RELATIVE COMPACTIFICATION OF SEMIABELIAN NÉRON MODELS

I. NAKAMURA
HOKKAIDO UNVERSITY

Abstract. Let R be a complete discrete valuation ring, $k(\eta)$ its fraction field, $S = \operatorname{Spec} R$, $(G_\eta, \mathcal{L}_\eta)$ a polarized abelian variety over $k(\eta)$ with $\mathcal{L}_\eta$ symmetric ample cubical and $\mathcal{G}$ the Néron model of G_η over S. Suppose that $\mathcal{G}$ is semiabelian over S. Then there exists a *unique* relative compactification $(P, \mathcal{N})$ of $\mathcal{G}$ such that (α) P is Cohen-Macaulay with $\operatorname{codim}_P(P \setminus \mathcal{G}) = 2$ and (β) $\mathcal{N}$ is ample invertible with $\mathcal{N}_{|\mathcal{G}}$ cubical and $\mathcal{N}_\eta = \mathcal{L}_\eta^{\otimes n}$ for some positive integer n. We study the totally degenerate case in [MN24], while we study in [N24] the partially degenerate case and then the case where R is a Dedekind domain. Most remarkable is that the compactification satisfying the conditions (α) and (β) is uniquely determined by $(G_\eta, \mathbf{Z}\mathcal{L}_\eta)$.

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1. Introduction

This report is based in part on a joint work with K. Mitsui. The preprints [MN24] and [N24] on the same subject appeared in arXiv.

Let R be a complete discrete valuation ring (abbr. CDVR), $k(\eta)$ (resp. $k(0)$) the fraction (resp. residue) field of R, $S := \operatorname{Spec} R$, and η (resp. 0) the generic (resp. closed) point of S.

Let $(G_\eta, \mathcal{L}_\eta)$ be a polarized abelian variety over $k(\eta)$ with $\mathcal{L}_\eta$ symmetric ample cubical, $\mathcal{G}$ the Néron model of G_η and $G := \mathcal{G}^0$ the identity component of $\mathcal{G}$. We say that G is semiabelian if G_0 is an extension of an abelian variety by a torus.

By subdividing fans associated with torus embeddings over S, [K98] proved that there are relative compactifications of a semiabelian Néron model $\mathcal{G}$ which are regular, but rarely satisfy the condition (b) of Theorem 1.1 below. Those constructed by [K98] are not codimension two compactifications in general.

In contrast with it, our compactification (which we wish to call a *cubical compactification*) of $\mathcal{G}$ is Cohen-Macaulay but not regular in general. However it is a codimension two compactification of $\mathcal{G}$ and perhaps its most remarkable feature is the uniqueness as in Theorem 1.2.

Thus we have a very simple picture:

- for a given $(G_\eta, \mathcal{L}_\eta)$, there exists a *unique Néron model* $\mathcal{G}$ of it;
- if G is semi-stable (=semi-abelian), then there exists a *unique cubical compactification* of $\mathcal{G}$.

A triple $(P, i, \mathcal{N})$ is called a *relative compactification of* $\mathcal{G}$ (abbr. *compactification of* $\mathcal{G}$) (extending $(G_\eta, \mathcal{L}_\eta)$) if

(rc1) P is an irreducible proper flat S-scheme;
(rc2) $i : \mathcal{G} \hookrightarrow P$ is an open immersion with $P_\eta = i(\mathcal{G}_\eta) = i(G_\eta)$;
(rc3) $\mathcal{N}$ is an ample invertible $\mathcal{O}_P$-module with $i^*\mathcal{N}_\eta \in \mathbf{Z}_{>0}\mathcal{L}_\eta$.

We prove the following theorems in [MN24] and [N24]:

Theorem 1.1. [MN24]+[N24] *If $\mathcal{G}$ is semiabelian over S, then there exists a relative compactification $(P, i, \mathcal{N})$ of $\mathcal{G}$ such that*

(a) *P is Cohen-Macaulay;*
(b) $i(\mathcal{G}) = P \setminus \operatorname{Sing}(P_0)$ *with* $\operatorname{codim}_P \operatorname{Sing}(P_0) \geq 2$ *where* $\operatorname{Sing}(P_0)$ *denotes the singular locus of P_0;*
(c) *$i^*\mathcal{N}$ is cubical with $i_*i^*\mathcal{N} = \mathcal{N}$;*
(d) *G acts on P so that $i_{|G}$ is G-equivariant.*

Theorem 1.2. [N24] *If $(P', i', \mathcal{N}')$ is another relative compactification of $\mathcal{G}$ subject to (a)-(c), then $P \simeq P'$.*

To be more precise, we prove:

Theorem 1.3. [N24] *Let $(G, \mathcal{L})$ be a semi-abelian S-scheme, $\mathcal{G}$ its Néron model and $(P, i, \mathcal{N})$ a cubical compactification of $\mathcal{G}$ extending $(G_\eta, \mathcal{L}_\eta)$. Then*

1. $P \simeq \operatorname{Proj} A(\mathcal{G}, i^*\mathcal{N})$ *with* $i_*i^*\mathcal{N} = \mathcal{N}$*;*
2. P *is uniquely determined by* $(G_\eta, \mathbf{Z}\mathcal{L}_\eta)$*;*
3. $\mathcal{N}$ *is uniquely determined by* $(G_\eta, \mathcal{L}_\eta)$ *up to positive multiples;*
4. *any* $k(\eta)$*-automorphism* h_η *of* G_η *with* $h_\eta^*\mathcal{L}_\eta \simeq \mathcal{L}_\eta$ *extends to an* S*-automorphism* g *of* P *with* $g(\mathcal{G}) = \mathcal{G}$ *and* $g^*\mathcal{N} = \mathcal{N}$*.*

The following is an over-Dedekind-domain version of the above:

Theorem 1.4. [N24] *Let D be a Dedekind domain, K the fraction field of D and $S = \operatorname{Spec} D$. Let $(G, \mathcal{L})$ be a semiabelian S-scheme over S, $\mathcal{L}$ a symmetric ample cubical invertible sheaf on G and $\mathcal{G}$ a Néron model of G. Then there exists a relative compactification $(P, i, \mathcal{N})$ of $\mathcal{G}$ extending $(G_K, \mathcal{L}_K)$ such that*

(a) P *is Cohen-Macaulay;*
(b) $i(\mathcal{G}) = P \setminus \operatorname{Sing}(P/S)$ *with* $\operatorname{codim}_P \operatorname{Sing}(P_S) \geq 2$ *where* $\operatorname{Sing}(P/S)$ *denotes the union of the singular loci of closed fibers of* P *over* S*;*
(c) $i^*\mathcal{N}$ *is ample cubical invertible;*
(d) G *acts on* P *so that* $i_{|G}$ *is* G*-equivariant.*

Moreover the relative compactification $(P, i, \mathcal{N})$ satisfying (a)-(c) is unique up to isomorphism in the sense of Theorem 1.2.

2. Construction of a relatively complete model

2.1. **Set-up.** Let R be a CDVR, $S = \operatorname{Spec} R$, and η (resp. 0) the generic point (resp. closed point) of S. $k(\eta)$ the fraction field of R and $k(0)$ the residue field of R by the maximal ideal I of R. Suppose that we are given a g-dimensional polarized abelian variety $(G_\eta, \mathcal{L}_\eta)$ over $k(\eta)$ with $\mathcal{L}_\eta$ ample cubical and symmetric. Let G be the connected Néron model of G_η. By choosing a suitable finite extension of $k(\eta)$, we may assume that G_0 is an extension of a g'-dimensional abelian variety A_0 over $k(0)$ by a split torus $\mathbf{G}_{m,k(0)}^{g''}$.

Throughout this report we assume $g = g''$, $A_0 = 0$ and $G_0 \simeq \mathbf{G}_{m,k(0)}^{g}$.

2.2. **Degeneration data.**

Theorem 2.1. [FC90] *Suppose that $A_0 = 0$ and G_0 is a split $k(0)$-torus T_0. Let X be a free $\mathbf{Z}$-module of rank g such that $T_0 \simeq \operatorname{Spec} k(0)[X]$. Then there exist a submodule Y of X of finite index, a function $a : Y \to k(\eta)^\times$ and a bilinear function $b : Y \times X \to k(\eta)^\times$* [1] *such that*

[1] To be exact, $b(y + y', x) = b(y, x)b(y', x)$ and $b(y, x + x') = b(y, x)b(y, x')$.

(i) *$a(0) = 1$, $a(y) = a(-y)$ $(\forall y \in Y)$;*
(ii) *$b(y,z) = b(z,y) = a(y+z)a(y)^{-1}a(z)^{-1}$ $(\forall y, z \in Y)$;*
(iii) *$b(y,y) \in I$ $(\forall y \in Y \setminus \{0\})$, and for every $n \geq 0$, $a(y) \in I^n$ for all but finitely many $y \in Y$;*
(iv) *$\Gamma(G_\eta, \mathcal{L}_\eta)$ is identified with the $k(\eta)$-vector subspace of formal Fourier series $\theta = \sum_{x \in X} \sigma_x(\theta) w^x$ such that*

$$\sigma_{x+y}(\theta) = a(y)b(y,x)\sigma_x(\theta),\ \sigma_x(\theta) \in k(\eta)\ (\forall x \in X, \forall y \in Y).$$

Remark 2.2. Furthermore by taking a finite extension of $k(\eta)$ and the integral closure of R in $k(\eta)$ if necessary, we may assume:

there exist a function $a : X \to k(\eta)^\times$ and a bilinear function $b : X \times X \to k(\eta)^\times$ extending a and b in Theorem 2.1, which satisfy instead of (i)-(ii)

(i') $a(0) = 1$, $a(y) = a(-y)$ $(\forall y \in X)$;
(ii') $b(y,z) = b(z,y) = a(y+z)a(y)^{-1}a(z)^{-1}$ $(\forall y, z \in X)$.

2.3. **Construction.** We start with the stronger version of Theorem 2.1 with (i') and (ii'). Then we can construct a flat projective S-scheme P extending $(G_\eta, \mathcal{L}_\eta)$ as follows [AN99]:

- we define a graded algebra $R^\sharp := R[a(x)w^x\vartheta; x \in X]$;
- $\tilde{P}^\sharp :=$ the normalization of $\mathrm{Proj}(R^\sharp)$ is a torus embedding over S locally of finite type;
- $\tilde{P}^{\sharp,\wedge}$: the formal completion of $R^\sharp$;
- the quotient $\tilde{P}^{\sharp,\wedge}/Y$ is algebraizable, where Y acts on $R^\sharp$ by
$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta;$$
- the algebraization $P^\sharp$ of $\tilde{P}^{\sharp,\wedge}/Y$ is flat projective with $P^\sharp \supset G$,;
- $(G_\eta, \mathcal{L}_\eta) \simeq (P^\sharp_\eta, \mathcal{O}_{P^\sharp}(1)_\eta)$.
- but $\mathcal{G} \not\subset P^\sharp$ in general.

See [AN99]. The above $(P, \mathcal{L})$ is the normalization of a projectively stable quasi-abelian scheme over S in [N99].

2.4. **The complex case.** Over the complex number field $\mathbf{C}$, $a(x)$ and $b(y,x)$ in the above are familiar objects:

- $D = \{s \in \mathbf{C}; |s| < 1\}$, $G = \mathbf{C}^g \times D/\mathbf{Z}^g + \mathbf{Z}^g T$;
- $a(x) = \mathbf{e}(x^tTx/2)$, $b(x,y) = \mathbf{e}(x^tTy)$;
- let $T = T_0(s) + B\log s/(2\pi i)$, with ${}^tT = T$, where $T_0(s)$ holomorphic on D, $B = \mathrm{val}_s(b) \in M_{g\times g}(\mathbf{Z})$ even positive definite;
- if $T_0 \equiv 0$, then $a(x) = s^{B(x,x)/2}$, $b(x,y) = s^{B(x,y)}$;
- we choose $R = \mathbf{C}[[s]]$ the formal power series ring of one variable to formulate the problem in the algebraic manner.

3. Construction of P

3.1. **Definitions.** Let $\mathcal{G}$ be the Néron model $\mathcal{G}$ of G. Next we construct a new relatively complete model $\tilde{P}$ so that P : the algebraization of $\tilde{P}^{\wedge}/Y$ may contain $\mathcal{G}$. Let $\Phi := \mathcal{G}_0/G_0$ be the component group of $\mathcal{G}$. Recall

$$\Phi \simeq X^{\vee}/\beta(Y)$$

where $\beta : Y \to X^{\vee}$ defined by $\beta(y)(x) = B(x, y)$. We choose $0 \in \Psi$ ($\subset X^{\vee}$) an arbitrary set of representatives of Φ. Let $N := |\Phi|$.

Definition 3.1. We define

$$\theta_{+} := \prod_{u\in\Psi} \vartheta_{v},\ \theta_{-} := \prod_{u\in\Psi} \vartheta_{-v},$$
$$\theta := \theta_{+}\theta_{-}.$$

Lemma 3.2. *Let $\mu := \beta^{-1}\det(B) : X^{\vee} \to Y$, and $N = |\Phi|$. By taking a finite extension of $k(\eta)$ if necessary, there exists an extension b^{e} of b to $X^{\vee} \times X$ such that $b^{e}(\beta(y), x) = b(y, x)$ $(\forall y \in Y, \forall x \in X)$. Set $\epsilon(u) := b^{e}(u, \mu(u))$ $(u \in X^{\vee})$. Then*

- $\delta_{u}^{*}\theta = \epsilon(u)w^{2\mu(u)}\theta$;
- *$\epsilon(u)$ is independent of the choice of Ψ;*
- $\epsilon(u + v) = \epsilon(u)\epsilon(v)b^{e}(u, \mu(v))^{2}$ $(\forall u, v \in X^{\vee})$;
- *$\delta_{\beta(y)} = S_{y}$ $(\forall y \in Y)$; this definition induces an isomorphism:*

$$\{\delta_{v}; v \in X^{\vee}\}/\{S_{y}; y \in Y\} \simeq X^{\vee}/\beta(Y) \simeq \Phi.$$

Definition 3.3. Now we define an algebra

$$R^{\dagger} := R[\epsilon(u)b^{e}(u, \alpha)w^{\alpha+2\mu(u)}\theta; \alpha \in \Sigma, u, v \in X^{\vee}], \tag{1}$$

where Σ is the set of integral points of some bounded convex polytope (Voronoi polytope) of $X_{\mathbf{R}}$ with integral vertices. See § 4.

3.2. **Construction of P.** Our construction of P proceeds as follows:

- $\tilde{P}$:= the normalization of $\mathrm{Proj}(R^{\dagger})$ is a torus embedding over S locally of finite type;
- $\tilde{P}^{\wedge}$: the formal completion of $\tilde{P}$;
- $X^{\vee}$ acts on $R^{\dagger}$ by

  $$\delta_{v}^{*}(\epsilon(u)b^{e}(u, \alpha)w^{\alpha+2\mu(u)}\theta) = \epsilon(u + v)b^{e}(u + v, \alpha)w^{\alpha+2\mu(u+v)}\theta;$$

- $\delta_{\beta(y)} = S_{y}$ $(\forall y \in Y)$;
- the quotient $\tilde{P}^{\wedge}/Y$ is algebraizable;
- the algebraization P of $\tilde{P}^{\wedge}/Y$ is flat projective;
- $(G_{\eta}, \mathcal{L}_{\eta}^{\otimes 2N}) \simeq (P_{\eta}, \mathcal{O}_{P}(1)_{\eta})$.

To achieve $P \supset \mathcal{G}$, we need another step in § 4 of choosing a suitable multiple of $\mathcal{O}_P(1)$.

Remark 3.4. The algebra $R^\sharp$ represents the sheaf $\mathcal{L}$, while the algebra $R^\dagger$ may represent the tensor product over $P^\wedge$:

$$\bigotimes_{u\in\Psi}\delta_v^*(\mathcal{L}^\wedge)^\times.$$

4. Voronoi polytopes

4.1. **Voronoi polytopes.** The relatively complete model $\tilde{P}$ is an S-scheme of locally of finite type, which is a torus embedding over S described by a certain fan consisting of infinitely many cones. It appears in the standard manner, which we do not write down here. See [MN24, 7.4]. Instead, we will define Voronoi polytopes, which helps us to understand the S-scheme $\tilde{P}$ to some extent.

By Th. 2.1 (ii'), we have a symmetric positive definite bilinear form

$$B(x,y)=\mathrm{val}_s\, b(x,y)\quad (x,y\in X)$$

associated with a totally degenerate semiabelian S-scheme $(G,\mathcal{L})$.

Definition 4.1. Let $l\in \mathbf{Z}_{>0}$. We define a distance $\|z\|=\sqrt{B(z,z)}$ on $X_{\mathbf{R}}$. For any $c\in X^\vee$, we define a Voronoi polytope $\Sigma_l(c)$ by

$$\Sigma_l(c)=\{x\in X_{\mathbf{R}};\|x-2l\mu(u)\|\geq\|x-2l\mu(c)\|\ (\forall u\in X^\vee)\}.$$

4.2. **Properties.** We summarize the properties of Voronoi polytopes:

- for $(G,\mathcal{L}^{\otimes l})$, we have lB and $\Sigma_l(c)$;
- $\Sigma_l(c)$ is a bounded convex polytope with $\Sigma_{ll'}(0)=l'\Sigma_l(0)$;
- $\Sigma_l(c)=\Sigma_l(0)+2l\mu(c)$;
- $X_{\mathbf{R}}=\bigcup_{c\in X^\vee}\Sigma_l(c)$;
- $\Sigma_l(c)$ $(c\in X^\vee)$ and their faces form a poly. decomp. of $X_{\mathbf{R}}$;
- $\Sigma_l:=\Sigma_l(0)\cap X$ is a finite subset of X with $\Sigma_l+2l\mu(X^\vee)=X$;
- there exists $l_0\in\mathbf{N}$ such that $\Sigma_{l_0}(0)$ is integral (Definition 4.2);
- if $B=E_8$, then $l_0=30$;
- if $g=2$, then $l_0=1$.

Definition 4.2. A convex polytope Δ in $X_{\mathbf{R}}$ is said to be *integral* if

(i) Δ is the convex closure of $\Delta\cap X$ and $\Delta=-\Delta\ni 0$;
(ii) $\Delta\cap X$ contains a basis of X.

Note that $l_0=1$ iff $\Sigma_1(0)$ is integral.

If Δ is integral, then $\Delta\cap X$ is a star of [M72, 2.2].

4.3. **Cubical compactifications.**

Theorem 4.3. *Let $R^\dagger$ be the algebra (1) with $\Sigma := \Sigma_1(0) \cap X$. If $\Sigma_1(0)$ is integral, then by choosing $\Sigma := \Sigma_1(0) \cap X$, P is a cubical compactification of $(G_\eta, \mathcal{L}_\eta)$ such that $\mathcal{G} \subset P$.*

If $\Sigma_l(0)$ is integral, then we take $\mathcal{L}^{\otimes l}$ for $\mathcal{L}$ and define the algebra $R_l^\dagger$ for $R^\dagger$ with due modification. See [MN24, 9.1] for the detail.

Lemma 4.4. *Let P be a cubical compactification of $(G_\eta, \mathcal{L}_\eta)$. Then P_0 is the union of normal torus embeddings, each of which is (isomorphic to) the one associated with the same Voronoi polytope $\Sigma_l(0)$.*

See [N24, 9.10-9.12].

Remark 4.5. In general, we need pullbacks of P in § 2 and faithfully flat descents of P in § 3 in order to construct cubical compactifications in Theorems 1.1-1.4. See [MN24, §§ 9/11-12] and [N24, §§ 6/8].

5. The properties of P and P_0

5.1. **The properties.** Assume $\Sigma_1(0)$ is integral. The following are some standard properties of P and P_0 proved in [N24, §§ 9-10]:

- P_0 is reduced, Cohen-Macaulay with trivial dualizing sheaf;
- the semi-universal covering $\tilde{P}_0$ of P_0 is stratified into closed subschemes $\bar{Z}(\Delta)$, each $\bar{Z}(\Delta)$ being associated with a (unique) closed face Δ of some $\Sigma(c)$ ($c \in X^\vee$) and $\dim_k \bar{Z}(\Delta) = \dim_{\mathbf{R}} \Delta$;
- $H^q(P, \mathcal{O}_P)$ is an R-free module of rank $\binom{g}{q}$;
- $H^q(P, \mathcal{L}^{\otimes m}) = H^q(P_0, \mathcal{L}_0^{\otimes m}) = 0$ ($\forall q > 0, \forall m > 0$);
- $\Gamma(P, \mathcal{L}^{\otimes m})$ is a free R-module of rank $(2Nm)^g |X/Y|$;
- $\mathcal{L}^{\otimes m}$ is very ample on P for $m \geq 4g$.

5.2. **The closed subscheme $\bar{Z}(\Delta)$.**

Definition 5.1. Let Fan be the fan in the $(g+1)$-dimensional Euclidean space $\mathbf{R}_{\geq 0} f_0 \bigoplus X_{\mathbf{R}}^\vee$ associated with the torus embedding $\tilde{P}$ over S, where f_0 is a $\mathbf{Z}$-basis of $\mathbf{Z} \subset \mathbf{R}$. Fan consists of cones τ_Δ where Δ ranges over the set of all faces of Voronoi polytopes $\Sigma(c)$ ($c \in X^\vee$). Let $Z(\Delta)$ ($\simeq \mathbf{G}_{m,k(0)}^{\dim \Delta}$) be the relative interior of $\bar{Z}(\Delta)$ of P_0.

Now let us summarize the properties of $\bar{Z}(\Delta)$:

- $H^q(\bar{Z}(\Delta), \mathcal{L}_{\bar{Z}(\Delta)}^{\otimes m}) = 0$ ($\forall q \geq 1, \forall m \geq 1$);
- $H^0(\bar{Z}(\Delta), \mathcal{L}_{\bar{Z}(\Delta)}^{\otimes m})$ is spanned by monomials w^x ($x \in m\Delta \cap X$);
- $\mathcal{L}_0^{\otimes m}$ is very ample on $\bar{Z}(\Delta)$ if $m \geq \dim \Delta$.

5.3. **Limits of K-rational points.**

Definition 5.2. Let f_0 be a $\mathbf{Z}$-basis of $\mathbf{Z}$, f_i ($i \in [1, g]$) a $\mathbf{Z}$-basis of $X^\vee$, and m_i the $\mathbf{Z}$-basis of X dual to f_i. We define

$$\mathrm{Cut}(\tau_\Delta) := -f_0 + (\tau_\Delta \cap (f_0 \times X^\vee_{\mathbf{R}})).$$

For example, $\Delta = \Sigma(c)$, then $\mathrm{Cut}(\tau_\Delta) = -c$. If $\Delta = \bigcap_{c\in\Lambda} \Sigma(c)$ (Λ: maximal), then $\mathrm{Cut}(\tau_\Delta) =$ the convex closure of $-c$ ($c \in \Lambda$).

Definition 5.3. Let s be a uniformizer of R, v_s the valuation of R with $v_s(s) = 1$ and define an *absolute value* $|\cdot|$ *of* $k(\eta)$ by $|a| = e^{-v_s(a)}$ for $a \in k(\eta)$. Then v_s and $|\cdot|$ can be uniquely extended to $\Omega :=$ the algebraic closure of $k(\eta)$. Note that $|a| \geq 0$, and $|a| = 0$ iff $a = 0$.

Definition 5.4. Let $Q \in \tilde{P}(\Omega)$ and $\log|w^x(Q)| := \log|Q^*(w^x)| \in \mathbf{Q}$ ($x \in \tilde{X}$). Since $w^{m_0}(Q) = s$, we define

$$\log(Q) = -\sum_{i=0}^{g} (\log|w^{m_i}(Q)|) f_i \in \tilde{X}^\vee_{\mathbf{Q}},$$

$$\mathrm{cutlog}(Q) = -f_0 + \log(Q) = -\sum_{i=1}^{g} (\log|w^{m_i}(Q)|) f_i \in X^\vee_{\mathbf{Q}}.$$

Let K be an algebraic extension of the field $k(\eta)$. The following describes the limits of K-rational points Q [N24, § 9.6].

Theorem 5.5. *Let K be a finite extension of $k(\eta)$ and $Q \in G(K)$. Then* $\lim Q \in Z(\Delta)$ *iff* $\mathrm{cutlog}(Q) \in \mathrm{Cut}(\tau_\Delta)^0$.

Note that Δ is uniquely determined by Q in Theorem 5.5.

6. Examples

6.1. **Two dimensional case.** We recall known compactifications in dimension two. We consider the following case:

- X is a union of two $\mathbf{P}^1$ with 3 double points, which is a stable curve of genus two embedded in $\mathcal{X}$;
- $\mathcal{X}$ is a proper regular surface over S with $\mathcal{X}_0 = X$;
- $(G_\eta, \mathcal{L}_\eta)$ is the Jacobian variety of $\mathcal{X}_\eta$;
- $\mathcal{G}$ is the Néron model G_η, and $G := \mathcal{G}^0$;
- $G_0 \simeq \mathbf{G}^2_{m,k(0)}$ is the generalized Jacobian of X.

Let P be a relative compactification of $\mathcal{G}$. Then P_0 is

$$\begin{cases} 3\nabla & \text{[N77] over } \mathbf{C}/\text{Non-cubical},\\ 2\Delta & \text{[N75] [Namikawa76][AN99]},\\ 2\Delta+\nabla & \text{[M72]},\\ 2\Delta,\ 2\Delta+\nabla & \text{[OS79]},\\ 3\nabla & \text{[MN24][N24]/Cubical} \end{cases}$$

where $\Delta = \mathbf{P}^2$, $\nabla = Q_{p_1,p_2,p_3}(\mathbf{P}^2)$.

6.2. **A smallest cubical compactification** $P^\natural$**.** Let $X = \mathbf{Z}m_1 + \mathbf{Z}m_2 \simeq \mathbf{Z}^2$, $m_3 = -(m_1+m_2)$, $\Sigma := \{0, \pm m_i; i \in [1,3]\}$, $\Psi := \{0, \pm m_1\}$ and $N := |\Psi| = 3$. We define $R^\natural$ as follows:

$$R^\natural = R[\epsilon_+(u)b^e(u,\alpha)w^{\alpha+\mu(u)}\theta_+; u \in X^\vee, \alpha \in \Sigma]$$

where

$$\theta_+ := \prod_{y\in\Psi}\vartheta_y,\ \epsilon_+(u) := s^{u_1^2+u_1u_2+u_2^2},\ b^e(u,y) = s^{u_1y_1+u_2y_2},$$
$$\beta(y) := (2y_1-y_2)f_1 + (-y_1+2y_2)f_2,$$
$$\mu(u) := (2u_1+u_2)m_1 + (u_1+2u_2)m_2,$$
$$E_+(u) := v_s\epsilon_+(u) = u(\mu(u))/2 = u_1^2+u_1u_2+u_2^2,$$
$$u = u_1f_1+u_2f_2,\ y = y_1m_1+y_2m_2,$$
$$\Sigma(0) := \{x \in X_{\mathbf{R}}; E_+(u)+u(x) \geq 0\ (\forall u \in X^\vee)\}.$$

It turns out

$$\Sigma(0) = \text{the convex closure of } \Sigma,\ \Sigma = \Sigma(0)\cap X.$$

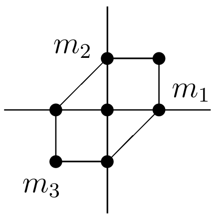

FIGURE 1. Σ and $\Sigma(0)$

The S-scheme $P^\natural$ is a cubical compactification of $\mathcal{G}$ which is constructed by starting from $\tilde{P}^\natural := \operatorname{Proj} R^\natural$ through the process in § 3.2. By § 5, $\bar{Z}(\Sigma(c))$ is the torus embedding associated with the Voronoi polytope $\Sigma(c)$, which is ∇. Thus we see that $P_0^\natural$ is the union of 3 copies of ∇.

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Department of Mathematics, Hokkaido University, Sapporo, 060, Japan

E-mail address: `nakamura@math.sci.hokudai.ac.jp`