On the cone conjecture for log Calabi–Yau mirrors of Fano 3-folds

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The Kinosaki Symposium
Thursday, Oct 24, 2024

<u>Kawamata-Morrison cone conjecture (1993)</u>

X: Calabi-Yau 3-fold

Then (i) $\mathrm{Aut}(X) \curvearrowright \mathrm{Nef}^e(X)$ with a rational polyhedral fundamental domain (R.P.F.D.);

(ii) $\mathrm{PsAut}(X) \curvearrowright \mathrm{Mov}^e(X)$ with an R.P.F.D.

<u>Important idea</u>

$f: \underbrace{Y}_{\substack{\text{sm proj} \\ \text{3 fold}}} \longrightarrow \mathbb{P}^1$ a K3 fibration s.t. $-K_Y = f^*\mathcal{O}(1)$

Z: flop of Y

Then the extremal rays of $\overline{\mathrm{Curv}}(Z)$ in the $K_Z < 0$ region is either of (numbering from Mori's Classification of Extremal Rays of $\overline{\mathrm{Curv}}(Y)$, see Kollár-Mori)

- Type (1): blowup of a smooth curve Γ
- Type (6): conic bundle

<u>Thm</u> [L.]

Y: sm. proj. 3-fold with K3 fibration $f: Y \to \mathbb{P}^1$ s.t. $-K_Y = f^*\mathcal{O}(1)$

Then (i) $\mathrm{PsAut}(Y) \curvearrowright \{$Type (6) codim. 1 faces of $\mathrm{Mov}^e(Y)\}$ with finitely many orbits;

(ii) $\mathrm{PsAut}(Y) \curvearrowright \{$Type (1) codim. 1 faces of $\mathrm{Mov}^e(Y)\}$ with finitely many orbits if $H^3(Y,\mathbb{C}) = 0$.

<u>Rem</u> If $H^3(Y,\mathbb{C}) = 0$, then in the case of Type (1) : genus $g(T) = 0$.

<u>Conj</u> $\mathrm{PsAut}(Y) \curvearrowright \{$Type (1) codim. 1 faces of $\mathrm{Mov}^e(Y)\}$ with finitely many orbits $\forall\, g \geq 0$.

By a result of Kawamata, we know

<u>Thm</u> $\mathrm{PsAut}(Y) \curvearrowright \{$ codim 1 faces of $\mathrm{Mov}^e(Y)$ containing $-K_Y\}$ w/ finitely many orbits.

<u>Rem</u> <u>Conj</u> + <u>Thm</u> (Kawamata) $\Rightarrow$ $\mathrm{PsAut}(Y) \curvearrowright \{$ all codim 1 faces of $\mathrm{Mov}^e(Y)\}$ w/ finitely many orbits.

<u>Pf of Main Thm</u>

<u>Kawamata-Morrison-Totaro Cone conjecture</u> (Proved by Totaro when $\dim Y = 2$)

(Y,Δ) : Klt and $K_Y + \Delta = 0$

Then (i) $\mathrm{Aut}(Y,\Delta) \curvearrowright \mathrm{Nef}^e(Y)$ w/ RPFD;

(ii) $\mathrm{PsAut}(Y,\Delta) \curvearrowright \mathrm{Mov}^e(Y)$ w/ RPFD.

Consider Y CY3 with K3 fibration $f: Y \to \mathbb{P}^1$ s.t. $-K_Y = f^*\mathcal{O}(1)$

F: general fiber of f

$F_1, F_2 \sim F$: sm. fibers

$\Delta = \frac{1}{2}F_1 + \frac{1}{2}F_2$

Then (Y,Δ) : Klt CY.

<u>Rem</u> Under our assumptions, such Y arise naturally as mirrors to $\underbrace{\text{sm. Fano 3folds}}_{\text{105 deformation types}}$.

[Birkar-Cascini-Hacon-McKernan] $\Rightarrow$ $\mathrm{Mov}(Y) = \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQM}}} \mathrm{Nef}(Z)$ is locally rational polyhedral in $\mathrm{Int}(\mathrm{Mov}(Y))$.

Under our assumptions, the only SQMs are flops / compositions of flops.

Let $S := \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQMs}}} \{\text{codim.1 faces of } \mathrm{Nef}(Z) \text{ in the boundary of } \mathrm{Mov}(Y)\} \twoheadrightarrow \{\text{codim.1 faces of } \mathrm{Mov}^e(Y)\}$.

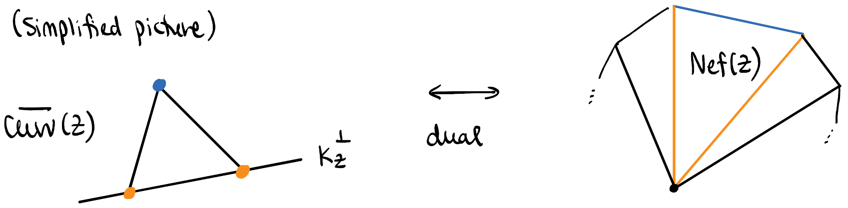
$\updownarrow$

$\bigcup_{\substack{Y \dashrightarrow Z \\ \text{flops}}} \{\text{extremal rays of } \overline{\mathrm{Curv}}(Y)\}$ $\quad \overline{\mathrm{Curv}}(Y) = \mathrm{Nef}(Z)^*$ $\Big]$ Strategy: classify these extremal rays

<u>Mori's Cone Theorem</u>

Z. sm. proj. var.

(Simplified picture)



$\Bigg\}$ $\mathrm{Mov}(Y) = \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQM}}} \mathrm{Nef}(Z)$

- Blue extremal rays in $K_Z < 0$ region: either Type (1) or Type (6)
- Orange extremal rays on $K_Z^{\perp}$: use Kawamata's Theorem

Type (6) Conic bundle $g : Z \to S$

$$\begin{tikzcd} Y \arrow[r, dashed, "\text{flops}"'] & Z \arrow[r, "g"] \arrow[d, "\text{K3 fibration}"] & S \\ & \mathbb{P}^1 & \end{tikzcd}$$

- $h := g|_F : \underbrace{F}_{\text{generic fiber}} \to S$ is finite, deg 2

- $\exists$ involution i corresponding to h: $\begin{matrix} F \circlearrowleft i \\ \downarrow \\ S : \text{smooth} \end{matrix}$

$$\Longrightarrow \begin{cases} (1)\ \mathrm{Fix}(i) = \bigcup \text{sm. curves} \Rightarrow S \text{ is rat'l surf.} \\ (2)\ \mathrm{Fix}(i) = \emptyset \Rightarrow S : \text{Enriques surf.} \quad (\text{cone conj holds (done)}) \end{cases}$$

Suppose S: rat'l surf. Then run MMP on S.

$\pi : S \to \underbrace{\bar{S}}_{\mathbb{P}^2 \text{ or } \mathbb{F}_n\ (0 \le n \le 4,\ n \ne 1)}$

Now have $F \xrightarrow{h} S \xrightarrow{\pi} \bar{S} \xrightarrow{\phi} \hat{S}$, $\theta := \pi \circ h : F \to \bar{S}$

Let $L := \theta^* M$, where

If $\bar{S} = \mathbb{P}^2$: take $\phi = \mathrm{id}$ and $M = H$ a hyperplane class

$\Rightarrow L^2 = 2$

If $\bar{S} = \mathbb{F}_n$: take $\phi : \mathbb{F}_n \to \underbrace{\mathbb{P}(1,1,n)}_{\hat{S}}$ the contraction of the neg. section of $\mathbb{F}_n$ and M the positive section of $\mathbb{F}_n$.

$\Rightarrow L^2 = 2M^2 = 2n \le 8$

Thm [Sterk]

F : K3 surf.

L: nef line bundle, $L^2 = 2k$ for some fixed $k \in \mathbb{N}$

Then $\mathrm{Aut}(F) \curvearrowright$ {all such L} w/ finitely many orbits.

$\rightsquigarrow$ $\mathrm{Aut}(\underbrace{F}_{\text{generic fiber}}) \curvearrowright$ {Type (6) faces of $\mathrm{Mov}^e(Y)$} w/ finitely many orbits.

and $\mathrm{Aut}(F_\eta) \subset \mathrm{PsAut}(Y)$.

Type (1) B.U. of sm. curve T or contraction of a ruled surf. E.

$\mathbb{P}^1$-bdl $g: E \to T$ (trivial bdl)

$\Rightarrow \forall$ general fiber F of K3 fibr. f., $T \subset F$

Assume $H^3(Y, \mathbb{C}) = 0$. Then $H^1(T) = 0$ so $T \cong \mathbb{P}^1$.

By Adjunction Formula, $T^2 = -2$.

Thm [Sterk, Kawamata]

S: K3 surf / k (field of char 0)

Then $\mathrm{Aut}(S) \curvearrowright \mathrm{Nef}^e(S)$ w/ a R.P.F.D.

In part., $\mathrm{Aut}(S) \curvearrowright$ {all (-2)-curves $C \subset S$} w/ finitely many orbits.

Let Y_η : the generic fiber of K3 fibr. $f: Y \to \mathbb{P}^1$

Then,

$$\mathrm{Aut}(Y_\eta) \curvearrowright \{(-2)\text{-curves } C \subset Y_\eta\} \text{ w/ finitely many orbits.}$$

$\Rightarrow$ $\mathrm{PsAut}(Y_\eta) \curvearrowright \{(-2)\text{-curves } C \subset Y_\eta\}$ w/ finitely many orbits.

$\Rightarrow$ $\mathrm{PsAut}(Y) \curvearrowright \{(-2)\text{-curves } C \subset Y_\eta\}$ w/ finitely many orbits.

$\square$