Proper Fredholm submanifolds and affine Kac-Moody symmetric spaces

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1 Introduction

Symmetric spaces, polar representations and isoparametric submanifolds in Euclidean space are closely related. For example, the isotropy representations of symmetric spaces are polar (i.e. there exists an affine subspace which meets every orbit orthogonally). Conversely, Dadok [1] showed that any polar representation on a Euclidean space is orbit equivalent to the isotropy representation of a symmetric space. It is known that any principal orbit of a polar representation is an isoparametric submanifold of the Euclidean space [15] (i.e. the normal bundle is flat and the principal curvatures along any parallel normal vector field are constant). Conversely, Thorbergsson [19] showed that an irreducible compact full isoparametric submanifold of a Euclidean space with codimension at least 3 is an orbit of a polar representation.

It is a natural question whether there are analogous relations in infinite dimensions. Palais and Terng [11, 16] began the study of submanifolds in Hilbert spaces in 1980s. They introduced a suitable class of submanifolds in Hilbert spaces, namely, namely *proper Fredholm* (PF) submanifolds. They also introduced the related class of Lie group actions on Hilbert manifolds, namely *proper Fredholm* (PF) actions. They gave examples of polar PF actions on Hilbert spaces by considering the gauge transformations on the space of L^2-connections. In particular they showed that principal orbits of those actions are isoparametric PF submanifolds in the Hilbert space. Their example was later extend by Pinkall and Thorbergsson [12] and Terng [17]. As a consequence, many examples of polar PF actions on Hilbert spaces and isoparametric PF submanifolds in Hilbert spaces were obtained. Looking at those examples, Terng [17] suspected the existence of infinite dimensional symmetric spaces related to affine Kac-Moody algebras. Those symmetric spaces are nowadays called *affine Kac-Moody symmetric spaces* and known as the infinite dimensional analogues of finite dimensional Riemannian symmetric spaces.

In this article, we give a brief survey on PF submanifolds in Hilbert spaces, polar PF actions on Hilbert spaces and affine Kac-Moody symmetric spaces. Moreover, we focus on affine Kac-Moody symmetric spaces of group type and make clear their relation to the canonical isomorphism between path spaces introduced in [12, 10].

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2 PF submanifolds and polar PF actions

Let N be a Hilbert manifold smoothly immersed in a separable Hilbert space V. Assume that N has finite codimension in V. The end point map $Y : T^{\perp}N \to V$ is defined by $Y(p,\xi) := p + \xi$. Then N is called *proper Fredholm* (PF) [11, 16] if Y is a Fredholm map and the restriction of Y to the normal disc bundle of any finite radius is a proper map. The Fredholm condition implies that the shape operators are compact self-adjoint operators, which allows us to compute the principal curvatures of N. The proper condition implies that the squared distance function from a point to N satisfies the Palais-Smale condition, which allows us to apply the infinite dimensional Morse theory to $N \subset V$. A PF submanifold N of V is called *isoparametric* if $T^{\perp}N$ is flat and the principal curvatures in the direction of any parallel normal vector field are constant.

Let $\mathcal{L}$ be a Hilbert Lie group acting on a separable Hilbert space V. The $\mathcal{L}$-action on V is called *proper* if the map $\mathcal{L} \times V \to V \times V$, $(g,p) \mapsto (g \cdot p, p)$ is a proper map and called *Fredholm* if for each $p \in V$ the map $\mathcal{L} \to V$, $g \mapsto g \cdot p$ is a Fredholm map. If the $\mathcal{L}$-action on V is isometric, proper and Fredholm (PF), then every $\mathcal{L}$-orbit is a PF submanifold of V. An isometric PF action of $\mathcal{L}$ on V is called *polar* if there exists an affine subspace W of V which meets every $\mathcal{L}$-orbit orthogonally. It follows that every principal orbit of a polar PF action on V is an *isoparametric* PF submanifold of V.

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. We fix an $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$ and equip G with the corresponding bi-invariant Riemannian metric. Let $\mathcal{G} := H^1([0,1], G)$ denote the path group of all Sobolev H^1-paths from $[0,1]$ to G and $V_{\mathfrak{g}} := H^0([0,1], \mathfrak{g})$ the path space (separable Hilbert space) of all H^0-maps (i.e. L^2-maps) from $[0,1]$ to $\mathfrak{g}$. Then $\mathcal{G}$ acts on $V_{\mathfrak{g}}$ by the gauge transformations:

$$g * u := gug^{-1} - g'g^{-1}.$$

This action is isometric and PF. However, this action is transitive. Thus we let U be a closed subgroup of $G \times G$ and set

$$P(G,U) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in U\},$$

which acts on $V_{\mathfrak{g}}$ by the gauge transformations. This action is isometric, PF and not transitive in general. Every orbit of the $P(G,U)$-action is a PF submanifold of $V_{\mathfrak{g}}$. Moreover, it was shown that if the action of U on G defined by $(b,c) \cdot a := bac^{-1}$ hyperpolar, then the $P(G,U)$-action on $V_{\mathfrak{g}}$ is polar [17]. Here, an isometric action is called *hyperpolar* if there exists a connected complete flat submanifold which meets every orbit orthogonally.

There are many examples of hyperpolar actions on G (see [7, 8, 9] for details):

(i) The adjoint action: The action of ΔG on G is hyperpolar, where ΔG denotes the diagonal subgroup of $G \times G$.

(ii) The sigma-action: For each $\sigma \in \mathrm{Aut}(G)$, the action of $G(\sigma)$ on G is hyperpolar, where $G(\sigma) := \{(b, \sigma(b)) \mid b \in G\}$.

(iii) The isotropy action: For a compact symmetric pair (G,K), the actions of K on G/K and of $K \times K$ on G are hyperpolar.

(iv) Hermann action: For two compact symmetric spaces (G,K) and (G,H), the actions of H on G/K and of $H \times K$ on G are hyperpolar.

(v) Cohomogeneity one actions on G (or G/K) are hyperpolar.

From these examples, one can obtain many examples of polar $P(G,U)$-actions on $V_{\mathfrak{g}}$ and thus obtain many examples of homogeneous isoparametric PF submanifolds in $V_{\mathfrak{g}}$.

In this way, there are many examples of polar PF actions on Hilbert spaces and isoparametric PF submanifolds in Hilbert spaces. Looking at those examples, Terng [17] suspected the existence of infinite dimensional symmetric spaces related to affine Kac-Moody algebras. This subject was also studied by Heintze, Palais, Terng and Thorbergsson [7]. However, at that time, the precise definition of those symmetric spaces was not given due to functional analytic difficulties to inherent in affine Kac-Moody groups. On the other hand, Heintze and Liu [6] showed a remarkable result on homogeneity of isoparametric PF submanifolds in Hilbert spaces. This result extends the result of Thorbergsson [19] in the finite dimensional Euclidean case and enhance the relation between polar PF actions and isoparametric PF submanifolds. Afterward, Heintze and Popsecue [4, 13] started to study affine Kac-Moody groups and symmetric spaces in the frame work of tame Fréchet manifolds [3] and showed their fundamental properties. In the next section we will review their formulation and basic facts.

3 Affine Kac-Moody symmetric spaces

In this section we review affine Kac-Moody algebras, groups and symmetric spaces. For more details, see [14, 13, 4].

Let G be a simply connected compact simple Lie group with Lie algebra $\mathfrak{g}$ and σ an automorphism of G. The differential of σ is still denoted by σ. Denote by $\langle\cdot,\cdot\rangle$ the inner product of $\mathfrak{g}$ which is the negative of the Killing form of $\mathfrak{g}$. The loop algebra

$$L(\mathfrak{g},\sigma)=\{u:\mathbb{R}\to\mathfrak{g}\mid u\in C^\infty,\ u(t+2\pi)=\sigma(u(t)) \text{ for all } t\}$$

is a Lie algebra with pointwise bracket. We equip the inner product $\langle u,v\rangle_{L^2}=\int_0^{2\pi}\langle u(t),v(t)\rangle dt$ with $L(\mathfrak{g},\sigma)$. Denote by ω_λ the cocycle defined by $\omega_\lambda(u,v)=\lambda\langle u',v\rangle_{L^2}$ for $\lambda\in\mathbb{R}\backslash\{0\}$. An *affine Kac-Moody algebra* is a Lie algebra

$$\hat{L}(\mathfrak{g},\sigma):=L(\mathfrak{g},\sigma)+\mathbb{R}c+\mathbb{R}d,$$

where the bracket is defined by

$$\begin{aligned}
&[u,v]=[u,v]+\omega_\lambda(u,v)c,\\
&[d,u]=u',\\
&[c,x]=0,
\end{aligned}$$

where $u,v\in L(\mathfrak{g},\sigma)$ and $x\in\hat{L}(\mathfrak{g},\sigma)$. It has the center $\mathbb{R}c$ and the derived algebra $\tilde{L}(\mathfrak{g},\sigma):=L(\mathfrak{g},\sigma)+\mathbb{R}c$. If $\sigma_1,\sigma_2\in\operatorname{Aut}\mathfrak{g}$ are conjugate by an inner automorphism then the corresponding affine Kac-Moody algebras are isomorphic. Thus we can assume that σ has finite order. We define the Lorentzian inner product on $\hat{L}(\mathfrak{g},\sigma)$ by

$$\langle u+\alpha c+\beta d, v+\gamma c+\delta d\rangle=\langle u,v\rangle_{L^2}+\alpha\delta+\beta\gamma.$$

Clearly $c,d\perp L(\mathfrak{g},\sigma)$, $\|c\|=\|d\|=0$ and $\langle c,d\rangle=1$. It follows that $\langle[x,y],z\rangle=\langle x,[y,z]\rangle$ for $x,y,z\in\hat{L}(\mathfrak{g},\sigma)$.

The twisted loop group

$$L(G,\sigma) = \{g : \mathbb{R} \to G \mid g \in C^\infty,\ g(t+2\pi) = \sigma(g(t)) \text{ for all } t\}$$

with pointwise multiplication is a Fréchet Lie group with Lie algebra $L(\mathfrak{g},\sigma)$. The cocycle ω_λ defines a left-invariant closed 2-form on $L(G,\sigma)$ and moreover defines a central extension $\tilde{L}(G,\sigma)$ of $L(G,\sigma)$ by the circle S^1 for discrete values of λ [14]. $\tilde{L}(G,\sigma)$ has Lie algebra $\tilde{L}(\mathfrak{g},\sigma)$. There exists a unique λ_0 such that $\tilde{L}(G,\sigma)$ is simply connected. An *affine Kac-Moody group* $\hat{L}(G,\sigma)$ is a Fréchet Lie group defined by

$$\hat{L}(G,\sigma) := S^1 \ltimes \tilde{L}(G,\sigma).$$

Here the S^1-action on $\tilde{L}(G,\sigma)$ is induced by the action on $L(G,\sigma)$ by shifting the parameter of loops. $\hat{L}(G,\sigma)$ is a 2-torus bundle over $L(G,\sigma)$ and has Lie algebra $\hat{L}(\mathfrak{g},\sigma)$. We equip the bi-invariant Lorentzian metric on $\hat{L}(G,\sigma)$. Then $\hat{L}(G,\sigma)$ is a symmetric space where a reflection at the identity is given by $g \mapsto g^{-1}$.

For an involutive automorphism $\hat{\rho}$ of $\hat{G} = \hat{L}(G,\sigma)$ we consider the quotient $\hat{G}/\hat{K}$ by the fixed point subgroup $\hat{K} = \hat{G}^{\hat{\rho}}$. The differential of $\hat{\rho}$ is still denoted by $\hat{\rho}$. The Lie algebra $\hat{\mathfrak{g}} = \hat{L}(\mathfrak{g},\sigma)$ is decomposed into the (± 1)-eigenspaces $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} + \hat{\mathfrak{m}}$. Restricting the inner product on $\hat{\mathfrak{g}}$ to $\hat{\mathfrak{m}}$ we equip the $\hat{G}$-invariant metric with $\hat{G}/\hat{K}$. Then $\hat{G}/\hat{K}$ is a symmetric space where a reflection at $e\hat{K}$ is given by $\hat{g}\hat{K} \mapsto \rho(\hat{g})\hat{K}$.

From the structure theory of involutions of affine Kac-Moody algebras [7, 4, 5] there are essentially two kinds of involutions, namely

(1) $\hat{\rho}$ satisfies $\hat{\rho}(c) = c$, $\hat{\rho}(d) = d$ and $\hat{\rho}(u)(t) = \rho(u(t))$ where $\rho \in \operatorname{Aut}\mathfrak{g}$, $\rho^2 = \mathrm{id}$ and $\sigma\rho = \rho\sigma$,

(2) $\hat{\rho}$ satisfies $\hat{\rho}(c) = -c$, $\hat{\rho}(d) = -d$, $\hat{\rho}(u)(t) = \rho(u(-t))$ where $\rho \in \operatorname{Aut}\mathfrak{g}$, $\rho^2 = \mathrm{id}$ and $\sigma\rho = \rho\sigma^{-1}$.

We will always consider the latter one, called the involution of the *second kind*, so that the extension from $L(G,\sigma)$ to $\hat{L}(G,\sigma)$ is not canceled in the quotient.

By definition an *affine Kac-Moody symmetric space* is either an affine Kac-Moody group $\hat{G}$ (the group type) or the symmetric space $\hat{G}/\hat{K}$ with respect to an involution $\hat{\rho}$ of the second kind. Note that $\hat{G}$ can be written as the quotient $\widehat{G \times G}/(\widehat{G \times G})^{\hat{\rho}}$ where $\widehat{G \times G} = \hat{L}(G \times G, \sigma \times \sigma^{-1})$ is a slight generalization of an affine Kac-Moody group and $\hat{\rho}$ the involution of the second kind defined by

$$\hat{\rho}(c) = -c, \quad \hat{\rho}(d) = -d, \quad \hat{\rho}(u,v)(t) = (v(-t), u(-t)). \tag{3.1}$$

It was shown that $\hat{G}$ and $\hat{G}/\hat{K}$ are tame Fréchet manifolds, where an inverse function theorem is available [3, 13]. The unique existence theorem of the Levi-Civita connection and the conjugacy theorem of finite dimensional maximal flats are verified for affine Kac-Moody symmetric spaces [13]. The concept of duality of symmetric spaces is extended to affine Kac-Moody symmetric spaces based on the theory of complex Kac-Moody groups [2]. The classification of affine Kac-Moody symmetric spaces is essentially equivalent to the classification of involutions of affine Kac-Moody algebras up to conjugation [5].

4 The isotropy representation

The isotropy representations of affine Kac-Moody symmetric spaces are closely related to polar PF actions on Hilbert spaces.

Let $\pi : \tilde{L}(G,\sigma) \to L(G,\sigma)$ denote the projection. For each $\tilde{g} \in \tilde{L}(G,\sigma)$ we write $g = \pi(\tilde{g})$. The *adjoint action* of $\hat{L}(G,\sigma) = S^1 \ltimes \tilde{L}(G,\sigma)$ on $\hat{\mathfrak{g}} = \hat{L}(\mathfrak{g},\sigma)$ is defined by

$$\begin{aligned}
\mathrm{Ad}(\tilde{g})c &= c,\\
\mathrm{Ad}(\tilde{g})d &= d - g'g^{-1} - \frac{1}{2}\|g'g^{-1}\|^2 c,\\
\mathrm{Ad}(\tilde{g})u &= gug^{-1} + \langle g'g^{-1}, gug^{-1}\rangle c
\end{aligned}$$

for $\tilde{g} \in \tilde{L}(G,\sigma)$ and

$$\mathrm{Ad}(e^{is}) = c, \quad \mathrm{Ad}(e^{is}) = d, \quad \mathrm{Ad}(e^{is})u = u_s$$

for $e^{is} \in S^1$ [14]. Here $u_s(t) := u(s+t)$. For the involution $\hat{\rho}$ of the second kind the canonical decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{k}} + \hat{\mathfrak{m}}$ is given by

$$\begin{aligned}
&\hat{\mathfrak{k}} = \{u \in L(\mathfrak{g},\sigma) \mid \rho(u(-t)) = u(t)\},\\
&\hat{\mathfrak{m}} = \{u + \alpha c + \beta d \mid u \in L(\mathfrak{g},\sigma),\ \rho(u(-t)) = -u(t),\ \alpha,\beta \in \mathbb{R}\}.
\end{aligned}$$

The adjoint action of $\hat{G}$ on $\hat{\mathfrak{g}}$ induces the action of $\hat{K}$ on $\hat{\mathfrak{m}}$, which is called the *isotropy representation* of $\hat{G}/\hat{K}$. In the group case we define the isotropy representation of $\hat{G}$ to be the induced action of $L(G,\sigma)$ on $\hat{\mathfrak{g}}$.

Since the adjoint action preserves the inner product and the d-coefficient it leaves invariant the two-sheeted hyperboloid $\{x \in \hat{L}(\mathfrak{g},\sigma) \mid \langle x,x\rangle = -1\}$, the hyperplane $\{u + \alpha c + d \mid u \in L(\mathfrak{g},\sigma)\}$ and hence their intersection

$$\mathrm{Hor}(\hat{\mathfrak{g}}) = \left\{ d + u - \frac{\|u\|^2+1}{2}c \mid u \in L(\mathfrak{g},\sigma)\right\},$$

which is geometrically interpreted as a horosphere of codimension 2. For $x = d+u-\frac{\|u\|^2+1}{2}c$ we have

$$(e^{is},\tilde{g})\cdot x = \left(d + g*u - \frac{\|g*u\|^2+1}{2}c\right)_s,$$

where $g*u = gug^{-1} - g'g^{-1}$ is the gauge transformation. Thus via the isometry

$$\Gamma : L(\mathfrak{g},\sigma) \to \mathrm{Hor}(\hat{\mathfrak{g}}), \qquad u \mapsto d + u - \frac{\|u\|^2+1}{2}c$$

$\hat{L}(G,\sigma)$ acts on $L(\mathfrak{g},\sigma)$ by the gauge transformations.

Recall that two isometric actions of A_1 on X_1 and of A_2 on X_2 are called *essentially equivalent* ([14, p. 167]) if there exist an injective homomorphism $\phi : A_1 \to A_2$ and an injective isometry $\varphi : X_1 \to X_2$ which have dense images and satisfy $\varphi(a\cdot p) = \phi(a)\cdot\varphi(p)$ for $a \in A_1$ and $p \in X_1$. For $r > 0$ we set $\mathcal{G}^r = H^1([0,r],G)$, $V^r_{\mathfrak{g}} = H^0([0,r],\mathfrak{g})$ and

$$P(G,L)^r = \{g \in \mathcal{G}^r \mid (g(0),g(r)) \in L\}$$

for a closed subgroup L of $G\times G$. Similarly we can define the $P(G,L)^r$-action on $V^r_{\mathfrak{g}}$ by gauge transformations.

The following two propositions show the close relation between affine Kac-Moody symmetric spaces and polar PF actions ([17, p. 148], [7, Proposition 4.14]).

Proposition 4.1 (Terng [17])**.** *Let $\hat{G} = \hat{L}(G,\sigma)$ be an affine Kac-Moody symmetric space of group type. Then the isotropy representation restricted to* $\mathrm{Hor}(\hat{\mathfrak{g}})$ *is essentially equivalent to the $P(G, G(\sigma))^{2\pi}$-action on $V_{\mathfrak{g}}^{2\pi}$.*

Proof. The completion of $L(G,\sigma)$ with respect to the H^1-metric is

$$\begin{aligned} &\{g : \mathbb{R} \to G \mid g \in H^1,\ g(t+2\pi) = \sigma(g(t)) \text{ for all } t\} \\ \cong\ &\{g : [0,2\pi] \to G \mid g \in H^1,\ g(2\pi) = \sigma(g(0))\}. \end{aligned}$$

Moreover the completion of $L(\mathfrak{g},\sigma)$ with respect to the H^0-metric is

$$\begin{aligned} &\{u : \mathbb{R} \to \mathfrak{g} \mid u \in H^0,\ u(t+2\pi) = \sigma(u(t)) \text{ for all } t\} \\ \cong\ &\{u : [0,2\pi] \to \mathfrak{g} \mid u \in H^0\}. \end{aligned}$$

This proves the proposition. □

Proposition 4.2 (Heintze-Palais-Terng-Thorbergsson [7])**.** *Let $\hat{G}/\hat{K}$ be an affine Kac-Moody symmetric space. Then the isotropy representation restricted to* $\mathrm{Hor}(\hat{\mathfrak{g}}) \cap \hat{\mathfrak{m}}$ *is essentially equivalent to the $P(G, G^{\rho} \times G^{\sigma\rho})^{\pi}$-action on $V_{\mathfrak{g}}^{\pi}$. Here the inner product of $V_{\mathfrak{g}}^{\pi}$ is defined by $\langle u, v\rangle := 2\int_0^{\pi} \langle u(t), v(t)\rangle dt$.*

Proof. The completion of $\hat{K}$ with respect to the H^1-metric is

$$\begin{aligned} &\{g : \mathbb{R} \to G \mid g \in H^1,\ g(t+2\pi) = \sigma(g(t)),\ \rho(g(-t)) = g(t)\} \\ \cong\ &\{g : [0,2\pi] \to G \mid g \in H^1,\ g(2\pi) = \sigma(g(0)),\ \rho(\sigma^{-1}g(2\pi - t)) = g(t)\} \\ \cong\ &\{g : [0,\pi] \to G \mid g \in H^1,\ \rho(\sigma^{-1}g(0)) = \sigma(g(0)),\ \rho(\sigma^{-1}g(\pi)) = g(\pi)\} \\ =\ &\{g : [0,\pi] \to G \mid g \in H^1,\ \sigma^{-1}\rho\sigma^{-1}g(0) = g(0),\ \rho\sigma^{-1}(g(\pi)) = g(\pi)\} \\ =\ &\{g : [0,\pi] \to G \mid g \in H^1,\ \rho g(0) = g(0),\ \sigma\rho(g(\pi)) = g(\pi)\}. \end{aligned}$$

The completion of $\Gamma^{-1}(\hat{\mathfrak{m}})$ with respect to the H^0-metric is

$$\begin{aligned} &\{u : \mathbb{R} \to \mathfrak{g} \mid u \in H^0,\ u(t+2\pi) = \sigma(u(t)),\ \rho(u(-t)) = -u(t)\} \\ \cong\ &\{u : [0,2\pi] \to \mathfrak{g} \mid u \in H^0,\ \rho(\sigma^{-1}u(2\pi - t)) = -u(t)\} \\ \cong\ &\{u : [0,\pi] \to \mathfrak{g} \mid u \in H^0\}. \end{aligned}$$

This proves the proposition. □

These facts are summarized in the following table:

		Group type
Affine Kac-Moody symmetric space	$\hat{L}(G,\sigma)/\hat{L}(G,\sigma)^{\hat{\rho}}$	$\hat{L}(G,\sigma)$
Isotropy representation	$P(G, H \times K)^{\pi} \curvearrowright V_{\mathfrak{g}}^{\pi}$ $(H := G^{\rho},\ K := G^{\sigma\rho})$	$P(G, G(\sigma))^{2\pi} \curvearrowright V_{\mathfrak{g}}^{2\pi}$
Finite dimensional counterpart	$H \curvearrowright G/K$ Hermann action	$G(\sigma) \curvearrowright G$ sigma-action

5 The group type

We will now focus on the case of group type.

Proposition 5.1. *Let $\widehat{G \times G}/(\widehat{G \times G})^{\hat{\rho}}$ be the affine Kac-Moody symmetric space isomorphic to $\hat{G}$. Then the isotropy representation restricted to the horosphere is essentially equivalent to the $P(G \times G, G(\sigma) \times \Delta G)^{\pi}$-action on $V^{\pi}_{\mathfrak{g}\oplus\mathfrak{g}}$.*

Proof. Recall that the involution $\hat{\rho}$ was defined by (3.1). We consider another involution $\hat{\tau}$ of the second kind defined by

$$\hat{\tau}(c) = -c, \quad \hat{\tau}(d) = -d, \quad \hat{\tau}(u,v)(t) = (\sigma^{-1}v(-t), \sigma u(-t)). \tag{5.1}$$

Note that $\hat{\rho}$ and $\hat{\tau}$ are conjugate and thus the corresponding quotients are isomorphic. Then by the similar argument as in Proposition 4.2 it follows that the isotropy representation of $\widehat{G \times G}/(\widehat{G \times G})^{\hat{\tau}}$ restricted to the horosphere is essentially equivalent to the $P(G \times G, G(\sigma) \times \Delta G)^{\pi}$-action on $V^{\pi}_{\mathfrak{g}\oplus\mathfrak{g}}$. □

In the study of polar PF actions on Hilbert spaces, the author [10] studied natural isomorphisms between path spaces. The idea of the isomorphisms is essentially due to Pinkall and Thorbergsson [12]. More precisely, we consider two maps $\Omega : \mathcal{G}^{2\pi} \to \mathcal{G}^{\pi}$ and $\Upsilon : V^{2\pi}_{\mathfrak{g}} \to V^{\pi}_{\mathfrak{g}\oplus\mathfrak{g}}$, which are respectively defined by

$$\Omega(g) = (g(t), g(2\pi - t)), \quad \Upsilon(u) = (u(t), -u(2\pi - t)).$$

It was shown that the $P(G, G(\sigma))^{2\pi}$-action $V^{2\pi}_{\mathfrak{g}}$ is conjugate to the $P(G\times G, G(\sigma)\times\Delta G)^{\pi}$-action on $V^{\pi}_{\mathfrak{g}\oplus\mathfrak{g}}$ via (Ω, Υ) [10, Corollary 3.3], that is, the diagram

$$\begin{array}{ccc} P(G, G(\sigma))^{2\pi} & \xrightarrow{\Omega} & P(G \times G, G(\sigma) \times \Delta G)^{\pi} \\ \curvearrowright & & \curvearrowright \\ V^{2\pi}_{\mathfrak{g}} & \xrightarrow{\Upsilon} & V^{\pi}_{\mathfrak{g}\oplus\mathfrak{g}} \end{array} \tag{5.2}$$

commutes. This together with Propositions 4.1 and 5.1 implies:

Corollary 5.2. *Let $\hat{G} = \hat{L}(G, \sigma)$ be an affine Kac-Moody symmetric space of group type and $\widehat{G \times G}/(\widehat{G \times G})^{\hat{\rho}}$ the quotient isomorphic to $\hat{G}$. Then their isotropy representations restricted to the horospheres are essentially equivalent to the $P(G, G(\sigma))^{2\pi}$-action on $V^{2\pi}_{\mathfrak{g}}$ and the $P(G \times G, G(\sigma) \times \Delta G)^{\pi}$-action on $V^{\pi}_{\mathfrak{g}\oplus\mathfrak{g}}$ respectively and these are conjugate via the canonical isomorphisms Ω and Υ.*

This corollary suggests that the isomorphism $\hat{G} \cong \widehat{G \times G}/(\widehat{G \times G})^{\hat{\rho}}$ induces the canonical isomorphisms (Ω, Υ). Let us show this more explicitly. By conjugacy we can replace the involution $\hat{\rho}$ with $\hat{\tau}$. Consider the map

$$\lambda : \widehat{G \times G} \to \hat{G}$$

whose differential is

$$d\lambda : \widehat{\mathfrak{g} \oplus \mathfrak{g}} \to \hat{\mathfrak{g}}, \quad (u(t), v(t)) + \alpha c + \beta d \mapsto (u(t) - \sigma^{-1}v(-t)) + \alpha c + \beta d.$$

The inverse image $\lambda^{-1}(\hat{e})$ of the identity $\hat{e}$ is $(\widehat{G \times G})^{\hat{\tau}}$. Thus it induces the isomorphism

$$\Lambda : \widehat{G \times G}/(\widehat{G \times G})^{\hat{\tau}} \to \hat{G}.$$

There is an isomorphism between the isotropy subgroups

$$\varphi : L(G, \sigma) \to (\widehat{G \times G})^{\hat{\tau}}, \quad g(t) \mapsto (g(t), \sigma(g(-t))).$$

The canonical decomposition $\widehat{\mathfrak{g} \oplus \mathfrak{g}} = \hat{\mathfrak{k}} + \hat{\mathfrak{m}}$ with respect to $\hat{\tau}$ is given by

$$\begin{aligned}\hat{\mathfrak{k}} &= \{(u(t), \sigma(u(-t))) \mid u \in L(\mathfrak{g}, \sigma)\},\\ \hat{\mathfrak{m}} &= \{(u(t), -\sigma u(-t)) + \alpha c + \beta d \mid u \in L(\mathfrak{g}, \sigma),\ \alpha, \beta \in \mathbb{R}\}.\end{aligned}$$

There is an isomorphism between the linear subspaces

$$\psi : \hat{\mathfrak{g}} \to \hat{\mathfrak{m}}, \quad u(t) + \alpha c + \beta d \mapsto (u(t), -\sigma(u(-t))) + \alpha c + \beta d.$$

We define the inner product of $\widehat{\mathfrak{g} \oplus \mathfrak{g}}$ by

$$\langle (u_1, u_2) + \alpha c + \beta d, (v_1, v_2) + \gamma c + \delta d \rangle = \frac{1}{2}(\langle u_1, u_2 \rangle_{L^2} + \langle v_1, v_2 \rangle_{L^2}) + \alpha\delta + \beta\gamma.$$

Then the isotropy representations of $L(G, \sigma)$ on $\hat{\mathfrak{g}}$ and of $(\widehat{G \times G})^{\hat{\tau}}$ on $\hat{\mathfrak{m}}$ are conjugate via φ and ψ. Moreover ψ induces the isometry

$$\psi : \mathrm{Hor}(\hat{\mathfrak{g}}) \to \mathrm{Hor}(\widehat{\mathfrak{g} \oplus \mathfrak{g}}) \cap \hat{\mathfrak{m}},$$

which induces

$$\psi : L(\mathfrak{g}, \sigma) \to \Gamma^{-1}(\hat{\mathfrak{m}}), \quad u(t) \mapsto (u(t), -\sigma(u(-t))).$$

Since $\sigma(g(-t)) = g(2\pi - t)$ and $\sigma(u(-t)) = u(2\pi - t)$ the diagrams

$$\begin{array}{ccc} L(G, \sigma) & \xrightarrow{\varphi} & \widehat{G \times G}^{\hat{\rho}} \\ \downarrow & & \downarrow \\ P(G, G(\sigma))^{2\pi} & \xrightarrow{\Omega} & P(G \times G, G(\sigma) \times \Delta G)^{\pi} \end{array} \tag{5.3}$$

and

$$\begin{array}{ccc} L(\mathfrak{g}, \sigma) & \xrightarrow{\psi} & \Gamma^{-1}(\hat{\mathfrak{m}}) \\ \downarrow & & \downarrow \\ V_{\mathfrak{g}}^{2\pi} & \xrightarrow{\Upsilon} & V_{\mathfrak{g} \oplus \mathfrak{g}}^{\pi} \end{array} \tag{5.4}$$

are commutative, where the vertical arrows denote the injective maps with dense images given in Propositions 4.1 and 4.2. In this way the isomorphism Λ induces (Ω, Υ).

Recall that the sigma-action $G(\sigma) \curvearrowright G$ can be viewed as a Hermann action $G(\sigma) \curvearrowright (G \times G)/\Delta G$. In fact, the diagram

$$\begin{array}{ccc} G(\sigma) & \xrightarrow{\mathrm{id}} & G(\sigma) \\ \curvearrowright & & \curvearrowright \\ G & \xleftarrow{\phi} & (G \times G)/\Delta G \end{array} \tag{5.5}$$

commutes, where $\phi(a,b) := ab^{-1}$. To consider the relation between this diagram and (5.2), we recall the concept of the *parallel transport map*. This is an equivariant Riemannian submersion $\Phi : V_{\mathfrak{g}} \to G$ defined by $\Phi(u) = g(1)$ for $g \in P(G, \{e\}\times G)$ satisfying $g^{-1}g' = u$. It follows that the diagram

$$\begin{array}{ccc} \mathcal{G} & \curvearrowright & V_{\mathfrak{g}} \\ \Psi^G \downarrow & & \Phi \downarrow \\ G\times G & \curvearrowright & G \end{array}$$

commutes, where Ψ^G is defined by $\Psi^G(g) := (g(0), g(1))$. If a compact symmetric space G/K with projection $p : G \to G/K$ is given, we consider the composition $\Phi_K := p \circ \Phi : V_{\mathfrak{g}} \to G \to G/K$, which is called the parallel transport map over G/K. It follows that the diagram

$$\begin{array}{ccc} P(G, G\times K) & \curvearrowright & V_{\mathfrak{g}} \\ p^G \circ \Psi^G \downarrow & & \Phi_K \downarrow \\ G & \curvearrowright & G/K \end{array}$$

commutes, where $p^G : G\times G \to G$ denotes the projection onto the first component. For the interval $[0,r]$ for each $r > 0$, we can similarly define $\Phi^r : V_{\mathfrak{g}}^r \to G$, $\Phi_K^r : V_{\mathfrak{g}}^r \to G/K$ and so on. In particular the following diagrams are commutative:

$$\begin{array}{ccc} P(G, G(\sigma))^{2\pi} & \curvearrowright & V_{\mathfrak{g}}^{2\pi} \\ (\Psi^G)^{2\pi} \downarrow & & \Phi^{2\pi} \downarrow \\ G(\sigma) & \curvearrowright & G \end{array}$$

and

$$\begin{array}{ccc} P(G\times G, G(\sigma)\times \Delta G)^{\pi} & \curvearrowright & V_{\mathfrak{g}\oplus\mathfrak{g}}^{\pi} \\ p^{G\times G} \circ (\Psi^{G\times G})^{\pi} \downarrow & & \Phi_{\Delta G}^{\pi} \downarrow \\ G(\sigma) & \curvearrowright & (G\times G)/\Delta G. \end{array}$$

Furthermore, from the properties of (Ω, Υ), the following diagrams are commutative [10, Theorem 3.2]:

$$\begin{array}{ccc} P(G, G(\sigma))^{2\pi} & \xrightarrow{\Omega} & P(G\times G, G(\sigma)\times \Delta G)^{\pi} \\ {\scriptstyle (\Psi^G)^{2\pi}}\Big\downarrow & & {\scriptstyle p^{G\times G}\circ(\Psi^{G\times G})^{\pi}}\Big\downarrow \\ G(\sigma) & \xrightarrow{\mathrm{id}} & G(\sigma) \end{array} \tag{5.6}$$

and

$$\begin{array}{ccc} V_{\mathfrak{g}}^{2\pi} & \xrightarrow{\Upsilon} & V_{\mathfrak{g}\oplus\mathfrak{g}}^{\pi} \\ {\scriptstyle \Phi^{2\pi}}\Big\downarrow & & {\scriptstyle \Phi_{\Delta G}^{\pi}}\Big\downarrow \\ G & \xleftarrow{\phi} & (G\times G)/\Delta G\,. \end{array} \tag{5.7}$$

In this way, (Ω, Υ) are closely related to (id, ϕ) via the parallel transport map.

As a consequence, the isomorphism Λ between $\hat{G}$ and $(\widehat{G \times G})^{\hat{\tau}}$ induces the canonical isomorphisms (Ω, Υ) and these are closely related to the isomorphisms (id, ϕ). These relations are summarized in the following table.

Affine Kac-Moody symmetric space	$\hat{G} = \hat{L}(G, \sigma)$	$\overset{\Lambda}{\cong}$	$\widehat{G \times G} / (\widehat{G \times G})^{\hat{\tau}}$
Isotropy representation	$P(G, G(\sigma))^{2\pi} \curvearrowright V_{\mathfrak{g}}^{2\pi}$	$\overset{(\Omega,\Upsilon)}{\cong}$	$P(G \times G, G(\sigma) \times \Delta G)^{\pi} \curvearrowright V_{\mathfrak{g}\oplus\mathfrak{g}}^{\pi}$
Finite dimensional counterpart	$G(\sigma) \curvearrowright G$	$\overset{(\mathrm{id},\phi)}{\cong}$	$G(\sigma) \curvearrowright (G \times G)/\Delta G.$

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