Resolution of some Singularities via Jet Spaces

Meral Tosun, Galatasaray University

Joint work with B.Karadeniz Şen, C. Plénat

Kinosaki Algebraic Geometry Symposium, Kyoto 2024

October 25, 2024

Characterization of Rational Singularities

Definition

Let X be a germ of (normal) surface in $\mathbb{C}^n$.

Let $\pi : \tilde{X} \to X$ be a resolution.

X has a rational singularity at 0 if $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$.

Let $E = \cup_{i=1}^r E_i$ be the exceptional fibre of π.

Characterization of Rational Singularities

Definition implies

(1) Dual graph Γ of π is a tree.

(2) For all positive divisors $Y = \sum_{i=1}^{r} a_i E_i$ supported on E, we have

$p_a(Y) := \frac{1}{2}[Y \cdot Y + \sum_{i=1}^{n} a_i(w_i - 2)] + 1 \leq 0$ for all positive divisors Y.

Characterization of Rational Singularities

(3) The smallest positive divisor Y satisfying $(Y \cdot E_i) \leq 0$ for each i, called the Artin cycle of π, has the arithmetic genus 0.

(4) The Artin cycle Z satisfies $Z \cdot Z = -m$ where m is the multiplicity of X at the singularity 0.

Rational singularities of multiplicity 2

The rational double point (RDP) singularities are defined in $\mathbb{C}^3$ by

$A_n : f(x, y, z) = xy - z^{n+1},\ n \in \mathbb{N},\ n \geq 1$

$D_n : f(x, y, z) = z^2 - x(y^2 + x^{n-2}),\ n \in \mathbb{N},\ n \geq 4$

$E_6 : f(x, y, z) = z^2 + y^3 + x^4$

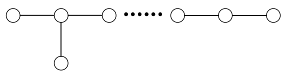
$E_7 : f(x, y, z) = x^2 + y^3 + yz^3$

$E_8 : f(x, y, z) = z^2 + y^3 + x^5$

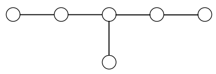
Rational singularities of multiplicity 2 (Du Val 1934)

A_n

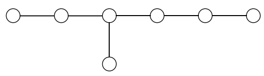
D_n



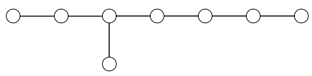
E_6



E_7



E_8



Meral Tosun, Galatasaray University Joint work with B.Karadeniz Şen, C. Plénat Kinosaki Algebraic Geometry Symposium, Kyoto 2024
Resolution of some Singularities via Jet Spaces

Rational singularities of multiplicity 3 (G.N.Tjurina, 1968)

The rational triple point (RTP) singularities of surfaces are defined in $\mathbb{C}^4$ by

RTP-type	Equations	RTP-type	Equations
$A_{k-1,\ell-1,m-1}$ $k,\ell,m \geq 1$	$xw - y^m w - y^{\ell+m} = 0$ $zw + y^\ell z - y^k w = 0$ $xz - y^{m+k} = 0$	$C_{k-1,\ell+1}$ $k \geq 1,\ \ell \geq 2$	$xz - y^k w = 0$ $w^2 - x^{\ell+1} - xy^2 = 0$ $zw - x^\ell y^k - y^{k+2} = 0$
$B_{k-1,n}$ $n = 2\ell > 3$	$xz - y^{k+\ell} - y^k w = 0$ $w^2 + y^\ell w - x^2 y = 0$ $zw - xy^{k+1} = 0$	$B_{k-1,n}$ $n = 2\ell - 1 \geq 3$	$xz - y^k w = 0$ $zw - xy^{k+1} - y^{k+\ell} = 0$ $w^2 - x^2 y - xy^\ell = 0$
D_{k-1} $k \geq 1$	$xz - y^{k+2} - y^k w = 0$ $zw - x^2 y^k = 0$ $w^2 + y^2 w - x^3 = 0$	F_{k-1} $k \geq 1$	$xz - y^k w = 0$ $zw - x^2 y^k - y^{k+3} = 0$ $w^2 - x^3 - xy^3 = 0$
H_n $n = 3k$	$z^2 - xw = 0$ $zw + y^k z - x^2 y = 0$ $w^2 + y^k w - xyz = 0$	H_n $n = 3k+1$	$z^2 - xy^{k+1} - xyw = 0$ $zw - x^2 y = 0$ $w^2 + y^k w - xz = 0$
H_n $n = 3k - 1$	$z^2 - xw = 0$ $zw - x^2 y - xy^k = 0$ $w^2 - y^k z - xyz = 0$	$E_{6,0}$	$z^2 - yw = 0$ $zw + y^2 z - x^2 y = 0$ $w^2 + y^2 w - x^2 z = 0$
$E_{0,7}$	$z^2 - yw = 0$ $zw - x^2 y - y^4 = 0$ $w^2 - x^2 z - y^3 z = 0$	$E_{7,0}$	$z^2 - yw = 0$ $zw + x^2 z - y^3 = 0$ $w^2 + x^2 w - y^2 z = 0$

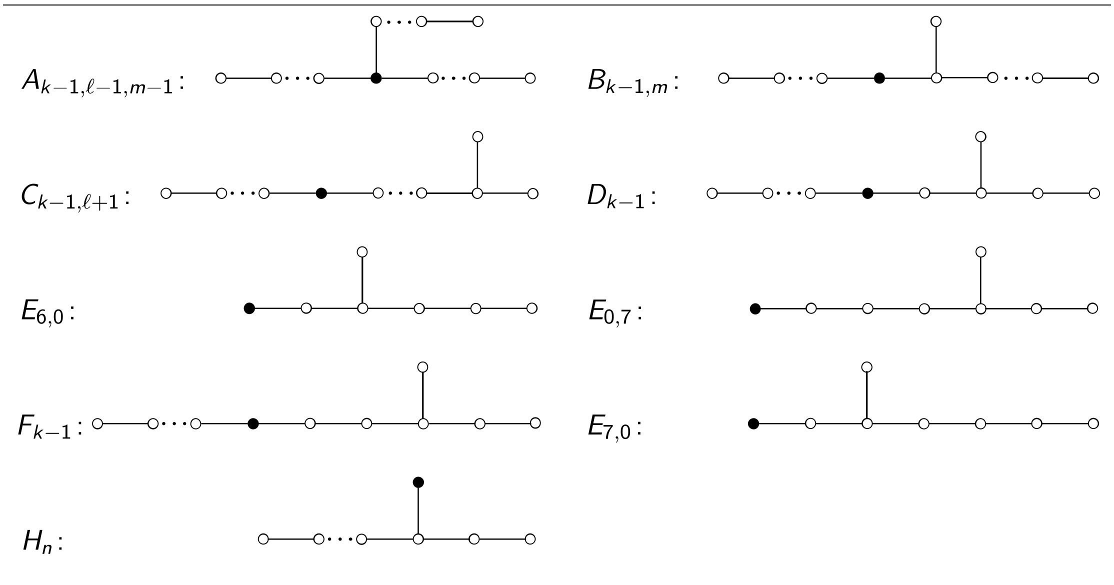
Non-isolated *RTP*-singularities (Altintas Sharland, Cevik, –)

Proposition - Definition

A (suitable) projection of each of the Tjurina's equations into $\mathbb{C}^3$ is one of the nonisolated hypersurface given by the equations below, called the nonisolated forms of RTP- singularities.

RTP-types	Equations	RTP-types	Equations
$A_{k,\ell,m},\ k,\ell,m > 1$	• $y^{3m+3} + xy^{m+1}z - xz^2 - z^3 = 0$ • $y^{k+\ell+m+3} + y^{2k+2}z + y^{k+1}z^2 + xy^{k+1}z + xz^2 - z^3 = 0$ • $y^{3k} + y^{2k+m+\ell-2} - 2y^{\ell+k}z - xy^k z + y^m z^2 + xz^2 - z^3 = 0$ • $y^{2k+m} + y^{k+m}z + y^{\ell+k}z + xy^k z - y^k z^2 + y^\ell z^2 + xz^2 - z^3 = 0$	$C_{n,m}$ $n \geq 3, m \geq 2$	$x^{n-1}y^{2m+2} + y^{2m+4} - xz^2 = 0$
$B_{k,n}, n \geq 2$ $k = 2r-1, r \geq 1$	$x^{2n+3}z - x^r y^2 - y^2 z = 0$	$B_{k,n}, n \geq 2$ $k = 2r, r \geq 1$	$x^{n+r+2}y - x^{2n+3}z + y^2 z = 0$
$D_n, n \geq 1$	$x^{2n+2}y^2 - x^{n+3}z + yz^2 = 0$	$F_{k-1}, k \geq 2$	$y^{2k+3} + x^2 y^{2k} - xz^2 = 0$
$H_n, n \geq 1$ $n = 3k$	$z^3 + xy^k z + x^3 y = 0$	$H_n, n \geq 1$ $n = 3k+1$	$z^3 + xy^{k+1}z + x^3 y^2 = 0$
$H_n, n \geq 1$ $n = 3k-1,$	$z^3 + x^2 y(x + y^{k-1}) = 0$	E_{60}	$z^3 + y^3 z + x^2 y^2 = 0$
E_{07}	$z^3 + y^5 + x^2 y^2 = 0$	E_{70}	$z^3 + x^2 yz + y^4 = 0$

Dual graphs of RTP-singularities (M.Artin, 1966)



Embedded resolutions, Jets

Let $X \subset Z$.

An embedded resolution of X is a proper birational morphism $\mu : Y \longrightarrow Z$ such that

Y is nonsingular,

the strict transform of X by μ is nonsingular and transversal to the exceptional locus of μ.

Singularities, Arcs, Jets

Definition

Let X be variety in $\mathbb{C}^N$.

An arc on $X \subset \mathbb{C}^n$ is a formal parametrized curve given by

$$\gamma : Spec\mathbb{C}[[t]] \to X$$

The set of arcs on X is called the arc space of X, denoted by $J_\infty(X)$.

Singularities, Arcs, Jets

Consider the $\mathbb{C}$-algebra homomorphism

$$\varphi : \frac{\mathbb{C}[x_1 \dots x_n]}{\langle f \rangle} \to \mathbb{C}[[t]]$$

defined by $x_i \longmapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)} t + \cdots$

These images satisfy $f(\varphi(x_1), \dots, \phi(x_n)) = 0$

Meral Tosun, Galatasaray University Joint work with B.Karadeniz Şen, C. Plénat Kinosaki Algebraic Geometry Symposium, Kyoto 2024
Resolution of some Singularities via Jet Spaces

Singularities, Arcs, Jets

For $\mathbf{x}^{(j)} = \left(x_1^{(j)}, \dots, x_n^{(j)}\right)$, write

$$f\left(\phi\left(x_1\right), \dots, \phi\left(x_n\right)\right) = \sum_{j \geq 0} F^{(j)}\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(j)}\right) t^j$$

and look for $F^{(j)}\left(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(j)}\right) = 0$.

$$J_\infty(X) = Spec \frac{\mathbb{C}\left[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots\right]}{\langle F^{(j)} \rangle_{j \geq 0}}$$

Singularities, Arcs, Jets

When X is smooth $J_\infty(X)$ is irreducible.

Singularities, Arcs, Jets

To compute $F^{(j)}$'s:

Let $\mathbf{D}$ be a derivation on $\mathbb{C}\left[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right]$ defined by

$$\mathbf{D}x_i^{(j)} = x_i^{(j+1)}, \quad i = 1, \ldots, n \;\; \textit{and} \;\; j \in \mathbb{N}$$

Replacing $x_i^{(j)}$ by $x_i^{(j)}/j!$, we get

$$f\left(\phi\left(x_1\right), \ldots, \phi\left(x_n\right)\right) = \sum_{j \geq 0} \frac{F^{(j)}\left(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(j)}\right)}{j!} t^j$$

where $F^{(0)} = f$ and $F^{(j)}$ is recursively defined by $\mathbf{D}\left(F^{(j)}\right) = F^{(j+1)}$.

This gives the differential structure on $J_\infty(X)$.

Jet space

The mth jet space of X is

$$J_m(X) = Spec \frac{\mathbb{C}\left[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right]}{\langle F^{(j)} \rangle_{j=0,\ldots,m}}$$

Jet space

The mth jet space of X is

$$J_m(X) = Spec\frac{\mathbb{C}\left[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right]}{\langle F^{(j)}\rangle_{j=0,\ldots,m}}$$

Note that

$J_0(X) = X$.

$\pi_{m,m-1} : J_m(X) \longrightarrow J_{m-1}(X)$ says arcs and jets are related to geometry of singularity. ; in

particular, $\pi_{m,0} : J_m(X) \longrightarrow J_0(X)$.

$\psi_m : J_\infty(X) \longrightarrow J_m(X)$ arcs by their limit behaviours at singular points.

Singularities, Arcs, Jets

When $X = \mathbb{C}^n$,

$J_m(X) = Spec\mathbb{C}\left[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\right] = \mathbb{C}^{n(m+1)}$.

Jets - RTP-singularities

Consider $X = E_{60} : f(x, y, z) = z^3 + y^3 z + x^2 y^2$.

We have $Sing(X) = \{(x, 0, 0) \mid x \in \mathbb{C}\}$.

$m = 0 \Longrightarrow f(x(t), y(t), z(t)) = z_0^3 + y_0^3 z_0 - x_0^2 y_0^2 = F_0 \ (mod\ \ t)$.

$$J_0(X) := Spec(\frac{\mathbb{C}[x_0, y_0, z_0]}{\langle F_0 \rangle}) \subset \mathbb{C}^3$$

Ttwo irreducible components:

Over $Sing(X)$: $V(\langle y_0, z_0 \rangle) := C_0^1$

Over the coordinate axes contained in X: $V(\langle x_0, z_0 \rangle) := C_0^2$

Jets - RTP-singularities

$m = 1 \Longrightarrow f(x(t), y(t), z(t)) = F_0 + tF_1 \ (mod\ \ t^2)$

We get $F_1 := 3z_0^2 z_1 + 3y_0^2 y_1 z_0 + y_0^3 z_1 - 2x_0 x_1 y_0^2 - 2x_0^2 y_0 y_1$ and

$$J_1(X) := Spec(\frac{\mathbb{C}[x_0, y_0, z_0, x_1, y_1, z_1]}{\langle F_0, F_1 \rangle}) \subset \mathbb{C}^6$$

$$\pi_{1,0} \colon J_1(X) \to J_0(X)$$

$$(x_0, y_0, z_0, x_1, y_1, z_1) \mapsto (x_0, y_0, z_0)$$

Over $Sing(X)$: $\pi_{1,0}^{-1}(C_0^1) = V(\langle y_0, z_0 \rangle) := C_1^1$

Over the coordinate axes: $\pi_{1,0}^{-1}(C_0^2) = V(\langle x_0, z_0, z_1 \rangle) := C_1^2$

Singularities, Arcs, Jets

Nash problem: What are the infos in $J_\infty(X)$ which is common to all resolution of the variety.

Singularities, Arcs, Jets

Definition

An irreducible component of the exceptional divisor of a resolution of X is called essential if, in any other resolution, that component is an irreducible component of the exceptional divisor.

Singularities, Arcs, Jets

Definition

An irreducible component of the exceptional divisor of a resolution of X is called essential if, in any other resolution, that component is an irreducible component of the exceptional divisor.

Nash Problem, 1963

The number of the irreducible components of $J_\infty(X)$ passing through $Sing(X)$ equals the number of essential irreducible components of the exceptional fibre of a resolution of X.

Singularities, Arcs, Jets

Let $\pi : \tilde{X} \longrightarrow X$ be a resolution of X.

π induces $\pi_\infty : J_\infty(\tilde{X}) \longrightarrow J_\infty(X)$ defined by $\gamma \mapsto \pi_\infty(\gamma) = \pi \circ \gamma$.

$\psi_m : J_\infty(X) \longrightarrow J_m(X)$ and $\psi_m^{\tilde{X}} : J_\infty(\tilde{X}) \longrightarrow J_m(\tilde{X})$

$(\psi_0^{\tilde{X}})^{-1}(E) \longrightarrow {\psi_0}^{-1}(SingX)$

$(\psi_0^{\tilde{X}})^{-1}(E) = \cup_{i=1}^r (\psi_0^{\tilde{X}})^{-1}(E_i)$ has finite number of irreducible components.

$\{$Irreducible components of $\left(\psi_0^X\right)^{-1}(SingX)\} \longrightarrow \{$Essential divisors of X$\}$

Singularities, Arcs, Jets

In dimension 2

In 2012, J. Fernandez de Bobadilla and M. Pe Pereira gave an affirmative answer to the Nash Problem using topological arguments.

In 2016, De Fernex and De Campo solved the problem using algebra-geometric arguments.

Singularities, Arcs, Jets

In higher dimension

In 2003, S. Ishii and J. Kollar gave a positive answer for toric varieties of all dimensions.

Singularities, Arcs, Jets

Example (Ishii, Kollar, 2003)

The 4-dimensional hypersurface singularity over an algebraically closed field of characteristic

$\neq 2, 3$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$$

has only 1 irreducible family of arcs but 2 essential exceptional components.

So Nash map is not surjective.

Singularities, Arcs, Jets

Inverse Nash Problem (Lejeune-Jalabert, Reguera, Mourtada, Plénat, 2013)

Given the jet spaces $J_m(X)$, can we construct a resolution of X?

Singularities, Arcs, Jets

In 2018, C. Plénat and H. Mourtada constructed a toric resolution for RDP-singularities.

Singularities, Arcs, Jets

In 2018, C. Plénat and H. Mourtada constructed a toric resolution for RDP-singularities.

Here we answer it for RTP-singularities.

Embedded valuation set

A key fact is: the interpretation of a divisorial valuation on X as the order of annihilation along arcs in an irreducible component of some contact locus.

Embedded valuation set

Consider the canonical map

$$\psi_m : J_\infty(\mathbb{C}^3) \to J_m(\mathbb{C}^3)$$

and the set

$$Cont^p(f) = \left\{\gamma \in J_\infty(\mathbb{C}^3) \mid \quad ord_\gamma(f) = p\right\}, \quad p \in \mathbb{N}$$

since $codim\,(C_{m+1}) > codim\,(C_m)$.

Embedded valuation set

Ley $C_m \subset J_m(X)^0$ be an irreducible component.

Let η be the generic point of $\psi_m^{-1}(C_m) \cap Cont^{m+1}(f)$.

For $h \in \mathbb{C}[x, y, z]$, let $\nu_{C_m}(h)$ denotes $\nu_{C_m}(h) = ord_t h \circ \eta$.

Associate a vector with each C_m as

$$v_m = (\nu_{C_m}(x), \nu_{C_m}(y), \nu_{C_m}(z)) \in \mathbb{N}^3$$

.

Embedded valuation set

Definition

The elements of the set

$$EC(X) := \{C_m \subset J_m(X)^0 \mid m \geq 1,\ \pi_{m,m-1}(C_m) \subset C_{m-1},\ v_m \neq v_{m-1}\}$$

is called the essential components of X.

The elements of $EV(X) := \{v_m \mid C_m \in EC(X)\}$ is called the embedded valuations of X.

Jets - RTP-singularities

Consider $X \subset \mathbb{C}^3 : f(x, y, z) = z^3 + y^3 z + x^2 y^2$. Compute $J_m(X)$.

We have $Sing(X) = \{(x, 0, 0) \mid x \in \mathbb{C}\}$.

$m = 0 \Longrightarrow f(x(t), y(t), z(t)) = z_0^3 + y_0^3 z_0 - x_0^2 y_0^2 = F_0 \ (mod\ t)$.

$$J_0(X) := Spec(\frac{\mathbb{C}[x_0, y_0, z_0]}{\langle F_0 \rangle}) \subset \mathbb{C}^3$$

Consider two irreducible components:

Over $Sing(X)$: $V(\langle y_0, z_0 \rangle) := C_0^1 \longrightarrow \mathbf{v}_0^1 = (0, 1, 1)$.

Over the coordinate axes contained in X: $V(\langle x_0, z_0 \rangle) := C_0^2 \longrightarrow \mathbf{v}_0^2 = (1, 0, 1)$.

Jets - RTP-singularities

- $(0, 1, 1)$
- $(1, 0, 1)$

0

Jets - RTP-singularities

$m = 1 \Longrightarrow f(x(t), y(t), z(t)) = F_0 + tF_1 \ (mod\ t^2)$

where $F_1 := 3z_0^2 z_1 + 3y_0^2 y_1 z_0 + y_0^3 z_1 - 2x_0 x_1 y_0^2 - 2x_0^2 y_0 y_1$.

$$J_1(X) := Spec(\frac{\mathbb{C}[x_0, y_0, z_0, x_1, y_1, z_1]}{\langle F_0, F_1\rangle}) \subset \mathbb{C}^6$$

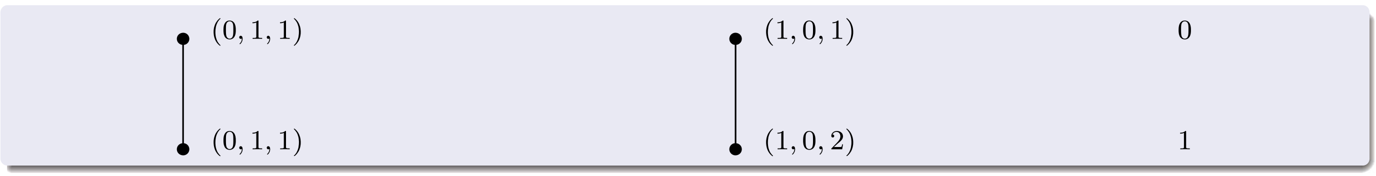
$$\pi_{1,0} \colon J_1(X) \to J_0(X)$$

$$(x_0, y_0, z_0, x_1, y_1, z_1) \mapsto (x_0, y_0, z_0)$$

Over $Sing(X)$: $\pi_{1,0}^{-1}(C_0^1) = V(\langle y_0, z_0\rangle) := C_1^1 \longrightarrow \mathbf{v}_1^1 = (0, 1, 1)$

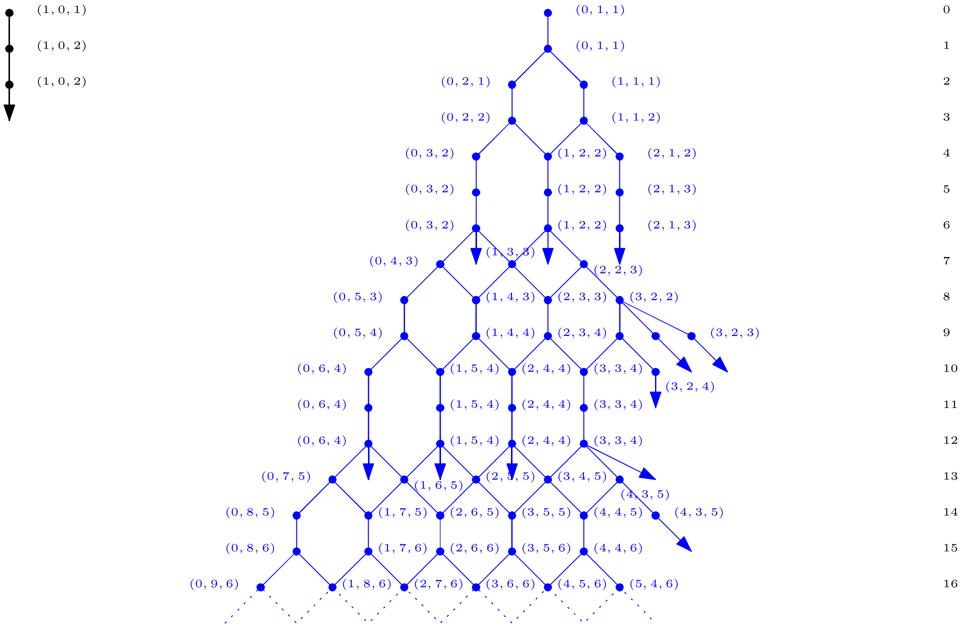
Over the coordinate axes: $\pi_{1,0}^{-1}(C_0^2) = V(\langle x_0, z_0, z_1\rangle) := C_1^2 \longrightarrow \mathbf{v}_1^2 = (1, 0, 2)$

Jet spaces



Jet graph - Example

$X \subset \mathbb{C}^3 : f(x, y, z) = z^3 + y^3 z + x^2 y^2.$



Relation between Jet graph - Dual Graph

Theorem (Koreeda 2021)

Let X be $D_n : f(x, y, z) = z^2 - xy^2 - x^{n-1} = 0$ for n=4.

Let $J_m^0(X) := \cup_{i=0}^n C_m^i$

Let $V = \{C_m^1, \ldots, C_m^n\}$.

Consider the subset $E \subseteq \{C_m^i \cap C_m^j \mid i, j \in \{1, \ldots, n\}$ with $i \neq j\}$ of the maximal elements for the inclusion relation.

Construct a graph Γ such that the vertices of Γ are elements of V, and there is given an edge between C_m^i and C_m^j if and only if $C_m^i \cap C_m^j \in E$.

Γ is the dual graph of the minimal resolution of the D_4-singularity.

Embedded resolutions, Jets

Construct an embedded toric resolution of RTP-singularities, we use their important property:

The RTP-singularities in $\mathbb{C}^4$ and in $\mathbb{C}^3$ are Newton non-degenerate.

Non-degeneracy - (Ehlers1975, Kouchnirenko 1976, Khovanskii 1977)

Definition - For hypersurfaces

Let X be a hypersuface in $\mathbb{C}^3$ defined by $f(\mathbf{x}) = \sum c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$.

Here $\mathbf{x}^{\mathbf{u}} = x_1^{u_1}x_2^{u_2}\cdots x_n^{u_n}$ with $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ for all $u_i \in \mathbb{N}$.

The support set of f is

$$supp(f) := \{\mathbf{u} \in \mathbb{R}^n \mid c_{\mathbf{u}} \neq 0\}.$$

The Newton polyhedron of X is

$$NP(X) := \overline{conv(\bigcup_{\mathbf{u}\in supp(f)} (\mathbf{u} + \mathbb{R}^n_{\geq 0}))}$$

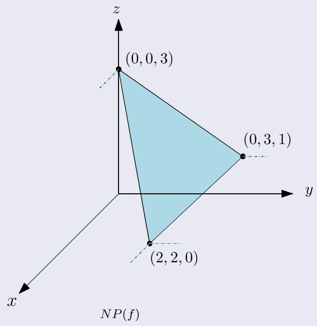
Non-degeneracy - (Ehlers1975, Kouchnirenko 1976, Khovanskii 1977)

Example - $E_{6,0}$

$X \subset \mathbb{C}^3 : f(x,y,z) = z^3 + y^3 z + x^2 y^2 = 0.$

$$supp(f) = \{(0,0,3),(0,3,1),(2,2,0)\}$$

The $NP(f)$ is



Newton non-degeneracy - (Ehlers1975, Kouchnirenko 1976, Khovanskii 1977)

Definition

Let F be a face of the $NP(X)$. The restriction of f to F is

$$f\mid_F := \sum_{\alpha \in supp(f) \cap F \subset \mathbb{Z}^n} c_\alpha \mathbf{x}^\alpha$$

Definition

A hypersurface $X \subset \mathbb{C}^3$ is called Newton non-degenerate if the variety $V(\langle f \mid_F \rangle)$ for each face F of $NP(X)$ has no singularities outside the coordinate hyperplanes.

Non-degeneracy-singularities

(Oka 1997) For complete intersection singularities, non-degeneracy is characterized in terms of the Newton polyhedra of a given set of generators of the ideal.

(Aroca, Morales, Shabbir 2018) For general case, non-degeneracy is characterized in terms of the initial ideals of the given ideal.

Dual of NP leads to a toric resolution

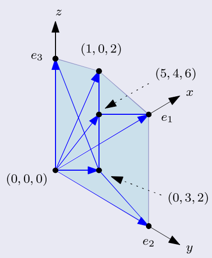
The set

$$DNP(f) := \{\mathbf{w} \in \mathbb{R}^n | \langle \mathbf{w}, \mathbf{v} \rangle \geq 0 \ \ \forall \mathbf{v} \in NP(f)\}$$

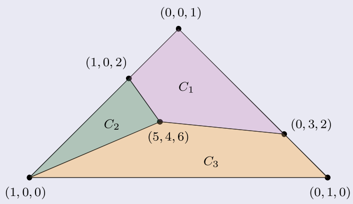
is called the dual Newton polyhedron of f.

Example

For $X \subset \mathbb{C}^3 : f(x, y, z) = z^3 + y^3 z + x^2 y^2$, the $DNP(f)$ is



$DNP(f)$



$DNP(f)$

Regular refinement of $DNP(X)$ - Determinant of a cone

Let $\sigma = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \rangle$ be a cone in $DNP(X)$.

We associate a matrix with σ as follows:

$$M_\sigma := [\mathbf{v}_1 \; \mathbf{v}_2 \ldots \mathbf{v}_r]$$

Consider the $r \times r$ minors M_r of M_σ.

Definition

Let M_r be the $r \times r$ minors of M_σ.

The determinant of σ is

$$det(\sigma) := gcd(det(M_r))$$

Regular refinement of $DNP(X)$

Definition

If $det(\sigma) = \pm 1$, σ is called regular.

Definition

$DNP(X)$ is regular if each cone in it is regular.

Newton non-degeneracy and resolution

Theorem (Varchenko, 1976 - Khovanskii, 1984)

Let $X \subset \mathbb{C}^3$ be Newton non-degenerate. The following properties are equivalent:

$i)$ A refinement $RDNP(X)$ of $DNP(X)$ is regular.

$ii)$ $\tilde{U}_{RDNP(X)}$ is the smooth toric variety associated to $RDNP(X)$ and the proper birational map π is an embedded toric resolution of X.

$$\begin{array}{ccc} \pi^{-1}(X) \cap \tilde{U}_{RDNP(X)} = \tilde{X} & \subset & \tilde{U}_{RDNP(X)} \\ \downarrow \pi & & \downarrow \pi_{RDNP(X)} \\ X & \subset & \mathbb{C}^3 \end{array}$$

The strict transform of $\{f = 0\}$ by $\pi_{RDNP(X)}$ is the Zariski closure of $(\pi^{-1}_{RDNP(X)}(\mathbb{C}^3 \cap \{f = 0\})$.

Embedded Resolution of RTP-singularities

Theorem (Karadeniz, Plénat, –, 2024)

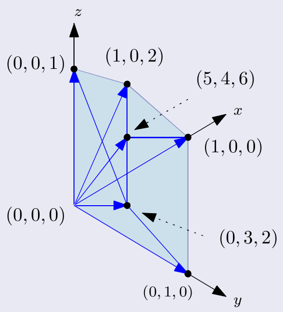
Let X be an RTP-singularity in $\mathbb{C}^3$.

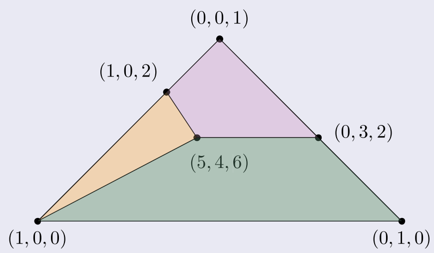
Then the refinement of $DNP(X)$ by the elements of $EV(X)$ give an embedded toric resolution and the abstract resolution graph is minimal.

To Prove the Theorem - Dual Newton Polyhedron

Example

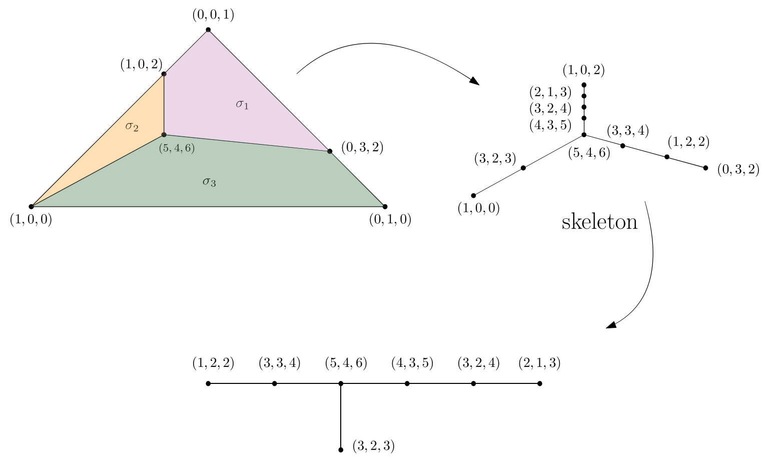
$X \subset \mathbb{C}^3 : f(x,y,z) = z^3 + y^3 z + x^2 y^2 = 0$. The dual of $NP(X)$ is





Abstract resolution

$X \subset \mathbb{C}^3 : f(x,y,z) = z^3 + y^3 z + x^2 y^2 = 0$.

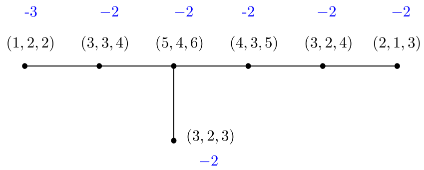


To Prove the Theorem - Oka's algorithm- Abstract Resolution

Definition

The number $c \in \mathbb{N}$ such that $c \cdot \mathbf{v} = \sum_{i=1}^{m} \mathbf{v}_i$ is called the weight of the vertex $\mathbf{v}$ in the graph

where each $\mathbf{v}_i$ is adjacent to $\mathbf{v}$ in the regular refinement.



Definition

A resolution of a surface is called minimal if there is no vertex with weight 1 in the graph.

To Prove the Theorem - Minimality of Toric Embedded Resolution

Definition (Bouvier, Gonzalez-Sprinberg, 1995)

Let $\sigma = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \rangle$ be 3-dimensional simplicial cone.

The smallest primitive vector on a 1-dimensional face of σ is called extremal vector.

Consider $\ell_\sigma : \mathbb{R}^n \to \mathbb{Q}$ such that $\ell_\sigma(\mathbf{v}_i) = 1$ with each extremal vector $\mathbf{v}_i$ for σ.

The set

$$p_\sigma := \sigma \cap \ell_\sigma^{-1}([0, 1])$$

is called the profile of σ.

Proposition - Definition

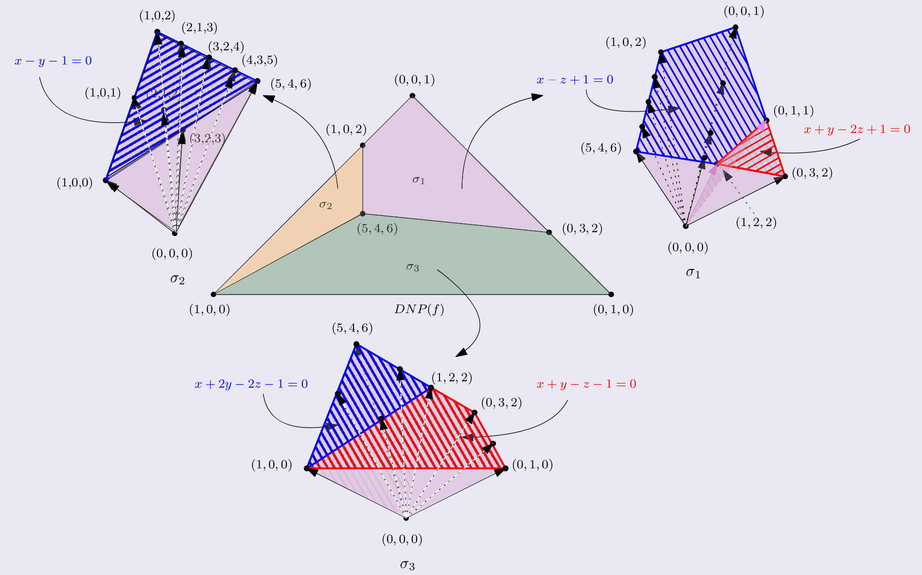
When σ is a simplicial cone, all extremal vectors lie on a unique hyperplane H.

When σ is a non-simplicial cone, all extremal vectors may lie on multiple hyperplanes

$H_1, H_2, \ldots, H_m$; we call them subprofiles.

Example - Profiles/Subprofiles

$X \subset \mathbb{C}^3 : f(x, y, z) = z^3 + y^3 z + x^2 y^2 = 0.$



To Prove the Theorem - Minimality of Toric Embedded Resolution

Theorem (Karadeniz, Plénat, –, 2024)

Let X be an RTP-singularity.

The elements of $EV(X)$ lie in the (sub)profiles of maximal dimensional cones of $DNP(X)$.

And, they are free over $\mathbb{Z}$.

To Prove the Theorem - Minimality of Toric Embedded Resolution

Remark

A minimal toric embedded resolution of singularities is not unique, but the divisorial valuations associated with the irreducible components of its exceptional fibre are the same for all minimal toric embedded resolutions.

Meral Tosun, Galatasaray University Joint work with B.Karadeniz Şen, C. Plénat Kinosaki Algebraic Geometry Symposium, Kyoto 2024
Resolution of some Singularities via Jet Spaces

Hilbert bases and EV(X)

Let $\sigma = \langle \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle \subset \mathbb{R}^d$, $\mathbf{v}_i \in \mathbb{Z}^d$ be cone.

A subset of integral vectors $\mathcal{H}(\sigma) \subset \sigma \cap \mathbb{Z}^d$ is called a Hilbert basis of σ iff

(i) each element of $\sigma \cap \mathbb{Z}^d$ can be written as a non-negative integer combination of elts of $\mathcal{H}(\sigma)$,

(ii) $\mathcal{H}(\sigma)$ has minimal cardinality with respect to all subsets of $\sigma \cap \mathbb{Z}^d$ for which (i) holds.

Corollary (Karadeniz, Plénat, –, 2024)

Let $\mathcal{H}(DNP(X)) = \cup \mathcal{H}(\sigma_i)$ where σ_i's is the 3-dimensional cones in $DNP(X)$.

The set $EV(X)$ and $\mathcal{H}(DNP(X))$ coincide.

To continue

Question 1) For any Newton non-degenerate singularity, does $\mathcal{H}(DNP(X))$ coincide EV(X) and give a toric embedded resolution?

Question 2) Is it true for all rational singularities that each element in H_σ lies inside p_σ where σ is a 3-dimensional cone in $DNP(X)$?

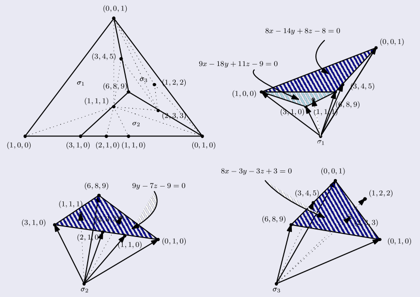
Hilbert bases and EV(X)

Example

$X : f(x,y,z) = y^3 + xz^2 - x^4$ in $\mathbb{C}^3$. It has elliptic singularities and is Newton non-degenerate.

One of the 3-dimensional cones of $DNP(X)$ is $\sigma_3 = \langle e_2, e_3, (6,8,9) \rangle$.

The element $(1,2,2) \in \mathcal{H}(\sigma_3)$ is outside of p_{σ_3}.



The set $\mathcal{H}(DNP(X))$ still give a toric embedded resolution of the singularity.

Hilbert bases and EV(X)

Remark

Let f defines an RDP-singularity.

Let $\mathcal{G}(X)$ be the Groebner fan of X.

The intersection $\mathcal{G}(X) \cap EV(X)$ is exactly the Hilbert basis of $DNP(X)$, so gives the minimal toric embedded resolution of the singularity.

This is not always true for RTP-singularities: We may find a vector in the intersection but it is not in Hilbert basis of $DNP(X)$. But this vector is not revealed in building the toric embedded resolution of the singularity. Hence $\mathcal{G}(X) \cap EV(X)$ also gives a toric embedded resolution of an RTP-singularity, which may not be minimal.

Embdedded toric resolution leads to a conjecture

Teissier's conjecture

Let $X \subset \mathbb{C}^n$ be a singular variety.

Does there exists an embedding $X \subset \mathbb{C}^N$ for $N \geq n$ and a toric structure on $\mathbb{C}^N$ such that X has an embedded toric resolution by one toric map.