Standard Models of dP4 fibrations in Positive Characteristic

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Corti defined the notion of standard models of del Pezzo fibrations and studied the existence of them over $\mathbb{C}$ with a fixed generic fibre in [Cor]. We proved that existence of standard models of del Pezzo fibrations of degree 4 in characteristic > 2. To show this, we use Kollár stability, which is introduced in [Kol] and [AFK].

We work over a field $k = \overline{k}$ with $\mathrm{char}(k) \neq 2$.

0 Standard models of dP fibrations

We say that $\pi : X \to C$ is a *del Pezzo fibration* (*dP fibration*) *of degree* d if π is a flat, projective morphism from a threefold X to a curve C, and the generic fibre X_K of π is a smooth del Pezzo surface of degree d over the function field K of C.

Definition 1. A dP fibration $\pi : X \to C$ of degree $d \geq 3$ is called a *standard model* if the following conditions are satisfied:

(1) X has only terminal singularities.
(2) π has integral fibres.
(3) $-K_X$ is a π-ample line bundle.

Question 2. Do there exist some standard models of del Pezzo fibrations as birational models for a given del Pezzo fibration $\pi : X \to C$?

Using techniques of descent, we only have to show that the existence of standard models $\pi : X \to C = \operatorname{Spec} R$ with a fixed generic fibre X_K, where R is the local ring of a point on the curve C.

1 Settings

We work over a DVR R with the fraction field K.
• V: space of quadrics in n variables $x_1, \ldots, x_n$ over R
• $M := Gr(2, V)$: The moduli of pencils
• $G := GL_n$: general linear group (group scheme over R)
G naturally acts on M by coordinate changes.
A *weight system* (ρ, E) is a pair consisting of $\rho = (w_1, \ldots, w_n) \in \mathbb{Z}^n$ and $E \in GL_n(R)$. Each weight system determines a K-point of G in the following form ;

$$E^{-1} \operatorname{diag}(t^{w_1}, \ldots, t^{w_n}) E.$$

Thus a weight system induces an action by the corresponding K-point of G.

2 Semistability over curves

In order to construct the standard models, we apply *Kollár stability*, which is introduced in [AFK], to the moduli of pencils M.

Definition 3. A pencil $f \in M(R)$ is *semistable* if the following properties are satisfied :

(1) The base locus X_K of the pencil f over K is smooth.
(2) For every weight system (ρ, M) with $\rho = (w_1, \ldots, w_n)$, the following inequality holds :

$$\mathrm{mult}_\rho(f) \leq \frac{4}{n} \sum_{i=1}^{n} w_i.$$

We use the following inequality ;

$$\mathrm{mult}_\rho(f) \geq val_t(\rho \cdot g) + val_t(\rho \cdot h),$$

where $\{g, h\}$ is any R-basis of the pencil f and

$$val_t(\rho \cdot g) := \max\{N \mid \forall \text{coeff. of } \rho \cdot g \text{ is divisible by } t^N\}.$$

Theorem 4. (1) *For any smooth* $(2, 2)$*-complete intersection* X_K *over* K*, there exists a semistable model* f *s.t. the generic fibre of the base locus* $X \to \operatorname{Spec} R$ *is* X_K.
(2) *Suppose* $n = 5$. *Let* $f \in M(R)$ *be a pencil which is semistable. Then the base locus* $X \subset \mathbb{P}^4_R$ *of the pencil* f *is a standard model of dP4 fibrations over* $\operatorname{Spec} R$.

3 Outline of the proof of Theorem 4

We show that the fibre X_0 of the closed point of R is integral.

Assume that X_0 is not integral. By the degree considerations, there are following two cases :

(1) X_0 contains a plane.
(2) X_0 is a union of two degree 2 surfaces Y_1, Y_2 (possibly $Y_1 = Y_2$ and X_0 is non-reduced).

Let $I \subset k[x_1, \ldots, x_5]$ be the ideal which defines X_0.
Case (1) : In some coordinates, $I \subset (x_1, x_2)$. Then f is destabilized by $\rho = (1, 1, 0, 0, 0)$
Case (2) : If $Y_1 \neq Y_2$ (resp. $Y_1 = Y_2$), since each degree 2 component is contained in some hyperplane ($\subset \mathbb{P}^4_k$), in some coordinates we may suppose $x_1 x_2 \in I$ (resp. $x_1^2 \in I$). In either case, f is destabilized by $\rho = (1, 0, 0, 0, 0)$.

References

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