Quasi-étale covers of Du Val del Pezzo surfaces and Zariski dense exceptional sets in dimension 2

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Abstract

In [1, 2], we study surfaces with Zariski dense exceptional sets in Manin's conjecture. By an MMP-type argument in [3], this reduces to the study of weak del Pezzo surfaces with the anticanonical polarization. In [1], we give examples of del Pezzo surfaces of degree 1 and Picard rank 1 with dense exceptional sets. In [2], we classify possibilities of such examples in the case of weak del Pezzo surfaces. The existence of such examples were known in dimension ≥ 3, but were unexpected in dimension 2. We focus on [2] in this poster.

Notation

We fix the following notation throughout the poster.

- F: a field of characteristic 0.
- S and T: Du Val del Pezzo surfaces (*i.e.*$-K_S$ is ample and S has canonical singularities) over F.
- $X \to S$ and $Y \to T$: minimal resolutions of S and T.
- $\pi : T \to S$: a quasi-étale cover (see the definition below).
- $N^1(X)$: the Néron-Severi space of X.
- $\rho(X) := \dim N^1(X)$: the Picard rank of X.
- $\Gamma(X)$: the dual graph of extremal curves on X.
- $S_d(\mathsf{ADE})$: the type of a Du Val del Pezzo surface of singularity type ADE (see the definition below). Some types are not determined by their singularity type, and we must distinguish between them.

Main results

Definition. We call a linear map $\sigma : N^1(X) \to N^1(Y)$ a **Cremona isometry** if it preserves intersection numbers, pseudo-effective cones, and canonical classes. We say X and Y are of the same **type** whenever such a σ exists. We denote the group of all such σ by $\mathrm{Cris}(X)$.

Definition. A **quasi-étale cover** $\pi : T \to S$ is a finite surjection which is étale in codimension 1.

Theorem 1. We classify all quasi-étale covers $\pi : T \to S$ up to types, determine the correspondence $\Gamma(Y) \dashrightarrow \Gamma(X)$.

Remark. The correspondence $\Gamma(Y) \dashrightarrow \Gamma(X)$ is induced by $f_* \circ g_*^{-1}$ from Proposition 6. This is in general not a function in each direction.

Corollary 2. Over any field of characteristic 0, we have

1. $\rho(Y) \leq \rho(X)$ when S is of degree ≥ 3;
2. $\rho(Y) < \rho(X)$ when S is of degree ≥ 4.

Conversely, we construct explicit examples of S of degree $1, 2$, such that $\rho(Y) > \rho(X)$, and examples of degree 3 such that $\rho(Y) \geq \rho(X)$.

Example 3. Manin's conjecture does not hold for the following surfaces with a proper closed exceptional set:

- $X^3 + 2XYW + XZ^2 - Y^2Z + ZW^2 = 0$ in $\mathbb{P}^3$.
- $W^2 + Z^3 + X^4Y^2 = 0$ in $\mathbb{P}(1,1,2,3)$, and so on...

Let Y^σ denote the twist of Y induced by $\sigma \in H^1(F, \mathrm{Aut}(T/S))$. As a direct application of [4]:

Corollary 4. Colliot-Thélène's conjecture holds for X if it holds for any Y^σ.

In future works, we will study whether this can provide new cases of the conjecture.

Manin's and Colliot-Thélène's conjectures

For now on, we focus on varieties over number fields.

Definition. Let X be a variety over a number field F. A subset $U \subseteq X(F)$ is called a **thin set** if

$$U := Z_0(F) \cup (\cup_i f_i(Z_i(F))),$$

where Z_0 is a proper closed subset of X, and $f_i : Z_i \to X$ are finitely many generically finite morphisms of degree ≥ 2.

Example. $\mathbb{P}^1(\mathbb{Q})$ is not thin; $\mathbb{P}^1(\mathbb{R})$ is thin.

Let X be a smooth rationally connected variety over a number field F.

Conjecture (Colliot-Thélène). $X(F)$ is dense in the Bruaer-Manin set $X(F_\Omega)^{\mathrm{Br_{nr}}(X)}$.

This conjecture implies that $X(F)$ is not thin and provides a positive answer to the inverse Galois problem.

Let $H_{\mathcal{L}}$ be the height function induced by a big and nef adelically metrized line bundle $\mathcal{L}$ on X. For any subset $U \subset X(F)$ and integer B, define $N(U, \mathcal{L}, B) := \#\{x \in U \mid H_{\mathcal{L}}(x) \leq B\}$.

Conjecture (Manin). There exists a thin subset $Z \subset X(F)$, such that

$$N(X(F) \backslash Z, \mathcal{L}, B) \sim cB^a (\log B)^{b-1}$$

as $B \to \infty$, where a and b are birational invariants of (X, L), and c is Peyre's constant which is related to $\#X(\mathbb{F}_q)$.

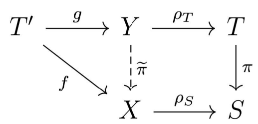
When X is weak Fano and $L \equiv -K_X$, we have that $a = 1$ and $b = \rho(X)$.

Some ingredients

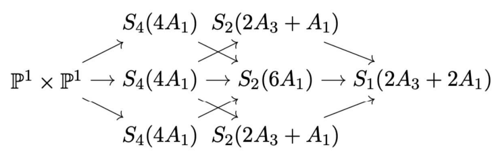
Proposition 5. Let E_i $(i \in I)$ be the (-2)-curves on X. Then

$$\pi_1(S_{\mathrm{smooth}}) \cong (\mathrm{Pic}(X) / \oplus_{i\in I} \mathbb{Z}E_i)_{\mathrm{tors}}.$$

Proposition 6 (a model of $\pi : T \to S$). Let $f : T' \to X$ be the normalization of X in the function field of T. Then the induced rational map $g : T' \dashrightarrow Y$ is regular, and $\mathrm{Exc}(g) = \mathrm{Ram}(f)$. Moreover, T' has cyclic quotient singularities.



Example 7. $S_1(2A_3 + 2A_1)$ is the unique type with $\pi_1(S_{\mathrm{smooth}}) \cong C_2 \times C_4$, the corresponding quasi-étale covers are



[1] Runxuan Gao, a Zariski dense exceptional set in Manin's Conjecture: dimension 2. *Research in Number Theory*, 9(2):42, 2023.

[2] Runxuan Gao, Quasi-étale covers of Du Val del Pezzo surfaces and Zariski dense exceptional sets in Manin's conjecture. *arXiv:2406.16240*, 2024.

[3] B. Lehmann and S. Tanimoto. On the geometry of thin exceptional sets in Manin's conjecture. *Duke Math. J.*, 166(15):2815–2869, 2017.

[4] Yonatan Harpaz and Olivier Wittenberg, supersolvable descent for rational points. *Algebra & Number Theory*, 18(4):787–814, 2024.