolds, we ask 3 versions of the Calabi problem:

1. Is the general member K-polystable?
   Does the family correspond to a non-empty component of $\mathrm{M}^{\mathrm{Kps}}_{n,V}$?
2. Which members are K-polystable?
3. What is the compactification of the associated component of $\mathrm{M}^{\mathrm{Kps}}_{n,V}$?
   What are the K-polystable limits of Fano manifolds in the family?

Calabi problem for del Pezzo surfaces

A smooth del Pezzo surface S_d of degree d is:

- (d=9) $\mathbb{P}^2$
- (d=8) $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up of $\mathbb{P}^2$ in a point
- (d=7) the blow up of $\mathbb{P}^2$ in two points
- (d=6) a divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1, 1, 1)$
- (d=5) a section of $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of codimension 4
- (d=4) a complete intersection of two quadrics in $\mathbb{P}^4$.
- (d=3) a cubic surface in $\mathbb{P}^3$
- (d=2) a quartic surface in $\mathbb{P}(1, 1, 1, 2)$
- (d=1) a sextic surface in $\mathbb{P}(1, 1, 2, 3)$

Calabi problem for del Pezzo surfaces

There are two smooth del Pezzo surfaces with non-reductive automorphisms:

- If $S_8 = \mathrm{Bl}_p\,\mathbb{P}^2$, then $\mathrm{Aut}(S_8) \cong \mathbb{G}_a^2 \rtimes \mathrm{PGL}_2(\mathbb{C})$
- If $S_7 = \mathrm{Bl}_{p,q}\,\mathbb{P}^2$, then $\mathrm{Aut}(S_7) \cong \big(\mathbb{B}_2 \times \mathbb{B}_2\big) \rtimes \boldsymbol{\mu}_2$, where $\mathbb{B}_2$ = Borel subgroup of $\mathrm{PGL}_2(\mathbb{C})$

Corollary (Matsushima obstruction)

If S is $\mathrm{Bl}_p(\mathbb{P}^2)$ or $\mathrm{Bl}_{p,q}(\mathbb{P}^2)$, $\mathrm{Aut}(S)$ is not reductive and S is not Kähler–Einstein (K-polystable).

Theorem (Tian)

A smooth del Pezzo surface that is not the blow up of $\mathbb{P}^2$ at one or two points is Kähler-Einstein (K-polystable).

This settles the first two versions of the Calabi problem for Fano surfaces.

Calabi problem for del Pezzo surfaces

- Families of dP surfaces of degree $d \geq 5$ have 0-dimensional moduli
- Associated components of $\mathrm{M}^{\mathrm{Kps}}_{2,d}$ for $d = 5, 6, 8$ and 9 are points
- For $d = 3, 4$, K and GIT-moduli spaces coincide

Theorem (Mabuchi-Mukai, Odaka-Spotti-Sun)

A K-polystable limit S of smooth del Pezzo surfaces of degree 4 is the intersection $Q \cap Q' \subset \mathbb{P}^4$ of simultaneaously diagonalisable quadrics

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \\ \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 = 0 \end{cases}$$

where no three of the λ_is are equal.
The K-moduli $\mathrm{M}^{\mathrm{Kps}}_{2,4} \simeq \mathbb{P}(1,2,3)$, and the locus of singular surfaces is cut out by the divisor $D = \{z_1^2 = 128 z_2\}$.
If $[S] \in \mathrm{M}^{\mathrm{Kps}}_{2,4}$, S is smooth, or has 2 or 4 A1 singularities.

Calabi problem for del Pezzo surfaces

Theorem (Odaka-Spotti-Sun)

A K-polystable limit S of smooth del Pezzo surfaces of degree 3 is either stable and has no worse than A1 singularities or is $X_3^T = \{x_0^3 - x_1x_2x_3 = 0\}$ and has 3 A2 singularities.
The K-moduli $\mathrm{M}_{2,3}^{\mathrm{Kps}} \simeq \mathbb{P}(1,2,3,4,5)$, and the locus of singular surfaces is cut out by $D = \{(z_1^2 - 64z_2)^2 = 2^{11}(8z_4 + z_1z_3)\}$.

Cases of degrees $d = 1$ or 2 are more subtle, as the natural GIT and K-moduli spaces are different.
The associated components $\mathrm{M}_{2,2}^{\mathrm{Kps}}$ and $\mathrm{M}_{2,1}^{\mathrm{Kps}}$ are 6 and 8-dimensional respectively and are explicitly described by Odaka-Spotti-Sun. Boundary points have worse than canonical singularities.

Calabi problem for degree 2 del Pezzo surfaces

- A smooth dP2 is a double cover $S \to \mathbb{P}^2$ branched over $F_4 \subset \mathbb{P}^2$
- If $[F_4] \in \overline{P_4}^{GIT}$, then F_4 is
  - stable if it has no worse than A1 or A2 singularities
  - strictly polystable if it is (a) a double conic $2C$ or (b) $C_1 \cup C_2$ where C_1, C_2 are tangent at 2 points and at least one of them is smooth.
- The $2:1$ cover $S \to \mathbb{P}^2$ branched along F_4 with $[F_4] \in \overline{P_4}^{GIT}$ is
  - non-normal hence non K-polystable if $F_4 = 2C$
  - K-polystable if F_4 is stable (S has no worse than A1 or A2 singularities) or $F_4 = C_1 \cup C_2$ (S has no worse than A3 singularities).

Calabi problem for degree 2 del Pezzo surfaces

Theorem (Odaka-Spotti-Sun, Ascher-DeVleming-Liu)

A K-polystable limit of smooth del Pezzo surfaces of degree 2 is

- *a double cover $S \to \mathbb{P}^2$ branched along $(F_4 = 0) \subset \mathbb{P}^2$ with $[F_4] \in \overline{P_4}^{GIT} \setminus [2C]$*

$$S_\infty = \{x_4^2 = f_4(x_1, x_2, x_3)\} \subset \mathbb{P}(1,1,1,2)$$

- *a double cover $S \to \mathbb{P}(1,1,4)$ branched along $\{z_3^2 = f_8(z_1, z_2)\}$ where f_8 is a binary octic.*

$$S_\infty = \{x_3^2 + x_4^2 = f_8(x_1, x_2)\} \subset \mathbb{P}(1,1,4,4)$$

The K-moduli space $\mathrm{M}_{2,2}^{\mathrm{Kps}}$ is a weighted blowup of $[2C] \in \overline{P_4}^{GIT}$.

- (Odaka) A K-polystable Fano variety is always normal.

Methods of proof

Valuative criterion (Fujita-Li)

$$X \text{ is K-polystable} \iff \beta(E) > 0 \text{ for all } E \text{ over } X$$

For E/X, $\beta(E) = A_X(E) - \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(-K_X - tE)dt$

- the stability threshold $\delta_X = \inf_{E/X} \frac{A_X(E)}{S_X(E)}$ is estimated with Abban-Zhuang's theory of admissible flags
- there are equivariant versions of the Fujita-Li criterion (Zhuang)
- X is divisorially unstable if there is a divisor $E \subset X$ such that $\beta(E) < 0$

Theorem (Fujita)

The general member of 26 families of smooth Fano 3-folds is divisorially unstable.

The Calabi problem for smooth Fano 3-folds

Have a complete answer to the first formulation of the Calabi problem:

Theorem (Fano project 21 (previously known $\sim$ 65 cases))

Let X be the general member of one of the 105 deformation families of Fano 3-folds. Then one of:

- *X belongs to family* MM_{2-26}*, or*
- *X belongs to one of the 26 deformation families of K-divisorially unstable Fano 3-folds, or*
- *X is K-polystable*

- 78 families with smooth K-polystable members
- Both K-polystable and non-K-polystable smooth Fano 3-folds in several families
- Have a complete answer for 58 out of 78 families with K-polystable members

K-moduli spaces of Fano 3-folds

Standard deformation theoretic argument: when non-empty, the dimension of the component of $\mathrm{M}^{\mathrm{Kps}}_{3,V}$ associated to the deformation family $\mathrm{MM}_{\rho-N}$ is

$$-\chi(X, T_X) \text{ for any } X \text{ in the family}$$

- As in the case of del Pezzo surfaces, birational classification is not modular - no details on isomorphic Fano varieties
- Parametrisation is a starting point for description of K-moduli spaces
- Many (44/105) families have 0-dimensional moduli
  - 20 of these consist of one smooth K-polystable object
  - 1 additional family with non-empty K-moduli space
  - Other 23 families have empty K-moduli
- 8 families have 1-dimensional moduli
  - 6 families with K-polystable general member
  - 2 families with empty K-moduli space

Geometry of family MM_{4-1}

Members of family MM_{4-1} are divisors $(1,1,1,1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by a linear combination of monomials of the form

$$x_1^i y_1^{1-i} x_2^j y_2^{1-j} x_3^k y_3^{1-k} x_4^l y_4^{1-l} \text{ for } i,j,k,l \in \{0,1\}$$

The birational geometry of X is illustrated by

$$\begin{tikzcd}
& X = (1,1,1,1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "pr_1"'] \arrow[dd, "pr_{23}"] \arrow[dr, "pr_{234}"] & \\
\mathbb{P}^1 & & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \supset \Gamma \arrow[dl] \\
& \mathbb{P}^1 \times \mathbb{P}^1 \supset \Delta &
\end{tikzcd}$$

Geometry of family MM_{4-1} - earlier results

- (Belousov-Loginov) Smooth members of family MM_{4-1} are K-polystable
- (Quantum Computing) Natural GIT picture associated to the action of

$$\Gamma_0 = (\mathrm{SL}_2(\mathbb{C}))^4 \text{ and } \Gamma = \Gamma_0 \rtimes \mathfrak{S}_4 \text{ on } V = H^0((\mathbb{P}^1)^4, \mathcal{O}(1,1,1,1))$$

Theorem

Up to Γ action, the equation $f \in V$ can be put in a normal form that is of one of 9 types.

Equations in these normal forms form sublinear systems, for example $\mathcal{G} \subset V$

$$\mathcal{G} = \Big\{ f = \frac{a+d}{2}(x_1x_2x_3x_4 + y_1y_2y_3y_4) + \frac{a-d}{2}(x_1x_2y_3y_4 + x_3x_4y_1y_2) + \\ + \frac{b+c}{2}(x_1x_3y_2y_4 + x_2x_4y_1y_3) + \frac{b-c}{2}(x_1x_4y_2y_3 + x_2x_3y_1y_4) \Big\}$$

Geometry of family MM_{4-1}

Let $f \in V$, then

1. If $f \in V$ and $\{f = 0\}$ smooth, then $\Gamma \cdot f \cap \mathcal{G} \neq \emptyset$
2. If $f \in \mathcal{G}$, then $\mathrm{Aut}(X)$ contains a subgroup μ_2^4 of $i^*(\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$,
3. If $f \in \mathcal{G}$, then $\{f = 0\}$ is reducible $\iff f \in \Gamma \cdot \{(0 : 0 : 0 : 1)\}$
4. If $f, f' \in \mathcal{G}$ and $X = \{f = 0\}, X' = \{f' = 0\}$ are irreducible, then $X \simeq X' \iff \Gamma \cdot f \cap \Gamma \cdot f' \neq \emptyset$

Theorem (CFFK'24)

- *The GIT quotient* $\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma \simeq \mathbb{P}(1, 3, 4, 6)$
- *The orbit* $\Gamma \cdot \mathbb{P}\mathcal{G}$ *is the set of GIT polystable points*
- *The subgroup* $\Gamma^{\mathcal{G}} = \{\sigma \in \Gamma | \sigma(\mathcal{G}) \subset \mathcal{G}\} \simeq W(F_4)$, *so*

$$\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma \simeq \mathbb{P}(1, 3, 4, 6) \simeq \mathbb{P}^3 /\!\!/ W(F_4)$$

- *Explicit description of* $\mathbb{P}\mathcal{G} \to \mathbb{P}(1, 3, 4, 6)$

Boundary of the GIT-moduli space

There is a stratification

$$\begin{aligned}(\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma)^{\mathrm{Sing}} &= (\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma)^{\mathrm{Red}} \sqcup (\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma)^{\mathrm{Curv}} \sqcup (\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma)^{6A_1} \\ &\sqcup (\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma)^{4A_1} \sqcup (\mathbb{P}V^{\mathrm{ss}} /\!\!/ \Gamma)^{2A_1}\end{aligned}$$

and if X_p is determined by $p = (a : b : c : d) \in \mathbb{P}\mathcal{G}$, then

- X_p is singular $\iff$ $(a^2 - b^2)(a^2 - c^2)(a^2 - d^2)(b^2 - c^2)(b^2 - d^2)(c^2 - d^2) = 0$
- X_p is reducible $\iff$ p maps to $[2 : 2 : 0 : 0] \in \mathbb{P}(1, 3, 4, 6)$

Theorem (CFFK'24)

X_p is K-polystable whenever it is irreducible

Another family of degenerations of MM_{4-1}

As in the case of del Pezzo surfaces of degree 2 want a K-polystable replacement for the reducible GIT polystable degeneration

$$(1,1,0,0)+(0,0,1,1)\subset \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$$

View this as a product of pairs

$$\left(C=(1,1)\subset \mathbb{P}^1\times\mathbb{P}^1\right)\rightsquigarrow (C_\infty\subset \mathbb{P}(1,1,2))$$

where C_∞ is the section at infinity in the cone over C.
Consider degenerations of the form $(2,2)\subset \mathbb{P}(112)\times\mathbb{P}(112)$ with equation

$$w_1w_2+\alpha s_1t_1s_2t_2+\beta(s_1^2s_2^2+t_1^2t_2^2)+\gamma(s_1^2t_2^2+t_1^2s_2^2)=0\}$$

parametrised by $\mathbb{P}^2(\alpha,\beta,\gamma)$

Theorem

X_p is K-polystable for all $p\in E$

K-polystable degenerations of MM_{4-1}

Theorem (CFFK'24)

A K-polystable limit of members of MM_{4-1} is one of

- *an irreducible $(1,1,1,1)\subset \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ of the form*
$$X=\{a(x_1x_2x_3x_4+y_1y_2y_3y_4)+b(x_1x_2y_3y_4+x_3x_4y_1y_2)+ \\ +c(x_1x_3y_2y_4+x_2x_4y_1y_3)+d(x_1x_4y_2y_3+x_2x_3y_1y_4)=0\} \quad (*)$$
for $(a:b:c:d)\in\mathbb{P}^3$.
- *an irreducible $(2,2)\subset \mathbb{P}(1,1,2)\times\mathbb{P}(1,1,2)$ of the form*
$$X=\{w_1w_2+\alpha s_1t_1s_2t_2+\beta(s_1^2s_2^2+t_1^2t_2^2)+\gamma(s_1^2t_2^2+t_1^2s_2^2)=0\} \quad (**)$$
for $(\alpha:\beta:\gamma)\in\mathbb{P}^2$.

K-moduli space

Theorem

The component of K-moduli space $\mathrm{M}^{\mathrm{Kps}}_{3,24}$ *associated to family* MM_{4-1} *is the blowup of* $\mathbb{P}(1,3,4,6)$ *at the smooth point* $\{[2:2:0:0]\}$ *with weights* $(1,2,3)$.

How to determine the weights?

- Direct study of 3-folds of 3-folds of the form (**)
- Gives rise to a second GIT quotient
- Show that the two GIT quotients "glue" to form the K-moduli space.