Holographic and symmetry breaking operators of holomorphic discrete series representations for real rank 3 cases

By

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Abstract

Let $(G,(G^\sigma)_0)$ be a symmetric pair of holomorphic type, and let $\mathcal{H}_\lambda(D)$ be a holomorphic discrete series representation of scalar type of G. Then the restriction $\mathcal{H}_\lambda(D)|_{(G^\sigma)_0}$ is decomposed multiplicity-freely into the Hilbert direct sum of countable holomorphic discrete series representations, and its branching law is given explicitly by the Hua–Kostant–Schmid–Kobayashi formula. Especially, there exist uniquely (up to constant multiple) the $(G^\sigma)_0$-intertwining operators (holographic operator, symmetry breaking operator) between $\mathcal{H}_\lambda(D)|_{(G^\sigma)_0}$ and each irreducible subrepresentation of $(G^\sigma)_0$. In this article, we treat the results on explicit construction of all intertwining operators for $\mathcal{H}_\lambda(D)|_{(G^\sigma)_0}$ when G and the associated symmetric subgroup $(G^{\sigma\theta})_0$ are both of real rank 3, of tube type and their non-compact parts are simple.

§ 1. Setting

The purpose of this article is to give results on explicit construction of $(G^\sigma)_0$-intertwining operators (holographic operators, symmetry breaking operators) appearing in the restriction of holomorphic discrete series representations, for the symmetric pairs $(G,(G^\sigma)_0)=(SU(3,3),SO^*(6))$, $(SO^*(12),SO^*(6)\times SO^*(6))$, $(E_{7(-25)},SU(2,6))$.

Throughout the paper, let $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$ and $d:=\dim_\mathbb{R}\mathbb{F}\in\{1,2,4,8\}$. For $x\in\mathbb{F}\otimes_\mathbb{R}\mathbb{C}$, let $\hat{x}$ denote the $\mathbb{F}$-conjugate, and $\overline{x}$ denote the complex conjugate. Let $\mathfrak{p}^\pm:=\mathrm{Herm}(3,\mathbb{F})\otimes_\mathbb{R}\mathbb{C}=\{X\in M(3,\mathbb{F})\mid X={}^t\hat{X}\}\otimes_\mathbb{R}\mathbb{C}$, and for $X\in\mathfrak{p}^\pm$, let $X^\sharp$ denote the

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adjugate element given by

$$\begin{pmatrix} a_1 & x_3 & \hat{x}_2 \\ \hat{x}_3 & a_2 & x_1 \\ x_2 & \hat{x}_1 & a_3 \end{pmatrix}^{\sharp} := \begin{pmatrix} a_2a_3 - x_1\hat{x}_1 & \hat{x}_2\hat{x}_1 - a_3x_3 & x_3x_1 - a_2\hat{x}_2 \\ x_1x_2 - a_3\hat{x}_3 & a_3a_1 - x_2\hat{x}_2 & \hat{x}_3\hat{x}_2 - a_1x_1 \\ \hat{x}_1\hat{x}_3 - a_2x_2 & x_2x_3 - a_1\hat{x}_1 & a_1a_2 - x_3\hat{x}_3 \end{pmatrix}.$$

For $X, Y \in \mathfrak{p}^{\pm}$, we consider the $\mathbb{C}$-bilinear form $(X|Y) := \mathrm{Re}_{\mathbb{F}} \operatorname{tr}(XY)$, and regard $\mathfrak{p}^+$ and $\mathfrak{p}^-$ as mutually dual spaces by this bilinear form. Also, for $X, Y \in \mathfrak{p}^{\pm}$ let $\det(X) := \frac{1}{3}(X|X^{\sharp})$, and

$$h(X,Y) := 1 - (X|Y) + (X^{\sharp}|Y^{\sharp}) - \det(X)\det(Y).$$

Using this, *the bounded symmetric domain* $D \subset \mathfrak{p}^+$ is defined by

$$D := (\text{Connected component of } \{X \in \mathfrak{p}^+ \mid h(X, \overline{X}) > 0\} \text{ containing } 0).$$

Let $(G, K) = (\mathrm{Bihol}(D)_0, \mathrm{Stab}(0)_0)$ be the identity components of the group of biholomorphisms on D and the stabilizer subgroup at 0. Then up to covering we have

$$(G,K) \simeq \begin{cases} (Sp(3,\mathbb{R}), U(3)) & (\mathbb{F} = \mathbb{R}), \\ (SU(3,3), S(U(3)\times U(3))) & (\mathbb{F} = \mathbb{C}), \\ (SO^*(12), U(6)) & (\mathbb{F} = \mathbb{H}), \\ (E_{7(-25)}, U(1)\times E_6) & (\mathbb{F} = \mathbb{O}), \end{cases}$$

and $D \simeq G/K$ becomes a Hermitian symmetric space. The complexified Lie algebra of G is written as $\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C} =: \mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-$ as vector spaces. Next, for $\lambda > 1 + 2d$, let $\langle \cdot, \cdot \rangle_{\mathcal{H}_\lambda(D)}$ be the *weighted Bergman inner product* on the space $\mathcal{O}(D) = \mathcal{O}_\lambda(D)$ of holomorphic functions on D given by

$$\langle f, g \rangle_{\mathcal{H}_\lambda(D)} := C_\lambda \int_D f(X)\overline{g(X)} h(X, \overline{X})^{\lambda - 2(1+d)} dX \qquad (f, g \in \mathcal{O}(D)),$$

where the normalizing constant C_λ is chosen so that $\|1\|^2_{\mathcal{H}_\lambda(D)} := \langle 1, 1 \rangle_{\mathcal{H}_\lambda(D)} = 1$ holds. Let $\mathcal{H}_\lambda(D) \subset \mathcal{O}_\lambda(D)$ be the corresponding Hilbert space (*weighted Bergman space*). Then its reproducing kernel is $h(X, \overline{Y})^{-\lambda}$, and there exist a map $\kappa \colon G \times D \to K^{\mathbb{C}}$ and a character $\chi \colon K^{\mathbb{C}} \to \mathbb{C}^{\times}$ such that the universal covering group $\widetilde{G}$ of G acts unitarily on $\mathcal{H}_\lambda(D)$ by

$$(\tau_\lambda(g)f)(X) := \chi(\kappa(g^{-1}, X))^{-\lambda} f(g^{-1}X). \tag{1.1}$$

$(\tau_\lambda, \mathcal{H}_\lambda(D))$ is called a holomorphic discrete series representation of $\widetilde{G}$.

For example, suppose $\mathbb{F}=\mathbb{C}$, so that $d=\dim_{\mathbb{R}}\mathbb{F}=2$, $\mathfrak{p}^{\pm}:=\mathrm{Herm}(3,\mathbb{C})\otimes_{\mathbb{R}}\mathbb{C}\simeq M(3,\mathbb{C})$. Then for $X,Y\in M(3,\mathbb{C})$ we have

$$h(X,Y)=1-\mathrm{tr}(XY)+\mathrm{tr}(X^{\sharp}Y^{\sharp})-\det(X)\det(Y)=\det(I-XY),$$
$$D=\{X\in M(3,\mathbb{C})\mid I-XX^{*}\text{ is positive definite}\}.$$

For $\lambda>1+2d=5$, the weighted Bergman inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_\lambda(D)}$ on $\mathcal{O}(D)=\mathcal{O}_\lambda(D)$ is given by

$$\langle f,g\rangle_{\mathcal{H}_\lambda(D)}=C_\lambda\int_D f(X)\overline{g(X)}\det(I-XX^{*})^{\lambda-6}dX,$$

and the universal covering group $\widetilde{G}$ of

$$G=SU(3,3):=\left\{g\in SL(6,\mathbb{C})\;\middle|\; g\begin{pmatrix}I_3&0\\0&-I_3\end{pmatrix}g^{*}=\begin{pmatrix}I_3&0\\0&-I_3\end{pmatrix}\right\}$$

acts on the weighted Bergman space $\mathcal{H}_\lambda(D)\subset\mathcal{O}_\lambda(D)$ by

$$\tau_\lambda\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}\right)f(X):=\det(cX+d)^{-\lambda}f\bigl((aX+b)(cX+d)^{-1}\bigr).$$

Next, let $(\mathbb{F},\mathbb{F}')=(\mathbb{C},\mathbb{R}),(\mathbb{H},\mathbb{C}),(\mathbb{O},\mathbb{H})$, and we consider the Cayley–Dickson extension $\mathbb{F}=\mathbb{F}'\oplus\mathbb{F}'j$ with an additional imaginary unit j. According to Yokota [20], we define an involution σ on $\mathfrak{p}^{+}=\mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}$ by, for $a_i\in\mathbb{C}$, $x_i,y_i\in\mathbb{F}'\otimes_{\mathbb{R}}\mathbb{C}$,

$$\begin{pmatrix}a_1&x_3+y_3j&\hat{x}_2-y_2j\\\hat{x}_3-y_3j&a_2&x_1+y_1j\\x_2+y_2j&\hat{x}_1-y_1j&a_3\end{pmatrix}\mapsto-\begin{pmatrix}a_1&x_3-y_3j&\hat{x}_2+y_2j\\\hat{x}_3+y_3j&a_2&x_1-y_1j\\x_2-y_2j&\hat{x}_1+y_1j&a_3\end{pmatrix},$$

so that $\mathfrak{p}_1^{+}:=(\mathfrak{p}^{+})^{\sigma}=\mathrm{Alt}(3,\mathbb{F}')^{\mathbb{C}}j\simeq(\mathbb{F}'^{3})^{\mathbb{C}}$, $\mathfrak{p}_2^{+}:=(\mathfrak{p}^{+})^{-\sigma}=\mathrm{Herm}(3,\mathbb{F}')^{\mathbb{C}}$, and extend this to the involutions on σ, $\sigma\theta$ on $G=\mathrm{Bihol}(D)_0$ by

$$\sigma(g):=\sigma\circ g\circ\sigma,\qquad \sigma\theta(g):=(-\sigma)\circ g\circ(-\sigma).$$

Then up to covering we have

$$(G,(G^{\sigma})_0,(G^{\sigma\theta})_0)=\begin{cases}(SU(3,3),SO^{*}(6),Sp(3,\mathbb{R}))&((\mathbb{F},\mathbb{F}')=(\mathbb{C},\mathbb{R})),\\(SO^{*}(12),SO^{*}(6)\times SO^{*}(6),U(3,3))&((\mathbb{F},\mathbb{F}')=(\mathbb{H},\mathbb{C})),\\(E_{7(-25)},SU(2,6),SU(2)\times SO^{*}(12))&((\mathbb{F},\mathbb{F}')=(\mathbb{O},\mathbb{H})),\end{cases}$$

and $D_1:=D^{\sigma}=D\cap\mathfrak{p}_1^{+}\simeq(G^{\sigma})_0/(K^{\sigma})_0$ also becomes a Hermitian symmetric space.

Next we consider the decomposition of $\mathcal{H}_\lambda(D)$ under $(\widetilde{G}^{\sigma})_0$. Let $\mathbb{Z}^r_{++}:=\{\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}^r\mid k_1\ge\cdots\ge k_r\ge0\}$. Then the space $\mathcal{P}(\mathfrak{p}_2^{+})$ of polynomials on $\mathfrak{p}_2^{+}$ is decomposed under $(K^{\sigma})_0=(K^{\sigma\theta})_0$ as

$$\mathcal{P}(\mathfrak{p}_2^{+})=\bigoplus_{\mathbf{k}\in\mathbb{Z}^3_{++}}\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^{+})$$

(Hua–Kostant–Schmid, [4, Theorem XI.2.4]), where

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) := \mathrm{Span}_{\mathbb{C}}\{\Delta_{\mathbf{k}}(k.X_2) \mid k \in (K^\sigma)_0\} \subset \mathcal{P}(\mathfrak{p}_2^+),$$
$$\Delta_{\mathbf{k}}(X_2) := (a_1)^{k_1-k_2}(a_1a_2 - x_3\hat{x}_3)^{k_2-k_3}\det(X_2)^{k_3} \in \mathcal{P}(\mathfrak{p}_2^+) \quad \left(X_2 = \begin{pmatrix} a_1 & x_3 & \hat{x}_2 \\ \hat{x}_3 & a_2 & x_1 \\ x_2 & \hat{x}_1 & a_3 \end{pmatrix} \right),$$

and according to this decomposition, by Kobayashi [9], for $\lambda > 1 + 2d$, $\mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0}$ is decomposed as

$$\mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0} = \sum_{\mathbf{k}\in\mathbb{Z}^3_{++}}^{\oplus} \mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)),$$

where $\mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)) \subset \mathcal{O}(D_1) \otimes \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$ is the Hilbert space defined by the inner product

$$\begin{aligned}&\langle f, g\rangle_{\mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))} \\ &:= C_{\lambda,\mathbf{k}} \int_{D_1} (f(X_1, B(X_1, \overline{X_1})X_2), g(X_1, X_2))_{\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+), X_2} h(X_1, \overline{X_1})^{\lambda-d} dX_1,\end{aligned}$$

where $(f(X_2), g(X_2))_{\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+), X_2} := \overline{g}\big(\frac{\partial}{\partial X_2}\big) f(X_2)\big|_{X_2=0}$ is the *Fischer inner product* on $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$ (see [4, Section XI.1]), $B\colon \mathfrak{p}_1^+ \times \mathfrak{p}_1^- \to \mathrm{End}_{\mathbb{C}}(\mathfrak{p}_2^+)$ is given later in (3.1), $d = \dim_{\mathbb{R}} \mathbb{F} = 2\dim_{\mathbb{R}} \mathbb{F}'$, and $C_{\lambda,\mathbf{k}}$ is chosen such that $\|f(X_2)\|_{\mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))} = |f(X_2)|_{\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)}$ holds for f independent of $X_1 \in D_1$, on which $(\widetilde{G}^\sigma)_0$ acts by

$$(\tau'_{\lambda,\mathbf{k}}(g)f)(X_1, X_2) := \chi(\kappa(g^{-1}, X_1))^{-\lambda} f(g^{-1}X_1, \kappa(g^{-1}, X_1)^{-1}X_2).$$

This is also called the holomorphic discrete series representation of $(\widetilde{G}^\sigma)_0$. Especially, if $\mathbf{k} = (k, k, k)$, then $\mathcal{H}_\lambda(D_1, \mathcal{P}_{(k,k,k)}(\mathfrak{p}_2^+)) \simeq \mathcal{H}_{\lambda+2k}(D_1)$ is of scalar type. Hence there exist uniquely (up to constant multiple) the $(\widetilde{G}^\sigma)_0$-intertwining operators (holographic operator/symmetry breaking operator)

$$\begin{aligned}\mathcal{F}^{\uparrow}_{\lambda,\mathbf{k}}&\colon \mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)) \longrightarrow \mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0},\\ \mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}}&\colon \mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0} \longrightarrow \mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)).\end{aligned}$$

The purpose of this paper is to construct these intertwining operators. We note that it is proved by Kobayashi–Pevzner [12] that symmetry breaking operators for holomorphic discrete series representations are always given by differential operators, that is, $\mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}}$ is of the form

$$(\mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}} f)(X_1) = F^{\downarrow}_{\lambda,\mathbf{k}}\left(\frac{\partial}{\partial X}\right) f(X)\bigg|_{X_2=0}$$

for some $F^{\downarrow}_{\lambda,\mathbf{k}}(Z) \in \mathcal{P}(\mathfrak{p}^-)_Z \otimes \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$, where $X = (X_1, X_2) \in \mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$.

Remark 1.1. (1) Previous works on differential symmetry breaking operators are given by, e.g., Rankin [19], Cohen [3], Peng–Zhang [18], Juhl [8], Ibukiyama–Kuzumaki–Ochiai [7], Kobayashi–Ørsted–Somberg–Souček [11], Kobayashi–Pevzner [12, 13], Kobayashi–Kubo–Pevzner [10] and the author [16].

(2) Previous works on holographic operators for symmetric pairs of holomorphic type are given by, e.g., Kobayashi–Pevzner [14] and the author [15].

(3) Previous works on Parseval–Plancherel type formulas for symmetric pairs of holomorphic type are given by, e.g., Hilgert–Krötz [5, 6], Ben Saïd [1, 2], Kobayashi–Pevzner [14] and the author [17] under different realizations of $\mathcal{H}_\lambda(D)$.

§ 2. Main theorems

We recall that $d = \dim_{\mathbb{R}} \mathbb{F} \in \{2, 4, 8\}$. For $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, let $(\lambda)_k := \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1)$, and let $X = (X_1, X_2) \in \mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$, $Z = (Z_1, Z_2) \in \mathfrak{p}^- = \mathfrak{p}_1^- \oplus \mathfrak{p}_2^-$. First we give the result on holographic operators. For proof see [15][1]

Theorem 2.1 ([15, Theorems 5.7 (5), 5.12]). *For $\lambda > 1 + 2d$, $\mathbf{k} \in \mathbb{Z}^3_{++}$, the holographic operator*

$$\mathcal{F}^{\uparrow}_{\lambda,\mathbf{k}} \colon \mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))_{(\widetilde{K}^\sigma)_0} \longrightarrow \mathcal{H}_\lambda(D)_{\widetilde{K}}|_{\mathfrak{g}^\sigma}$$

is given (up to constant multiple) by the infinite-order differential operator

$$(\mathcal{F}^{\uparrow}_{\lambda,\mathbf{k}} f)(X) = F^{\uparrow}_{\lambda,\mathbf{k}}\left(X_2; \frac{\partial}{\partial X_1}\right) f(X_1),$$

$$F^{\uparrow}_{\lambda,\mathbf{k}}(X_2; Z_1) = \sum_{\substack{0 \leq l_1 \leq k_1 - k_2 \\ 0 \leq l_2 \leq k_2 - k_3 \\ 0 \leq l_3}} \frac{1}{\left(\lambda + k_2 + k_3 - \frac{3}{4}d + 1\right)_{l_1}} \frac{1}{\left(\lambda + k_1 + k_3 - \frac{1}{2}d + 1\right)_{l_2}} \times \frac{1}{\left(\lambda + k_1 + k_2 - \frac{1}{4}d + 1\right)_{l_3}} f^{\uparrow}_{\mathbf{k},\mathbf{l}}(X_2; Z_1),$$

where $f^{\uparrow}_{\mathbf{k},\mathbf{l}}(X_2; Z_1) \in \mathcal{P}(\mathfrak{p}_2^+ \oplus \mathfrak{p}_1^-)_{(X_2, Z_1)} \otimes \mathrm{Hom}_{\mathbb{C}}(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+), \mathbb{C})$ is given later in (2.1).

Next we give the result on symmetry breaking operators. Its proof is given in a later section.

Theorem 2.2 ([17, Conjecture 7.2]). *For $\lambda > 1 + 2d$, $\mathbf{k} \in \mathbb{Z}^3_{++}$, the symmetry breaking operator*

$$\mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}} \colon \mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0} \longrightarrow \mathcal{H}_\lambda(D_1, \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))$$

[1] In [15, Theorem 5.12], the subscripts $(m_3, m_2, m_1), 2$ are wrong. The correct subscripts are $(m_3, m_2, m_1), d_2$.

is given (up to constant multiple) by the differential operator

$$(\mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}}f)(X_1)=F^{\downarrow}_{\lambda,\mathbf{k}}\left(\frac{\partial}{\partial X}\right)f(X)\bigg|_{X_2=0},$$

$$F^{\downarrow}_{\lambda,\mathbf{k}}(Z)=\sum_{\substack{0\le l_1\\ 0\le l_2\le k_1-k_2\\ 0\le l_3\le k_2-k_3\\ l_1+l_2\le k_3}}\frac{\left(-k_1-\frac{1}{2}d\right)_{l_2+l_3}}{\left(-\lambda-k_2-k_3+\frac{3}{4}d+1\right)_{l_1}}\frac{\left(-k_2-\frac{1}{4}d\right)_{l_1+l_3}}{\left(-\lambda-k_1-k_3+\frac{1}{2}d+1\right)_{l_2}}$$

$$\times\frac{(-k_3)_{l_1+l_2}}{\left(-\lambda-k_1-k_2+\frac{1}{4}d+1\right)_{l_3}}f^{\downarrow}_{\mathbf{k},\mathbf{l}}(Z_1,Z_2),$$

where $f^{\downarrow}_{\mathbf{k},\mathbf{l}}(Z_1,Z_2)\in\mathcal{P}(\mathfrak{p}_1^-\oplus\mathfrak{p}_2^-)_{(Z_1,Z_2)}\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$ *is given later in (2.2).*

Next we give the result on the Parseval–Plancherel type formula. For proof see [17]. We normalize $\mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}}$ such that its restriction to $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$,

$$\begin{array}{cccc}\mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}}\colon & \mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0} & \longrightarrow & \mathcal{H}_\lambda(D_1,\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))\\ & \cup & & \cup\\ \mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}}|_{\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)}\colon & \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+) & \longrightarrow & \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)\end{array}$$

is the identity on $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$.

Theorem 2.3 ([17, Corollary 6.7]). *For* $\lambda>1+2d$, $f\in\mathcal{H}_\lambda(D)$, *we have*

$$\|f\|^2_{\mathcal{H}_\lambda(D)}=\sum_{\mathbf{k}\in\mathbb{Z}^3_{++}}C(\lambda,\mathbf{k})\|\mathcal{F}^{\downarrow}_{\lambda,\mathbf{k}}f\|^2_{\mathcal{H}_\lambda(D_1,\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))},$$

$$C(\lambda,\mathbf{k})=\frac{\prod_{1\le i<j\le 3}\left(\lambda-\frac{d}{4}(i+j-2)\right)_{k_i+k_j}}{\prod_{1\le i<j\le 4}\left(\lambda-\frac{d}{4}(i+j-3)\right)_{k_i+k_j}}$$

$$=\frac{\left(\lambda+k_1+k_3-\frac{d}{4}\right)_{k_2-k_3}\left(\lambda+\max\{k_1,k_2+k_3\}-\frac{d}{2}\right)_{\min\{k_3,k_1-k_2\}}\left(\lambda+k_2-\frac{3}{4}d\right)_{k_3}}{(\lambda)_{k_1+k_2}\left(\lambda-\frac{d}{2}\right)_{\min\{k_1,k_2+k_3\}}(\lambda-d)_{k_3}},$$

where $k_4:=0$.

In the following we give the definition of $f^{\uparrow}_{\mathbf{k},\mathbf{l}}(X_2;Z_1)$, $f^{\downarrow}_{\mathbf{k},\mathbf{l}}(Z_1,Z_2)$. We identify $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-)\xrightarrow{\sim}\mathrm{Hom}_{\mathbb{C}}(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+),\mathbb{C})$ by $f(Y_2)\mapsto f\left(\frac{\partial}{\partial X_2}\right)\big|_{X_2=0}$, and take $\mathrm{K}_{\mathbf{k}}(X_2;W_2)\in(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)_{X_2}\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-)_{W_2})^{(K^\sigma)_0}$ corresponding to the identity. For $Z_1\in\mathfrak{p}_1^-$, since $(Z_1)^\sharp\in\mathfrak{p}_2^+$ is at most of rank 1, for $m\in\mathbb{Z}_{\ge0}$, $X_2\in\mathfrak{p}_2^+$, $Z_2\in\mathfrak{p}_2^-$ we have

$$((Z_1)^\sharp|(X_2)^\sharp)^m\in\mathcal{P}_{2m}(\mathfrak{p}_1^-)_{Z_1}\otimes\mathcal{P}_{(m,m,0)}(\mathfrak{p}_2^+)_{X_2},$$
$$((Z_1)^\sharp|Z_2)^m\in\mathcal{P}_{2m}(\mathfrak{p}_1^-)_{Z_1}\otimes\mathcal{P}_{(m,0,0)}(\mathfrak{p}_2^-)_{Z_2},$$

where $\mathcal{P}_m(\mathfrak{p}_1^-)$ is the set of homogeneous polynomials on $\mathfrak{p}_1^-$ of degree m. For $(l_1,l_2,l_3)\in(\mathbb{Z}_{\geq 0})^3$ let $(l_1,l_2,l_3)^\sharp := (l_2+l_3, l_1+l_3, l_1+l_2)$. Then from these we have

$$((Z_1)^\sharp|(X_2)^\sharp)^m \mathrm{K}_{\mathbf{k}}(X_2;W_2) \in \bigoplus_{\substack{|\mathbf{l}|=m,\ 0\leq l_3\\ 0\leq l_1\leq k_1-k_2\\ 0\leq l_2\leq k_2-k_3}} \mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^-)_{Z_1}\otimes\mathcal{P}_{\mathbf{k}+\mathbf{l}^\sharp}(\mathfrak{p}_2^+)_{X_2}\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-)_{W_2},$$

$$((Z_1)^\sharp|Z_2)^m \mathrm{K}_{\mathbf{k}}(X_2;Z_2) \in \bigoplus_{\substack{|\mathbf{l}|=m,\ 0\leq l_1\\ 0\leq l_2\leq k_1-k_2\\ 0\leq l_3\leq k_2-k_3}} \mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^-)_{Z_1}\otimes\mathcal{P}_{\mathbf{k}+\mathbf{l}}(\mathfrak{p}_2^-)_{Z_2}\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)_{X_2}.$$

Then by taking the orthogonal projection, we define $f^\uparrow_{\mathbf{k},\mathbf{l}}(X_2;Z_1) = f^\uparrow_{\mathbf{k},\mathbf{l}}(X_2;Z_1,W_2)$, $f^\downarrow_{\mathbf{k},\mathbf{l}}(Z_1,Z_2) = f^\downarrow_{\mathbf{k},\mathbf{l}}(Z_1,Z_2;X_2)$ by

$$\begin{aligned}
f^\uparrow_{\mathbf{k},\mathbf{l}}(X_2;Z_1,W_2) &:= \frac{(-1)^{|\mathbf{l}|}}{|\mathbf{l}|!}\operatorname{Proj}_{\mathbf{k}+\mathbf{l}^\sharp,X_2}\big(((Z_1)^\sharp|(X_2)^\sharp)^{|\mathbf{l}|}\mathrm{K}_{\mathbf{k}}(X_2;W_2)\big) \\
&\in \mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^-)_{Z_1}\otimes\mathcal{P}_{\mathbf{k}+\mathbf{l}^\sharp}(\mathfrak{p}_2^+)_{X_2}\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-)_{W_2} \\
&\simeq \mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^-)_{Z_1}\otimes\mathcal{P}_{\mathbf{k}+\mathbf{l}^\sharp}(\mathfrak{p}_2^+)_{X_2}\otimes\operatorname{Hom}_{\mathbb{C}}(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+),\mathbb{C}),
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
f^\downarrow_{\mathbf{k},\mathbf{l}}(Z_1,Z_2;X_2) &:= \frac{(-1)^{|\mathbf{l}|}}{|\mathbf{l}|!}\operatorname{Proj}_{\mathbf{k}+\mathbf{l},Z_2}\big(((Z_1)^\sharp|Z_2)^{|\mathbf{l}|}\mathrm{K}_{\mathbf{k}}(X_2;Z_2)\big)\det(Z_2)^{-|\mathbf{l}|} \\
&\in \mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^-)_{Z_1}\otimes\mathcal{P}_{\mathbf{k}-\mathbf{l}^\sharp}(\mathfrak{p}_2^-)_{Z_2}\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)_{X_2}.
\end{aligned}\tag{2.2}$$

In the following we consider the case $\mathbf{k}=(k,k,k)$. Then $\mathcal{P}_{(k,k,k)}(\mathfrak{p}_2^+)\simeq\mathbb{C}_{-2k}$ is 1-dimensional. We identify these via

$$\det(X_2)^k \mapsto c_k := \sqrt{\left(1+\frac{d}{2}\right)_k\left(1+\frac{d}{4}\right)_k(1)_k}.$$

Then we have

$$\mathrm{K}_{(k,k,k)}(X_2;W_2) = c_k^{-2}\det(X_2)^k\det(W_2)^k \in (\mathcal{P}_{(k,k,k)}(\mathfrak{p}_2^+)\otimes\mathcal{P}_{(k,k,k)}(\mathfrak{p}_2^-))^{(K^\sigma)_0},$$

and for $\mathbf{l}=(0,0,m)$ or $(m,0,0)$ we have

$$\begin{aligned}
f^\uparrow_{(k,k,k),(0,0,m)}(X_2;Z_1,W_2) &= \frac{c_k^{-2}}{m!}(-((Z_1)^\sharp|(X_2)^\sharp))^m\det(X_2)^k\det(W_2)^k, \\
f^\downarrow_{(k,k,k),(m,0,0)}(Z_1,Z_2;X_2) &= \frac{c_k^{-2}}{m!}\left(-\frac{((Z_1)^\sharp|Z_2)}{\det(Z_2)}\right)^m\det(X_2)^k\det(Z_2)^k.
\end{aligned}$$

Otherwise we have $f^\uparrow_{(k,k,k),\mathbf{l}}(X_2;Z_1,W_2)$, $f^\downarrow_{(k,k,k),\mathbf{l}}(Z_1,Z_2;X_2)=0$. Via the identification $\mathcal{P}_{(k,k,k)}(\mathfrak{p}_2^+)\simeq\mathbb{C}_{-2k}$ these become

$$\begin{aligned}
f^\uparrow_{(k,k,k),(0,0,m)}(X_2;Z_1) &= \frac{c_k^{-1}}{m!}(-((Z_1)^\sharp|(X_2)^\sharp))^m\det(X_2)^k, \\
f^\downarrow_{(k,k,k),(m,0,0)}(Z_1,Z_2) &= \frac{c_k^{-1}}{m!}\left(-\frac{((Z_1)^\sharp|Z_2)}{\det(Z_2)}\right)^m\det(Z_2)^k.
\end{aligned}$$

Hence we get the following.

Corollary 2.4 ([16, Theorem 8.6]). *For $\lambda > 1+2d$, $k \in \mathbb{Z}_{\geq 0}$, the intertwining operators are given by*

$$\mathcal{F}^{\uparrow}_{\lambda,k}\colon \mathcal{H}_{\lambda+2k}(D_1)_{(\widetilde{K}^\sigma)_0} \longrightarrow \mathcal{H}_\lambda(D)_{\widetilde{K}}|_{\mathfrak{g}^\sigma}, \qquad (\mathcal{F}^{\uparrow}_{\lambda,k}f)(X) = F^{\uparrow}_{\lambda,k}\left(X_2; \frac{\partial}{\partial X_1}\right) f(X_1),$$

$$\mathcal{F}^{\downarrow}_{\lambda,k}\colon \mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0} \longrightarrow \mathcal{H}_{\lambda+2k}(D_1), \qquad (\mathcal{F}^{\downarrow}_{\lambda,k}f)(X_1) = F^{\downarrow}_{\lambda,k}\left(\frac{\partial}{\partial X}\right) f(X)\bigg|_{X_2=0},$$

where

$$F^{\uparrow}_{\lambda,k}(X_2;Z_1) = c_k^{-1}\sum_{m=0}^{\infty} \frac{1}{\left(\lambda+2k-\frac{1}{4}d+1\right)_m m!}(-((Z_1)^\sharp|(X_2)^\sharp))^m \det(X_2)^k,$$

$$F^{\downarrow}_{\lambda,k}(Z) = c_k^{-1}\sum_{m=0}^{k} \frac{(-k)_m\left(-k-\frac{1}{4}d\right)_m}{\left(-\lambda-2k+\frac{3}{4}d+1\right)_m m!}\left(-\frac{((Z_1)^\sharp|Z_2)}{\det(Z_2)}\right)^m \det(Z_2)^k.$$

Their operator norms are given by

$$\|\mathcal{F}^{\uparrow}_{\lambda,k}\|_{op}^2 = \|\mathcal{F}^{\downarrow}_{\lambda,k}\|_{op}^{-2} = C(\lambda,(k,k,k)) = \frac{\left(\lambda+k-\frac{3}{4}d\right)_k}{(\lambda)_{2k}\left(\lambda-\frac{d}{2}\right)_k(\lambda-d)_k}.$$

§ 3. Jordan algebra structure on $\mathfrak{p}^+ = \mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}$

In this section, toward the proof of Theorem 2.2, we review the Jordan algebra structure on $\mathfrak{p}^+ = \mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}$. We recall that the pair $(V,\circ)$ of a vector space V and a bilinear product $\circ$ is called a Jordan algebra if $x\circ y = y\circ x$ and $x\circ((x\circ x)\circ y) = (x\circ x)\circ(x\circ y)$ hold for all $x,y\in V$. Then $\mathfrak{p}^+ = \mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}$ ($\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$) becomes a Jordan algebra by the product $X\circ Y = \frac{1}{2}(XY+YX)$. For a general Jordan algebra V, for $x,y\in V$, we define operators $P(x), P(x,y), B(x,y) \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$\begin{aligned} P(x)z &:= 2x\circ(x\circ z) - (x\circ x)\circ z,\\ P(x,y)z &:= (P(x+y)-P(x)-P(y))z,\\ B(x,y)z &:= z - P(x,z)y + P(x)P(y)z, \end{aligned} \tag{3.1}$$

and when V is simple, the *generic norm* $h\colon V\times V\to\mathbb{C}$ is defined by

$$h(x,y) := \mathrm{Det}(B(x,y)\colon V\to V)^{\mathrm{rank}\,\mathfrak{p}^+/2\dim\mathfrak{p}^+}.$$

Then for $\mathfrak{p}^+ = \mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}$ ($\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$), these are given by

$$\begin{aligned} P(X)Z &= XZX, \qquad P(X,Y)Z = XZY+YZX,\\ B(X,Y)Z &= (I-XY)Z(I-YX),\\ h(X,Y) &= 1-(X|Y)+(X^\sharp|Y^\sharp)-\det(X)\det(Y). \end{aligned}$$

On the other hand, for $\mathbb{F}=\mathbb{O}$, these formulas (except for $h(X,Y)$) do not hold. To include the $\mathbb{F}=\mathbb{O}$ case, we need the *Freudenthal product* given by

$$X\times Y:=(X+Y)^{\sharp}-X^{\sharp}-Y^{\sharp},$$

so that $X\times X=2X^{\sharp}$. Then for $\mathfrak{p}^+=\mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}$ ($\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$), we have

$$\begin{aligned} P(X)Z&=(X|Z)X-X^{\sharp}\times Z,\\ P(X,Y)Z&=(X|Z)Y+(Y|Z)X-(X\times Y)\times Z. \end{aligned}$$

Using $P(X)Z$, the differential of the action τ_λ on $\mathcal{H}_\lambda(D)$ (1.1) is given by, for $(A,B,C)\in\mathfrak{g}^{\mathbb{C}}=\mathfrak{p}^+\oplus\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{p}^-$,

$$(d\tau_\lambda(A,B,C)f)(X)=\left.\frac{d}{dt}\right|_{t=0}f(X-t(A+B.X-P(C)X))+\lambda(-d\chi(B)+(X|C))f(X).$$

Next we recall the decomposition $\mathfrak{p}^+:=\mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}=\mathrm{Alt}(3,\mathbb{F}')^{\mathbb{C}}j\oplus\mathrm{Herm}(3,\mathbb{F}')^{\mathbb{C}}=:\mathfrak{p}^+_1\oplus\mathfrak{p}^+_2$, where $\mathbb{F}=\mathbb{F}'\oplus\mathbb{F}'j$. We identify $\mathrm{Alt}(3,\mathbb{F}')^{\mathbb{C}}j\simeq(\mathbb{F}'^3)^{\mathbb{C}}$ by

$$(y_1,y_2,y_3)\mapsto\begin{pmatrix}0&y_3j&-y_2j\\-y_3j&0&y_1j\\y_2j&-y_1j&0\end{pmatrix},$$

and write $X=(X_1,X_2)=(\xi,x)\in\mathfrak{p}^+_1\oplus\mathfrak{p}^+_2=(\mathbb{F}'^3)^{\mathbb{C}}\oplus\mathrm{Herm}(3,\mathbb{F}')^{\mathbb{C}}$. Then under this identification we have

$$\begin{aligned} ((\xi,x)|(\zeta,z))&=2\,\mathrm{Re}_{\mathbb{F}'}(\xi{}^t\hat{\zeta})+\mathrm{Re}_{\mathbb{F}'}\mathrm{tr}(xz)=:2(\xi|\zeta)+(x|z),\\ (\xi,x)^{\sharp}&=(-\xi x,x^{\sharp}-{}^t\hat{\xi}\xi), \end{aligned}$$

(see Yokota [20]), and hence for $X=(\xi,x)$, $Z=(\zeta,z)$, we have

$$P((\xi,x))(\zeta,z)=(\xi{}^t\hat{\zeta}\xi+\zeta x^{\sharp}+(x|z)\xi-\xi xz,\ xzx+({}^t\hat{\xi}\xi)\times z+2(\xi|\zeta)x-x{}^t\hat{\xi}\zeta-{}^t\hat{\zeta}\xi x).$$

§ 4. Proof for symmetry breaking operators

In this section we give a proof of Theorem 2.2. To do this, we recall the *F-method* developed by Kobayashi–Pevzner [12]. We take a basis $\{E_\alpha\}$ of $\mathfrak{p}^+$, and let $\{Z_\alpha\}$ be the corresponding coordinate of $\mathfrak{p}^-$. Using this, we define a vector-valued differential operator $\mathcal{B}_\lambda\colon\mathcal{P}(\mathfrak{p}^-)\to\mathcal{P}(\mathfrak{p}^-)\otimes\mathfrak{p}^+$ by

$$(\mathcal{B}_\lambda f)(Z):=\frac{1}{2}\sum_{\alpha\beta}P(E_\alpha,E_\beta)Z\frac{\partial^2 f}{\partial Z_\alpha\partial Z_\beta}(Z)+\lambda\sum_\alpha E_\alpha\frac{\partial f}{\partial Z_\alpha}(Z).$$

This is called the *Bessel operator* (see [4, Section XV.2]), and this is independent of the choice of $\{E_\alpha\}$. Then we can show that, for $C\in\mathfrak{p}^-$, $-(\mathcal{B}_\lambda|C)$ coincides with the "algebraic Fourier transform" of

$$d\tau_\mu(0,0,C)f(X)=\left.\frac{d}{dt}\right|_{t=0} f(X+tP(X)C)+\mu(X|C)f(X)$$

for $\mu=2(1+d)-\lambda$, where we regard $(0,0,C)\in\mathfrak{g}^{\mathbb{C}}=\mathfrak{p}^+\oplus\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{p}^-$. Also let $(\mathcal{B}_\lambda)_1\colon\mathcal{P}(\mathfrak{p}^-)\to\mathcal{P}(\mathfrak{p}^-)\otimes\mathfrak{p}_1^+$ be the orthogonal projection of $\mathcal{B}_\lambda$ onto $\mathfrak{p}_1^+$. In terms of bases $\{\epsilon_\alpha\}\subset\mathfrak{p}_1^-$, $\{e_\alpha\}\subset\mathfrak{p}_2^-$, with the dual coordinates $\{2\zeta_\alpha\}$ of $\mathfrak{p}_1^+$ and $\{z_\alpha\}$ of $\mathfrak{p}_2^+$ with respect to the pairing $(X|Y)=2(\xi|\eta)+(x|y)$, $(\mathcal{B}_\lambda)_1$ is rewritten as

$$\begin{aligned}(\mathcal{B}_\lambda)_1&=\frac{1}{2}\sum_{\alpha\beta}(P(E_\alpha,E_\beta)Z)_1\frac{\partial^2}{\partial Z_\alpha\partial Z_\beta}+\lambda\sum_\alpha(E_\alpha)_1\frac{\partial}{\partial Z_\alpha}\\&=\frac{1}{8}\sum_{\alpha\beta}(\epsilon_\alpha{}^t\hat{\zeta}\epsilon_\beta+\epsilon_\beta{}^t\hat{\zeta}\epsilon_\alpha)\frac{\partial^2}{\partial\zeta_\alpha\partial\zeta_\beta}+\frac{1}{2}\sum_{\alpha\beta}\zeta(e_\alpha\times e_\beta)\frac{\partial^2}{\partial z_\alpha\partial z_\beta}\\&\quad+\frac{1}{2}\sum_{\alpha\beta}((e_\beta|z)\epsilon_\alpha-\epsilon_\alpha e_\beta z)\frac{\partial^2}{\partial\zeta_\alpha\partial z_\beta}+\frac{\lambda}{2}\sum_\alpha\epsilon_\alpha\frac{\partial}{\partial\zeta_\alpha}.\end{aligned}$$

Then the following holds.

Theorem 4.1 (F-method, Kobayashi–Pevzner [12])**.** *We have the isomorphisms*

$$\begin{aligned}&\operatorname{Hom}_{(\widetilde{G}^\sigma)_0}\big(\mathcal{O}_\lambda(D)|_{(\widetilde{G}^\sigma)_0},\mathcal{O}_\lambda(D_1,\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))\big)\\&\simeq\operatorname{Hom}_{(\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{p}^-)^\sigma}\big(\chi^\lambda\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-),\operatorname{ind}_{\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{p}^-}^{\mathfrak{g}^{\mathbb{C}}}(\chi^\lambda)\big)\\&\simeq\big\{F(Z)\in(\mathcal{P}(\mathfrak{p}^-)_Z\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))^{(K^\sigma)_0}\;\big|\;((\mathcal{B}_\lambda)_{1,Z}\otimes I_{\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)})F(Z)=0\big\}.\end{aligned}$$

Here, the isomorphism of the left hand side and the right hand side is given by taking the symbol of the differential operator.

Therefore, to prove Theorem 2.2, it suffices to verify

$$F^{\downarrow}_{\lambda,\mathbf{k}}(Z)\in(\mathcal{P}(\mathfrak{p}^-)_Z\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))^{(K^\sigma)_0},\tag{4.1}$$

$$((\mathcal{B}_\lambda)_{1,Z}\otimes I_{\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)})F^{\downarrow}_{\lambda,\mathbf{k}}(Z)=0.\tag{4.2}$$

Here, (4.1) is clear from $f^{\downarrow}_{\mathbf{k},\mathbf{l}}(Z_1,Z_2)\in(\mathcal{P}(\mathfrak{p}^-)_Z\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+))^{(K^\sigma)_0}$. Next we verify (4.2). In the following we write $Z=(Z_1,Z_2)=(\zeta,z)\in\mathfrak{p}_1^-\oplus\mathfrak{p}_2^-$. To verify (4.2), first we extend the definition of $F^{\downarrow}_{\lambda,\mathbf{k}}(Z)$ from $\mathbf{k}\in\mathbb{Z}^3_{++}$ to $\mathbf{k}\in\{\mathbf{k}\in\mathbb{C}^3\mid k_1-k_2,k_2-k_3\in\mathbb{Z}_{\geq0},\ k_3\in\mathbb{C}\}$. We recall that $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^\pm)=\mathcal{P}_{(k_1-k_3,k_2-k_3,0)}(\mathfrak{p}_2^\pm)\det(z)^{k_3}$ holds for $\mathbf{k}\in\mathbb{Z}^3_{++}$. Then this formula is available for $k_3\in\mathbb{C}$. For $k_1',k_2'\in\mathbb{Z}_{\geq0}$ with $k_1'\geq k_2'$, we fix a non-zero polynomial $\mathrm{K}_{(k_1',k_2',0)}(z)\in\mathcal{P}_{(k_1',k_2',0)}(\mathfrak{p}_2^-)$. Then for $k_3\in\mathbb{C}$ let $\mathrm{K}_{(k_1,k_2,k_3)}(z):=$

$\mathrm{K}_{(k_1-k_3,k_2-k_3,0)}(z)\det(z)^{k_3}\in\mathcal{P}_{(k_1,k_2,k_3)}(\mathfrak{p}_2^-)$ (We note that in the original definition we have $\mathrm{K}_{\mathbf{k}}(z)\in(\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)\otimes\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-)_z)^{(K^\sigma)_0}$, but the $(K^\sigma)_0$-invariance is needed only for the proof of (4.1), and for the proof of (4.2) we may forget the $(K^\sigma)_0$-invariance and may suppose $\mathrm{K}_{\mathbf{k}}(z)$ is a scalar-valued polynomial). Using this, we define the function $f^{\downarrow}_{\mathbf{k},\mathbf{l}}(Z_1,Z_2)=f^{\downarrow}_{\mathbf{k},\mathbf{l}}(\zeta,z)$ by

$$\begin{aligned}f^{\downarrow}_{\mathbf{k},\mathbf{l}}(\zeta,z)&:=\frac{(-1)^{|\mathbf{l}|}}{|\mathbf{l}|!}\operatorname{Proj}_{\mathbf{k}+\mathbf{l},Z_2}\bigl(((Z_1)^\sharp|Z_2)^{|\mathbf{l}|}\mathrm{K}_{\mathbf{k}}(Z_2)\bigr)\det(Z_2)^{-|\mathbf{l}|}\\&=\frac{1}{|\mathbf{l}|!}\operatorname{Proj}_{\mathbf{k}+\mathbf{l},z}\bigl(({}^t\hat{\zeta}\zeta|z)^{|\mathbf{l}|}\mathrm{K}_{\mathbf{k}}(z)\bigr)\det(z)^{-|\mathbf{l}|}\\&=\frac{1}{|\mathbf{l}|!}\operatorname{Proj}_{(k_1-k_3,k_2-k_3,0)+\mathbf{l},z}\bigl(({}^t\hat{\zeta}\zeta|z)^{|\mathbf{l}|}\mathrm{K}_{(k_1-k_3,k_2-k_3,0)}(z)\bigr)\det(z)^{k_3-|\mathbf{l}|}\\&\in\mathcal{P}_{2|\mathbf{l}|}(\mathfrak{p}_1^-)_\zeta\otimes\mathcal{P}_{\mathbf{k}-\mathbf{l}^\sharp}(\mathfrak{p}_2^-)_z,\end{aligned}$$

and for $\lambda\in\mathbb{C}$, $k_1-k_2,k_2-k_3\in\mathbb{Z}_{\ge0}$, $k_3\in\mathbb{C}$, let

$$\begin{aligned}F^{\downarrow}_{\lambda,\mathbf{k}}(Z)=\sum_{\substack{0\le l_1\\0\le l_2\le k_1-k_2\\0\le l_3\le k_2-k_3}}&\frac{\bigl(-k_1-\frac12 d\bigr)_{l_2+l_3}}{\bigl(-\lambda-k_2-k_3+\frac34 d+1\bigr)_{l_1}}\frac{\bigl(-k_2-\frac14 d\bigr)_{l_1+l_3}}{\bigl(-\lambda-k_1-k_3+\frac12 d+1\bigr)_{l_2}}\\&\times\frac{(-k_3)_{l_1+l_2}}{\bigl(-\lambda-k_1-k_2+\frac14 d+1\bigr)_{l_3}}f^{\downarrow}_{\mathbf{k},\mathbf{l}}(\zeta,z).\end{aligned}$$

If $k_3\in\mathbb{Z}_{\ge0}$, then only $l_1+l_2\le k_3$ terms remain, and this becomes a finite sum. Especially if $\mathrm{K}_{(k_1',k_2',0)}(z)\in(\mathcal{P}_{(k_1',k_2',0)}(\mathfrak{p}_2^+)\otimes\mathcal{P}_{(k_1',k_2',0)}(\mathfrak{p}_2^-)_z)^{(K^\sigma)_0}$, then $F^{\downarrow}_{\lambda,\mathbf{k}}(Z)$ coincides with the original one up to constant multiple.

Next we give the integral expression of $F^{\downarrow}_{\lambda,\mathbf{k}}(Z)$. Let $\mathfrak{n}^-\subset\mathfrak{p}^-$, $\mathfrak{n}_2^-\subset\mathfrak{p}_2^-$ be the Euclidean real forms and $\Omega\subset\mathfrak{n}^-$, $\Omega_2\subset\mathfrak{n}_2^-$ be the symmetric cones consisting of positive-definite matrices,

$$\begin{aligned}\mathfrak{p}^-&:=\mathrm{Herm}(3,\mathbb{F})^{\mathbb{C}}\supset\mathfrak{n}^-:=\mathrm{Herm}(3,\mathbb{F})\supset\ \Omega:=\mathrm{Herm}_+(3,\mathbb{F}),\\\mathfrak{p}_2^-&:=\mathrm{Herm}(3,\mathbb{F}')^{\mathbb{C}}\supset\mathfrak{n}_2^-:=\mathrm{Herm}(3,\mathbb{F}')\supset\Omega_2:=\mathrm{Herm}_+(3,\mathbb{F}').\end{aligned}$$

We recall that $d:=\dim_{\mathbb{R}}\mathbb{F}=2\dim_{\mathbb{R}}\mathbb{F}'$. For $\mu\in\mathbb{C}$, $\mathbf{k}\in\mathbb{C}^3$, let

$$\Gamma_d(\mu+\mathbf{k}):=(2\pi)^{\frac34 d}\prod_{i=1}^3\Gamma\biggl(\mu+k_i-\frac{d}{4}(i-1)\biggr).$$

Then the Laplace transform on Ω_2 satisfies the following.

Theorem 4.2 (Gindikin, [4, Lemma XI.2.3, Section IX.3]).
(1) For $\operatorname{Re}\mu>\frac{d}{2}$, $\mathbf{k}\in\mathbb{Z}^3_{++}$, $f(z)\in\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-)$, $w\in\Omega_2+\sqrt{-1}\mathfrak{n}_2^-$,

$$\int_{\Omega_2}e^{-(z|w)}f(z)\det(z)^{\mu-(1+\frac{d}{2})}dz=\Gamma_d(\mu+\mathbf{k})f(w^{-1})\det(w)^{-\mu}.$$

(2) For $\operatorname{Re}\mu > 1+d$, $\mathbf{k}\in\mathbb{Z}^3_{++}$, $f(z)\in\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^-)$, $a,z\in\Omega_2$,

$$\frac{\Gamma_d(\mu+\mathbf{k})}{(2\pi\sqrt{-1})^{3(1+\frac{d}{2})}}\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{(z|w)}f(w^{-1})\det(w)^{-\mu}dw = f(z)\det(z)^{\mu-(1+\frac{d}{2})}.$$

Then by applying the Laplace and the inverse Laplace transforms to

$$e^{-((Z_1)^\sharp|Z_2)}\mathrm{K}_{\mathbf{k}}(Z_2) = e^{({}^t\hat{\zeta}\zeta|z)}\mathrm{K}_{\mathbf{k}}(z) = \sum_{\substack{0\le l_1\\ 0\le l_2\le k_1-k_2\\ 0\le l_3\le k_2-k_3}} f^{\downarrow}_{\mathbf{k},\mathbf{l}}(\zeta,z)\det(z)^{|\mathbf{l}|}$$

suitably, we can show the following.

Proposition 4.3. *For* $\operatorname{Re}k_1 < -\frac{d}{2}$, $\operatorname{Re}(\lambda+k_1+k_2) < -\frac{d}{4}$, $Z=(\zeta,z)\in\Omega$, $a\in\Omega_2$, *we have*

$$F^{\downarrow}_{\lambda,\mathbf{k}}(Z) = \frac{\Gamma_d\bigl(-\lambda+\frac{3}{4}d+1-(k_2+k_3,k_1+k_3,k_1+k_2)\bigr)}{\Gamma_d((-k_3,-k_2,-k_1))}\int_{\Omega_2} e^{-(y|z)}\det(y)^{2\lambda+|\mathbf{k}|-d-1}$$
$$\times\left(\frac{1}{(2\pi\sqrt{-1})^{3(1+\frac{d}{2})}}\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{(x|y^\sharp)}\det(x)^{\lambda+|\mathbf{k}|-\frac{3}{4}d-1}e^{({}^t\hat{\zeta}\zeta|x^{-1})}\mathrm{K}_{\mathbf{k}}(x^{-1})\,dx\right)dy.$$

We put $\det(x)^{\lambda+|\mathbf{k}|-\frac{3}{4}d-1}\mathrm{K}_{\mathbf{k}}(x^{-1}) =: f(x)$. To prove (4.2), it suffices to verify

$$(\mathcal{B}_\lambda)_1\int_{\Omega_2} e^{-(y|z)}\det(y)^{2\lambda+|\mathbf{k}|-d-1}\left(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{(x|y^\sharp)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}f(x)\,dx\right)dy = 0.$$

First we consider $(\mathcal{B}_\lambda)_1 e^{-(y|z)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}$. In the following, we use the convention $\bigl([\frac{\partial}{\partial z}f]g\bigr)h = \frac{\partial f}{\partial z}gh + f\frac{\partial g}{\partial z}h$. Then we have

$$\begin{aligned}
&(\mathcal{B}_\lambda)_1 e^{-(y|z)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}\\
&=\Biggl[\frac{1}{8}\sum_{\alpha\beta}(\epsilon_\alpha{}^t\hat{\zeta}\epsilon_\beta+\epsilon_\beta{}^t\hat{\zeta}\epsilon_\alpha)\frac{\partial^2}{\partial\zeta_\alpha\partial\zeta_\beta}+\frac{1}{2}\sum_{\alpha\beta}\zeta(e_\alpha\times e_\beta)\frac{\partial^2}{\partial z_\alpha\partial z_\beta}\\
&\quad+\frac{1}{2}\sum_{\alpha\beta}((e_\beta|z)\epsilon_\alpha-\epsilon_\alpha e_\beta z)\frac{\partial^2}{\partial\zeta_\alpha\partial z_\beta}+\frac{\lambda}{2}\sum_\alpha\epsilon_\alpha\frac{\partial}{\partial\zeta_\alpha}\Biggr]e^{-(y|z)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}\\
&=\left(\zeta x^{-1}{}^t\hat{\zeta}\zeta x^{-1}+\left(1-\frac{d}{4}\right)\zeta x^{-1}+\zeta y^\sharp-((y|z)\zeta x^{-1}-\zeta x^{-1}yz)+\lambda\zeta x^{-1}\right)e^{-(y|z)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}\\
&=\left[\zeta x^{-1}{}^t\hat{\zeta}\zeta x^{-1}+\zeta y^\sharp+\left(\lambda-\frac{d}{4}+1\right)\zeta x^{-1}+\sum_\alpha((y|e_\alpha)\zeta x^{-1}-\zeta x^{-1}ye_\alpha)\frac{\partial}{\partial y_\alpha}\right]\\
&\qquad\times e^{-(y|z)}e^{({}^t\hat{\zeta}\zeta|x^{-1})},
\end{aligned}$$

and hence we get

$$\begin{aligned}
&(\mathcal{B}_\lambda)_1 \int_{\Omega_2} e^{-(y|z)} \det(y)^{2\lambda+|\mathbf{k}|-d-1} \biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{(x|y^\sharp)} e^{({}^t\hat{\zeta}\zeta|x^{-1})} f(x)\,dx \biggr) dy \\
&= \int_{\Omega_2} \biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} \biggl(\biggl[\zeta x^{-1}{}^t\hat{\zeta}\zeta x^{-1} + \zeta y^\sharp + \Bigl(\lambda - \frac{d}{4} + 1\Bigr)\zeta x^{-1} + \sum_\alpha ((y|e_\alpha)\zeta x^{-1} \\
&\quad - \zeta x^{-1} y e_\alpha) \frac{\partial}{\partial y_\alpha} \biggr] e^{-(y|z)} \biggr) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} e^{({}^t\hat{\zeta}\zeta|x^{-1})} f(x)\,dx \biggr) dy \\
&= \int_{\Omega_2} \biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{-(y|z)} \biggl[\zeta x^{-1}{}^t\hat{\zeta}\zeta x^{-1} + \zeta y^\sharp + \Bigl(\lambda - \frac{d}{4} + 1\Bigr)\zeta x^{-1} \\
&\quad - \sum_\alpha \frac{\partial}{\partial y_\alpha} ((y|e_\alpha)\zeta x^{-1} - \zeta x^{-1} y e_\alpha) \biggr] \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} e^{({}^t\hat{\zeta}\zeta|x^{-1})} f(x)\,dx \biggr) dy.
\end{aligned}$$

Next we consider $\sum_\alpha \frac{\partial}{\partial y_\alpha}((y|e_\alpha)\zeta x^{-1} - \zeta x^{-1} y e_\alpha) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)}$.

$$\begin{aligned}
&\sum_\alpha \frac{\partial}{\partial y_\alpha} ((y|e_\alpha)\zeta x^{-1} - \zeta x^{-1} y e_\alpha) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} \\
&= \biggl(3\Bigl(1 + \frac{d}{2}\Bigr)\zeta x^{-1} - \Bigl(1 + \frac{d}{2}\Bigr)\zeta x^{-1} \biggr) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} \\
&\quad + (2\lambda + |\mathbf{k}| - d - 1)\bigl((y|y^{-1})\zeta x^{-1} - \zeta x^{-1} y y^{-1}\bigr) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} \\
&\quad + \bigl((y|x \times y)\zeta x^{-1} - \zeta x^{-1} y(x \times y)\bigr) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} \\
&= (2 + d)\zeta x^{-1} \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} \\
&\quad + (2\lambda + |\mathbf{k}| - d - 1)(3\zeta x^{-1} - \zeta x^{-1}\bigr) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} \\
&\quad + \bigl(2(x|y^\sharp)\zeta x^{-1} - \zeta x^{-1}((x|y^\sharp)I - x y^\sharp)\bigr) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} \\
&= \bigl((4\lambda + 2|\mathbf{k}| - d)\zeta x^{-1} + (x|y^\sharp)\zeta x^{-1} + \zeta y^\sharp\bigr) \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)},
\end{aligned}$$

where at the 2nd equality we have used

$$(y|x \times y) = (x|y \times y) = 2(x|y^\sharp), \qquad x y^\sharp + y(x \times y) = (x|y^\sharp)I,$$

the latter of which follows from the polarization of $yy^\sharp = \det(y)I$. Hence we get

$$\begin{aligned}
&(\mathcal{B}_\lambda)_1 \int_{\Omega_2} e^{-(y|z)} \det(y)^{2\lambda+|\mathbf{k}|-d-1} \biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{(x|y^\sharp)} e^{({}^t\hat{\zeta}\zeta|x^{-1})} f(x)\,dx \biggr) dy \\
&= \int_{\Omega_2} e^{-(y|z)} \biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} \biggl[\zeta x^{-1}{}^t\hat{\zeta}\zeta x^{-1} + \zeta y^\sharp + \Bigl(\lambda - \frac{d}{4} + 1\Bigr)\zeta x^{-1} \\
&\quad - \sum_\alpha \frac{\partial}{\partial y_\alpha} ((y|e_\alpha)\zeta x^{-1} - \zeta x^{-1} y e_\alpha) \biggr] \det(y)^{2\lambda+|\mathbf{k}|-d-1} e^{(x|y^\sharp)} e^{({}^t\hat{\zeta}\zeta|x^{-1})} f(x)\,dx \biggr) dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega_2} e^{-(y|z)} \det(y)^{2\lambda+|\mathbf{k}|-d-1}\biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-}\biggl(\zeta x^{-1}{}^t\hat{\zeta}\zeta x^{-1}+\Bigl(\lambda-\frac{d}{4}+1\Bigr)\zeta x^{-1}\\
&\quad -(4\lambda+2|\mathbf{k}|-d)\zeta x^{-1}-(x|y^\sharp)\zeta x^{-1}\biggr)e^{(x|y^\sharp)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}f(x)\,dx\biggr)dy\\
&= \int_{\Omega_2} e^{-(y|z)} \det(y)^{2\lambda+|\mathbf{k}|-d-1}\biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-}\biggl(\biggl[\Bigl(-3\lambda-2|\mathbf{k}|+\frac{3}{4}d+1\Bigr)\zeta x^{-1}\\
&\quad -\sum_\alpha (x|e_\alpha)\zeta x^{-1}\frac{\partial}{\partial x_\alpha}\biggr]e^{(x|y^\sharp)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}\biggr)f(x)\,dx\biggr)dy\\
&= \int_{\Omega_2} e^{-(y|z)} \det(y)^{2\lambda+|\mathbf{k}|-d-1}\biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{(x|y^\sharp)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}\\
&\quad \times\biggl[\Bigl(-3\lambda-2|\mathbf{k}|+\frac{3}{4}d+1\Bigr)\zeta x^{-1}+\sum_\alpha\frac{\partial}{\partial x_\alpha}(x|e_\alpha)\zeta x^{-1}\biggr]f(x)\,dx\biggr)dy\\
&= \int_{\Omega_2} e^{-(y|z)} \det(y)^{2\lambda+|\mathbf{k}|-d-1}\biggl(\int_{a+\sqrt{-1}\mathfrak{n}_2^-} e^{(x|y^\sharp)}e^{({}^t\hat{\zeta}\zeta|x^{-1})}\\
&\quad \times\biggl[\Bigl(-3\lambda-2|\mathbf{k}|+\frac{9}{4}d+3\Bigr)\zeta x^{-1}+\zeta x^{-1}\sum_\alpha (x|e_\alpha)\frac{\partial}{\partial x_\alpha}\biggr]f(x)\,dx\biggr)dy=0,
\end{aligned}
$$

where the last equality follows from that $f(x)=\det(x)^{\lambda+|\mathbf{k}|-\frac{3}{4}d-1}\mathrm{K}_{\mathbf{k}}(x^{-1})$ is homogeneous of degree $3\lambda+2|\mathbf{k}|-\frac{9}{4}d-3$. Hence we have verified (4.2) for $\operatorname{Re}k_1<-\frac{d}{2}$, $\operatorname{Re}(\lambda+k_1+k_2)<-\frac{d}{4}$. Then by analytic continuation, this holds for all $\lambda,k_3\in\mathbb{C}$ (except for poles), and especially for $k_3\in\mathbb{Z}_{\geq 0}$. By Theorem 4.1, this proves Theorem 2.2.

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