Indices of smooth Calabi–Yau varieties

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$/\mathbb{C}$.

Definition (Calabi–Yau pairs)

A *Calabi–Yau pair* (X, B) consists of a normal projective variety X and a $\mathbb{Q}$-divisor $B \geq 0$ on X such that $K_X + B \sim_{\mathbb{Q}} 0$.
A *Calabi–Yau variety* X is a Calabi–Yau pair $(X, 0)$.

- Calabi–Yau curves $\iff$ (smooth) elliptic curves.
- smooth Calabi–Yau surfaces
  $\iff$ K3 surfaces, abelian surfaces, Enriques surfaces, bielliptic surfaces.

Definition (Indices of Calabi–Yau pairs)

(X, B): a Calabi–Yau pair. The *index* of (X, B) is

$$\operatorname{index}(X, B) = \min \{ m \in \mathbb{Z}_{\geq 1} \mid m(K_X + B) \sim 0 \}.$$

Definition (Sets of indices)

$n \in \mathbb{Z}_{\geq 1}$. $\Phi \subseteq [0, 1] \cap \mathbb{Q}$.

$$I_{\mathrm{lc}}(n, \Phi) = \left\{ \operatorname{index}(X, B) \;\middle|\; \begin{array}{l} (X, B)\text{: lc Calabi–Yau pair} \\ \dim X = n,\ \operatorname{Coeff}(B) \subseteq \Phi \end{array} \right\},$$

$$I_{\mathrm{klt}}(n, \Phi) = \left\{ \operatorname{index}(X, B) \;\middle|\; \begin{array}{l} (X, B)\text{: klt Calabi–Yau pair} \\ \dim X = n,\ \operatorname{Coeff}(B) \subseteq \Phi \end{array} \right\},$$

$$I_{\mathrm{term}}(n) = \{ \operatorname{index}(X) \mid X\text{: terminal Calabi–Yau variety of dimension } n \},$$

$$I_{\mathrm{sm}}(n) = \{ \operatorname{index}(X) \mid X\text{: smooth Calabi–Yau variety of dimension } n \}.$$

- $I_{\mathrm{sm}}(1) = \{1\}$, $I_{\mathrm{sm}}(2) = \{1, 2, 3, 4, 6\}$.
- $I_{\mathrm{sm}}(n) \subseteq I_{\mathrm{term}}(n) \subseteq I_{\mathrm{klt}}(n, \Phi) \subseteq I_{\mathrm{lc}}(n, \Phi)$.
- $\Phi \subseteq \Phi' \implies I_{\mathrm{klt}}(n, \Phi) \subseteq I_{\mathrm{klt}}(n, \Phi')$, $I_{\mathrm{lc}}(n, \Phi) \subseteq I_{\mathrm{lc}}(n, \Phi')$.

Conjecture (Index conjecture for Calabi–Yau pairs)

$n \in \mathbb{Z}_{\geq 1}$. $\Phi \subseteq [0, 1] \cap \mathbb{Q}$: a finite set.
Then the set $I_{\mathrm{lc}}(n, \Phi)$ is bounded (i.e., a finite set).
In other words, $\exists m \in \mathbb{Z}_{\geq 1}$,

$$\forall (X, B)\text{: lc Calabi–Yau pair of dimension } n \text{ with } \operatorname{Coeff}(B) \subseteq \Phi,$$
$$m(K_X + B) \sim 0.$$

- [JL21]: true in dimension $n \leq 3$, and partially in dimension $n = 4$.
- open in higher dimension.

Toward the proof, several steps are known:

Theorem

$n \in \mathbb{Z}_{\geq 1}$.

- *[Jia24] $I_{\mathrm{term}}(n)$ bounded $\implies I_{\mathrm{klt}}(n, \{0\})$ bounded.*
- *[Xu19] $I_{\mathrm{klt}}(n, \{0\})$ bounded $\implies I_{\mathrm{klt}}(n, \Phi)$ bounded for any finite Φ.*

Consider proving the index conjecutre for *klt* Calabi–Yau pairs, using *inductionon the dimension n*. Then we need to consider:

Question

$n \in \mathbb{Z}_{\geq 1}$. Does it hold that

$$I_{\mathrm{klt}}(n-1, \Phi) \text{ bounded} \implies I_{\mathrm{term}}(n) \text{ bounded}$$

for some Φ?

The main theorem partially answers to Question:

Theorem ([M])

$n \in \mathbb{Z}_{\geq 1}$. Then

$$I_{\mathrm{sm}}(n) \subseteq I_{\mathrm{klt}}(n-1, \Phi_{\mathrm{st}}),$$

where $\Phi_{\mathrm{st}} = \{ 1 - 1/b \mid b \in \mathbb{Z}_{\geq 1} \}$.

- The set Φ_{st} is not finite, but a DCC set.

Corollary ([M])

The set $I_{\mathrm{sm}}(4)$ is bounded.

Sketch of proof:

- Induction on the dimension n.

Let

- X: a smooth Calabi–Yau variety of dimension n,
- $m = \operatorname{index}(X)$.

Enough to show: $m \in I_{\mathrm{klt}}(n-1, \Phi_{\mathrm{st}})$.

Beauville–Bogomolov decomposition:

- $\exists$ a finite étale Galois cover $\tilde{X} = \prod_i Y_i \to X$,
- Y_i: a strict Calabi–Yau variety, an irreducible symplectic variety, or an abelain variety,
- $\exists g \in \operatorname{Aut}(\tilde{X}/X)$ of index m (i.e., $g^* \curvearrowright H^0(\tilde{X}, K_{\tilde{X}}) \simeq \mathbb{C}$ is of order m).

Case (1): $\tilde{X} = Y_1$.

- The action $g \curvearrowright \tilde{X}$ is free
  $\rightsquigarrow \varphi(m) \leq 2n$ (φ the Euler function)
  $\rightsquigarrow m \in I_{\mathrm{klt}}(n-1, \Phi_{\mathrm{st}})$ *by the following theorem*.

Case (2): $\tilde{X}$ decomposes into at least 2 components.

- Let $\tilde{X} = Y_1 \times Z$, where $Z := \prod_{i \neq 1} Y_i$.
- Assume g also decomposes as $g = g_1 \times g_Z$ (otherwise easy).
- If $g_1 \curvearrowright Y_1$ has a fixed point:
  the action $g_Z \curvearrowright Z$ is free
  $\rightsquigarrow$ *apply the induction to $Z/\langle g_Z \rangle$*
  $\rightsquigarrow m_Z \in I_{\mathrm{klt}}(n_Z - 1, \Phi_{\mathrm{st}})$
  $\rightsquigarrow m \in I_{\mathrm{klt}}(n-1, \Phi_{\mathrm{st}})$.
- If g_1 has no fixed point:
  $m_1 = \operatorname{index}(g_1)$ satisfies $\varphi(m_1) \leq 2n_1$ by the holomorphic Lefchetz formula
  $\rightsquigarrow m_1 \in I_{\mathrm{klt}}(n_1 - 1, \Phi_{\mathrm{st}})$ *by the following theorem*
  $\rightsquigarrow m \in I_{\mathrm{klt}}(n-1, \Phi_{\mathrm{st}})$.

Theorem ([M])

$n \in \mathbb{Z}_{\geq 3}$. Then $\{ m \in \mathbb{Z}_{\geq 1} \mid \varphi(m) \leq 2n \} \subseteq I_{\mathrm{klt}}(n-1, \Phi_{\mathrm{st}})$.

Sketch of proof:

- Induction on the dimension n
  $\rightsquigarrow$ we may assume $m = p^e$ is a prime power.
- Construct Calabi–Yau pairs using weighted projective spaces.
  E.g.,

$$(X, B) = \left(\mathbb{P}((p-1)^{(e-1)}, 1^{(p-1)}), \frac{p^e - 1}{p^e} H + \sum_{i=0}^{e-1} \frac{p^{i+1} - 1}{p^{i+1}} H_i \right),$$

$$H_i = \{x_i = 0\}, \quad H = \{x_0 + \cdots + x_{e-2} + x_{e-1}^{p-1} + \cdots + x_{e+p-3}^{p-1} = 0\}$$

  is a Calabi–Yau pair of dimension $p + e - 3$, of index p^e.

Toward the index conjecture

- Similar results for *terminal* Calabi–Yau varieties
  $I_{\mathrm{term}}(n) \subseteq I_{\mathrm{klt}}(n-1, \Phi_{\mathrm{st}})$?
- Index conjecutre for *lc* Calabi–Yau pairs.

[Jia24] Junpeng Jiao, *On structures and discrepancies of klt calabi–yau pairs*, 2024, arXiv: 2405.13321v2.

[JL21] Chen Jiang and Haidong Liu, *Boundedness of log pluricanonical representations of log Calabi–Yau pairs in dimension* 2, Algebra Number Theory **15** (2021), no. 2, 545–567.

[Xu19] Yanning Xu, *Some results about the index conjecture for log Calabi–Yau pairs*, 2019, arXiv: 1905.00297v2.