On Batyrev's theorem for modular quotient singularities

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Abstract

Batyrev's theorem is a generalized version of McKay correspondence over $\mathbb{C}$. However, its analog in positive characteristic does not always hold. In this poster, we show that analog of Batyrev's theorem holds for modular quotient singularities of a specific type.

Batyrev's theorem over $\mathbb{C}$

Theorem (Batyrev[1]). Let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$. Denote the associated quotient singularity by $X := \mathbb{C}^n/G$. If there exists a crepant resolution $f : Y \to X$, then

$$e(Y) = \#\mathrm{Conj}(G).$$

That is, the **Euler characteristic** of the crepant resolution equals the **number of conjugacy classes** in G.

Question. What happens in positive characteristic?

Non-modular vs modular

k: a perfect field of characteristic $p > 0$

$G \subseteq \mathrm{SL}(n, k)$: a finite group with no reflections

$X = \mathbb{A}^n_k/G$: the associated quotient singularity

Definition. G is **modular** if $p|\#G$. Otherwise, G is **non-modular**.

Fact. The analog of Batyrev's theorem holds in non-modular cases. For modular cases, both examples and counterexamples exist.

Examples and counterexamples

One can find results for modular cases below from Yasuda[6], Yamamoto[5] and Fan[2][3] respectively.

We always use H to denote a non-modular abelian subgroup of G in this poster.

$e(Y) = \#\mathrm{Conj}(G)$	$e(Y) \neq \#\mathrm{Conj}(G)$
C_p (any dimension)	S_3 $(p = n = 3)$
$H \rtimes C_3$ $(p = n = 3)$	$H \rtimes S_3$ $(p = n = 3)$
$H \rtimes C_2$ $(p = 2, n = 4)$	A_4 $(p = 2, n = 4)$

Our main result contains the left column in the table as its examples.

Wild McKay correspondence

To study Euler characteristic of crepant resolutions, we use the following version of wild McKay correspondence by Yasuda[7].

Let k be a finite field of characteristic $p > 0$, $K = k((t))$ be the local field of Laurent series with coefficients in k, and $G \subseteq \mathrm{SL}(n, k)$ be a finite group with no reflections. If the corresponding quotient singularity $X := \mathbb{A}^n/G$ has a crepant resolution $f : Y \to X$, then

$$\#Y(\mathbb{F}_q) = \frac{1}{\#G} \sum_{\rho: G_K \to G} q^{n-\mathbf{v}(\rho)}.$$

$k = \mathbb{F}_q$:the finite field of order $q = p^e$

$G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$: the absolute Galois group of K

$\rho : G_K \to G$: any continuous homomorphism

$\mathbf{v}$: a function (definition omitted) mapping any ρ to a rational number

Main theorem

Theorem (F[4]). Let $k = \overline{\mathbb{F}_p}$ be the algebraic closure of finite fields of characteristic $p > 0$, and $G \subseteq \mathrm{SL}(n, k)$ be a finite group with no reflections, such that

$$G \cong H \rtimes C_p,$$

where H is a non-modular abelian group and C_p denotes the p-cyclic group.

If $X := \mathbb{A}^n_k/G$ has a crepant resolution $f : Y \to X$, then

$$e(Y) = \#\mathrm{Conj}(G) = \#\{[\text{indecomposable } kG\text{-modules}]\}.$$

Remarks

- To use wild McKay correspondence to compute Euler characteristic, one has to consider any subgroup $G' \subseteq G$ and Galois extensions of K with Galois group isomorphic to G'.
- Under the assumption of main result, we have that

$$\#\{[\text{indecomposable } kG\text{-modules}]\} = \#\mathrm{Conj}(G)$$

can be computed easily because of the structure of G.

Sketch of proof

Step 1

Show that the main result holds when the center $C(G) = \{e\}$.

Step 2

Show that the main result holds when the center $C(G) = G$.

Step 3

Combine obtained results. For subgroups of a **general** G, they are all discussed as below.

- non-modular subgroups (from analogous Batyrev's theorem)
- non-abelian modular subgroups (from Step 1)
- abelian modular subgroups (from Step 2)

Finally one can show that the main result holds for any group with a structure as a semidirect product $H \rtimes C_p$.

References

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