Rational curves on coindex 3 Fano varieties

Fumiya Okamura

Graduate school of mathematics, Nagoya University

This poster is based on the paper [JO]. We describe the irreducible components of the moduli spaces of rational curves on smooth coindex 3 Fano varieties. This is a higher-dimensional analog of the study of smooth Fano 3-folds [BLRT]. In particular, we prove the moduli space of rational curves representing each numerical class is irreducible when the dimension is at least 5.

0 Settings

- X: Fano manifold/$\mathbb{C}$ of coindex 3, dimension $n \geq 4$
- *coindex* $:= n + 1 - \max\{r \in \mathbb{Z} \mid -K_X/r \in \mathrm{Pic}(X)\}$
- $H := -K_X/(n-2)$: *fundamental divisor*
- Coindex 3 Fano manifolds are classified by [Muk]

1 Main Theorems

Theorem 1. *Let* $n \geq 5$ *and* $\alpha \in NE(X)$
$\Longrightarrow$ $\mathrm{Mor}(\mathbb{P}^1, X, \alpha)$ *irreducible.*
Moreover, $\alpha \in \mathrm{Nef}_1(X) \setminus \partial NE(X)$
$\Longrightarrow$ $\mathrm{Mor}(\mathbb{P}^1, X, \alpha)$ *gen. para. emb. free curves.*

Theorem 2. *Let* $n = 4$ *and* $\alpha \in \mathrm{Nef}_1(X) \setminus \partial NE(X)$
Assume that $X \not\cong \mathbb{P}^1 \times V$ *(*V*: Fano 3-fold of even index)*

- $H.\alpha = 1$ *and* X *is general*
  $\Longrightarrow$ $\mathrm{Mor}(\mathbb{P}^1, X, \alpha)$ *smooth irreducible.*
- $H.\alpha \geq 2$
  $\Longrightarrow$ $\exists! M \subset \mathrm{Mor}(\mathbb{P}^1, X, \alpha)$ *gen. para. emb. free curves.*
  Any other component parametrizes either
  (i) non-dominant family of curves, or
  (ii) multiple covers of H*-lines.*

2 Strategy for the proof

(A) Distinguish between *accumulating* and *Manin* components

(B) Base case (spaces of H-lines)

(C) Induction step (*Movable bend-and-break*)

3 Accumulating components

$M \subset \mathrm{Mor}(\mathbb{P}^1, X, \alpha)$: irreducible component

$$\begin{array}{ccl} \mathcal{U} & \xrightarrow{s} & X\text{: evaluation morphism} \\ {\scriptstyle\pi}\big\downarrow & \text{universal family} & \\ M & & \end{array}$$

M is *accumulating*

$$\overset{\text{“def”}}{\Longleftrightarrow} \begin{cases} \text{(i) } s \text{ is non-dominant, or} \\ \text{(ii) general fiber of } s \text{ is not irreducible} \end{cases}$$

Accumulating comps can be detected by the a-inv.

Definition 3 (*a-invariant*)**.** *Y: proj var, L: nef, big divisor*

$$a(Y, L) := \inf\{t \in \mathbb{R} \mid K_{\tilde{Y}} + t\phi^* L\text{: pseudo-effective on } \tilde{Y}\},$$

which is indep. of the choice of a resolution $\phi \colon \tilde{Y} \to Y$.

Then, [LT] tells that:

$$\begin{aligned} \text{(i)} &\implies a(Z, -K_X|_Z) > 1, \text{ where } Z := \overline{s(\mathcal{U})}, \text{ and} \\ \neg\text{(i)}\wedge\text{(ii)} &\implies \text{Stein factorization } f\colon Y \to X \text{ of } s \text{ is an } a\text{-cover} \\ & \qquad (\text{i.e., } a(Y, -f^*K_X) = 1). \end{aligned}$$

Theorem 4. *For Z as above, one of the following holds:*

- *$n \leq 5$, $\rho(X) = 2$, and Z is the exceptional locus of an elementary divisorial contraction.*
- *$n = 4$, $(Z, H|_Z) \cong (\mathbb{P}^2, \mathcal{O}(1))$, and Z is contracted by an elementary fiber-type contraction.*

Theorem 5. *For $f \colon Y \to X$ as above, let $\phi \colon Y \dashrightarrow B$ be the Iitaka fib'n for $K_Y - f^*K_X$. Then one of the following holds:*

- *$n = 4$, ϕ is birat to a base change of a family of H-lines,*
- *$\rho(X) = 2$, ϕ is birat to a base change of a MFS.*

4 Base case

Theorem 6. *Let* $\alpha \in \mathrm{Nef}_1(X)$ *such* $H.\alpha = 1$.
X: general in moduli $\Longrightarrow$ $\mathrm{Mor}(\mathbb{P}^1, X, \alpha)$*: smooth irreducible.*

(*Idea*). $\exists \mathbb{G} \supset X$: ambient variety s.t. X is a C.I. of very ample divisors on $\mathbb{G}$. Consider the family $\mathcal{X} \to W$ of C.I.s on $\mathbb{G}$, and relate the irreducibility of $\mathrm{Mor}(\mathbb{P}^1, X, \alpha)$ and $\mathrm{Mor}(\mathbb{P}^1, \mathbb{G}, \alpha)$. $\square$

5 Movable Bend-and-Break

Theorem 7. *Suppose* $X \supset \nexists$*contractible divisor.*
$\alpha \in \mathrm{Nef}_1(X)$ *s.t.* $H.\alpha \geq 2$, $\Longrightarrow$ $\forall$*free curve* $\in \mathrm{Mor}(\mathbb{P}^1, X, \alpha)$ *deforms into a union of two free curves.*

MBB is the key tool for counting the number of Manin (non-accumulating) components, and one can conclude the main theorems.

References

[BLRT] R. Beheshti et al. *Moduli spaces of rational curves on Fano threefolds.* Adv. Math., 408 Part A:Article 108557, 2022.

[JO] E. Jovinelly and F. Okamura. *Rational Curves on Coindex 3 Fano Varieties.* arXiv:2409.00834, 2024.

[LT] B. Lehmann and S. Tanimoto. *Geometric Manin's conjecture and rational curves.* Compos. Math., 155(5):833–862, 2019.

[Muk] S. Mukai. *Biregular classification of Fano 3-folds and Fano manifolds of coindex* 3. Proc. Nat. Acad. Sci. U.S.A., 86.9 (1989), pp.3000–3002.