On controling the dynamical degrees of automorphisms of complex simple abelian varieties

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Classification of abelian varieties

Let X be an (projective) abelian variety over $\mathbb{C}$ and define $\mathrm{End}(X) := \{f : X \to X : \text{ morphism} \mid f \text{ preserves the group structure}\}$. If X is a simple abelian variety, then the structure of $\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is classified as below. (cf. [1, Chapter 5.5])

Classification of $\mathrm{End}_{\mathbb{Q}}(X)$

	$B = \mathrm{End}_{\mathbb{Q}}(X)$	K	d	e_0	restriction
Type I	K	totally real	1	e	$e \mid g$
Type II	totally indefinite quaternion algebra over K	totally real	2	e	$2e \mid g$
Type III	totally definite quaternion algebra over K	totally real	2	e	$2e \mid g$
Type IV	division ring with center K	CM-field	d	$\frac{e}{2}$	$\frac{d^2e}{2} \mid g$

Remark

$B := \mathrm{End}_{\mathbb{Q}}(X)$ is a division ring, $\phi : B \to B$ is a Rosati involution, K is a center of B, $K_0 := \{x \in K \mid \phi(x) = x\}$, $[B : K] = d^2$, $[K : \mathbb{Q}] = e$ and $[K_0 : \mathbb{Q}] = e_0$.

For each type of division ring B, for almost all cases, there is a construction of a complex simple abelian variety X such that $\mathrm{End}_{\mathbb{Q}}(X) = B$ (see below).

Dynamical degrees of endomorphisms of abelian varieties

Fix $f \in \mathrm{End}(X)$ for a g-dimenisonal complex abelian variety X. The dynamical degrees of f are defined as $\lambda_k(f) := \lim_{n\to+\infty} ||(f^n)^* : \mathrm{H}^{k,k}(X) \to \mathrm{H}^{k,k}(X)||^{\frac{1}{n}}$ for $0 \le k \le g$, where $||\cdot||$ is a norm for linear transformations and $\mathrm{H}^{k,k}(X)$ is the (k,k)-Dolbeault cohomology.
Writing $X = \mathbb{C}^g/\Lambda$ and f induces $\rho_f : \mathbb{C}^g \to \mathbb{C}^g$. Under this notation, the dynamical degrees of f can be calculated as $\lambda_k(f) := \prod_{i=1}^{k} |\rho_i|^2$ where $\{\rho_i\}_{1\le i\le g}$ are eigenvalues of ρ_f which are renumbering as $|\rho_1| \ge |\rho_2| \ge \cdots \ge |\rho_g| > 0$. By using the Lefschetz fixed point theorem, this value can be calculated more concretely.

Theorem (cf. [1, Chapter 13])

Assume X is simple and denote $B := \mathrm{End}_{\mathbb{Q}}(X)$ and define d, e, e_0 as above. Then, $\prod_{i=1}^{g}(1-\rho_i)(1-\overline{\rho_i}) = \mathrm{Nrd}_{B/\mathbb{Q}}(\mathrm{id}_X - f)^{\frac{2g}{de}}$, where $\mathrm{Nrd}_{B/\mathbb{Q}}$ is the reduced norm.

By applying this theorem for $n_X - f \in \mathrm{End}(X)$ $(n \in \mathbb{Z})$, we get the next theorem.

Theorem (cf. [5, Section 3], [3, Section 2.2])

Assume X is simple and $f \in \mathrm{End}(X)$. Let $P(x) \in \mathbb{Z}[x]$ be the minimal polynomial of f in $\mathrm{End}_{\mathbb{Q}}(X)$. Then, each root of $P(x)$ appears the same number of times in the set $\{\rho_1, \ldots, \rho_g, \overline{\rho_1}, \ldots, \overline{\rho_g}\}$

Thus, the calculation of the dynamical degree of endomorphisms on simple abelian varieties is returned to considering $f \in B$ as the element of the division ring.

Constructing simple abelian varieties from division ring

Let $(B, \mathcal{O}. \phi)$ be a triple of a division ring B of any type in the above table, a $\mathbb{Z}$-order $\mathcal{O}$ for B, and a positive anti-involution $\phi : B \to B$, which fixes K_0 in K.
The construction of a complex abelian variety X with $B \hookrightarrow \mathrm{End}_{\mathbb{Q}}(X)$ such that $\mathcal{O} \hookrightarrow \mathrm{End}(X)$ and ϕ is compatible with the Rosati involution of X, is explained in [1, Chapter 9], If certain conditions for d, e, and e_0 hold, then $B \hookrightarrow \mathrm{End}_{\mathbb{Q}}(X)$ can be an isomorphism, and now X is a simple abelian variety.

Realization of Salem number

A Salem number is a real algebraic integer λ greater than 1 where its other conjugates have modulus at most equal to 1, at least one having a modulus equal to 1.
There is a relation between the first dynamical degree and the Salem number.

Theorem ([2, Theorem 5.1(1)])

Let X be a compact Kähler surface and $f : X \dashrightarrow X$ be a bimeromorphic map with $\rho(f^*) > 1$ for the operator f^* on $\mathrm{H}^{1,1}(X)$ where ρ is the spectral radius. Then the operator f^* has exactly one eigenvalue $\lambda \in \mathbb{R}_{>0}$, of modulus $|\lambda| > 1$, and in fact $\lambda = \rho(f^*)$.

The realization of a Salem number as the first dynamical degree is one of the topics, but for the case of 2-dimensional abelian varieties, realizable Salem numbers have a degree at most 4, For higher-dimensional cases, a realization of a part of Salem numbers is proved in [3, Theorem 2.1 (i)].
We get all Salem numbes are realizable as the first dynamical degree of an automorphism of a simple abelian variety. It proceeds as follows (cf. [5, Section 5]).
Let λ be a Salem number of degree g, and let $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of $\sqrt{\lambda}$. Let γ be one of the roots of $f(X)$ which has modulus 1, and define $L := \mathbb{Q}(\gamma)$, $K := \mathbb{Q}(\gamma + \frac{1}{\gamma})$. Then, there exists an algebraic integer $a \in \mathcal{O}_K$ such that $K(\sqrt{a}) = L$, and a prime number p such that the quaternion algebra $\left(\frac{a,p}{K}\right)$ is divisional (cf. [3, Theorem 2.5, Lemma 2.11]). Existence of a positive anti-involution ϕ over $B := \left(\frac{a,p}{K}\right)$ is reduced to the existence of $c \in B \backslash K$ such that c^2 is totally negative (cf. [1, Theorem 5.5.3]), and this can be achieved.
Let $1, i, j, ij$ be the basis of $\left(\frac{a,p}{K}\right)$ such that $i^2 = a, j^2 = p, ij = -ji$, and then $\mathcal{O} := \mathcal{O}_L \oplus \mathcal{O}_L j$ is a $\mathbb{Z}$-order of $\left(\frac{a,p}{K}\right)$. Thus, we get a suitable complex simple abelian variety X (of Type II) from the triple $(B, \mathcal{O}, \phi)$ as above, and $\mathrm{End}(X)$ contains γ. The minimal polynomial of γ is $f(x)$, and the maximal absolute value of its roots is $\sqrt{\lambda}$. Thus, the first dynamical degree of the automorphism, which corresponds to $\gamma \in \mathrm{End}(X)$, is $\sqrt{\lambda}^2 = \lambda$.

Small first dynamical degrees with fixing dimension

For a g-dimenisonal simple abelian variety X and its automorphism, we consider the minimal polynomial of automorphism in endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(X)$. Now the first dynamical degree of the automorphism is the square of the maximal absolute value of the roots of the minimal polynomial. Since the degree of the minimal polynomial is at most $2g$ by the above table, there are only finitely many candidates of the minimal polynomial when considering the first dynamical degree is less than $1+\epsilon$ for any ϵ. Thus, the smallest first dynamcial degree of automorphisms, which are larger than 1, exists for each g, and it can be calculated by dividing by cases from Type I to Type IV. We denote the minimum value by $m(g)$, and the result for small g and the prime cases are as below.

Theorem ([6, Theorem B])

dimension g	*minimum value $m(g)$*
1	*(all 1)*
2	$4\cos^2(\pi/5) = 2.6180\cdots$
3	$4\cos^2(\pi/7) = 3.2469\cdots$
4	$2\cos(\pi/5) = 1.6180\cdots$
5	$4\cos^2(\pi/11) = 3.6825\cdots$
6	$2\cos(\pi/7) = 1.8019\cdots$
7	$4.0333\cdots$
8	$2\cos(\pi/5) = 1.6180\cdots$
9	$1.1509\cdots^2 = 1.3247\cdots$
10	$1.1762\cdots^2 = 1.3836\cdots$

Theorem ([6, Theorem A])

For a prime number p,

$$\begin{cases} m(p) = 4\cos^2\left(\frac{\pi}{2p+1}\right) & (\textit{if } 2p+1 \textit{ is also a prime}) \\ 4 + 4^{-2p-3} < m(p) < 4.3264 = (2.08)^2 & (\textit{otherwise}) \end{cases}$$

Small first dynamical degrees without fixing dimension

Without fixing the dimension of simple abelian varieties, there exists an automorphism, whose dynamical degree is as close to 1 as possible, but not 1. It proceeds as follows (cf. [5, Section 8]). Let $p > 3$ be a prime number and define $F := \mathbb{Q}\left(\zeta_p, \sqrt[p]{\zeta_p + \frac{1}{\zeta_p}}\right)$, $K = \mathbb{Q}(\zeta_p)$ and $K_0 = \mathbb{Q}\left(\zeta_p + \frac{1}{\zeta_p}\right)$. Now F/K is a cyclic extension of degree p, K_0 is a totally real number field, and its field extension K is a CM-field. There exists an element $u \in K^\times/N_{F/K}(F^\times)$ whose order is exactly p, and define $B = \oplus_{i=0}^{p-1} u^i F$. Let σ be the generator of $\mathrm{Gal}(F/K)$ and define the multiplication on B by $e \cdot u = u \cdot \sigma(e)$ for $e \in F$. By [4, Chapter 15.1, Corollary d], B is a division ring, which is of Type IV in the table.
Define $a := \zeta_p + \frac{1}{\zeta_p}$ and then the map

$$\begin{array}{cccc} \phi: & B & \to & B \\ & u^i\zeta_p^j\sqrt[p]{a^k} & \mapsto & u^i\zeta_p^{ik-j}\sqrt[p]{a^k} \ (0 \le i,k \le p-1,\ 0 \le j \le p-2) \end{array}$$

is an anti-involution over $\mathbb{Q}$ with fixing K_0 in K. By [1, Theorem 5.5.6], there exists a positive anti-involution on B.
Now by defining $\mathcal{O} = \oplus_{i=0}^{p-1} u^i \mathcal{O}_F$, the triple $(B, \mathcal{O}, \phi)$ induces a complex simple abelian variety X (of Type IV, dimension $\frac{p^2(p-1)}{2}$) with $\sqrt[p]{\zeta_p + \frac{1}{\zeta_p}} \in \mathrm{End}(X)$. The minimal polynomial of $\sqrt[p]{\zeta_p + \frac{1}{\zeta_p}}$ over $\mathbb{Q}$ is $\prod_{k=1}^{\frac{p-1}{2}}\left(x^p - \left(\zeta_p^k + \frac{1}{\zeta_p^k}\right)\right)$ and so the first dynamical degree of the corresponding automorphism is $\sqrt[p]{\left|\zeta_p^{\frac{p-1}{2}} + \frac{1}{\zeta_p^{\frac{p-1}{2}}}\right|^2}$. By $p \to \infty$, this value converges to 1.

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