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Self-closeness numbers of rational mapping spaces

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Abstract
For a closed connected oriented manifold M of dimension $2n$, it was proved by Møller and Raussen that the components of the mapping space from M to S^{2n} have exactly two different rational homotopy types. However, since this result was proved by the algebraic models for the components, it is unclear whether other homotopy invariants distinguish their rational homotopy types or not. The self-closeness number of a connected CW complex is the least integer k such that any of its self-maps inducing an isomorphism in π_* for $* \leq k$ is a homotopy equivalence, and there is no result on the components of mapping spaces so far. For a rational Poincaré complex X of dimension $2n$ with finite π_1, we completely determine the self-closeness numbers of the rationalized components of the mapping space from X to S^{2n} by using their Brown–Szczarba models. As a corollary, we show that the self-closeness number does distinguish the rational homotopy types of the components. Since a closed connected oriented manifold is a rational Poincaré complex, our result partially generalizes that of Møller and Raussen.

Keywords Self-closeness number · Mapping space · Rational homotopy theory · Brown–Szczarba model

Mathematics Subject Classification 55P10 · 55P62 · 55P15

1 Introduction

Given two spaces X and Y, we can associate the mapping space $\mathrm{Map}(X, Y)$. It is a classical problem in algebraic topology to classify the homotopy types of the path-components of $\mathrm{Map}(X, Y)$ for given X and Y. This problem dates back, at least to the work of Whitehead [25] in 1946, and there are many classification results for specific

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X and Y. For instance, [5, 8, 10, 25] studied the problem for $X = Y = S^n$. More recently, the problem was intensively studied in [9, 11, 13, 14, 20, 21, 24] when Y is the classifying space of a Lie group G, as the components are the classifying spaces of gauge groups by [4]. There are also results in [3, 26] for other X and Y. See [19] and its references for more details.

Let M be a closed connected oriented manifold of dimension n. We recall a classification result on the mapping space $\mathrm{Map}(M, S^n)$. By the Hopf degree theorem, the mapping degree gives a one-to-one correspondence between the path-components of $\mathrm{Map}(M, S^n)$ and $\mathbb{Z}$. Let $\mathrm{Map}(M, S^n; k)$ denote the path-component of degree k. In [6], for M with vanishing first Betti number, Hansen classified the homotopy types of the components of $\mathrm{Map}(M, S^n)$ such that $\mathrm{Map}(M, S^n, k)$ and $\mathrm{Map}(M, S^n; l)$ are homotopy equivalent if and only if one of the following conditions hold:

(1) $|k| = |l|$ for n even;
(2) the parity of k and l are equal for n odd but not equals to 1,3,7;
(3) any k and l for $n = 1, 3, 7$.

Now we consider the rational homotopy types of $\mathrm{Map}(M, S^n; k)$. Since the rationalization of an odd sphere is an H-space, all components $\mathrm{Map}(M, S^n; k)$ have the same rational homotopy type for n odd. Since the degree k self-map of S^n is a rational homotopy equivalence for $k \neq 0$, all components $\mathrm{Map}(M, S^n; k)$ but $\mathrm{Map}(M, S^n; 0)$ have the same rational homotopy type for n even. Then it remains to show whether or not $\mathrm{Map}(M, S^n; 0)$ and $\mathrm{Map}(M, S^n; 1)$ have the same rational homotopy type. Møller and Raussen [15] proved that they are not of the same rational homotopy type by considering their algebraic models, instead of specific homotopy invariants such as homology.

The *self-closeness number* of a connected CW complex X, denoted by $N\mathcal{E}(X)$, is defined to be the least integer k such that every self-map of X inducing an isomorphism in the homotopy groups of dimension $\leq k$ is a homotopy equivalence. The self-closeness number was introduced by Choi and Lee [2] in 2015, and there are several results on it [2, 12, 16–18, 23, 27]. However, there are few explicit computations. Li [12] and Oda and Yamaguchi [18] determined the self-closeness numbers for some special homogeneous spaces. Later, the author [23] computed those for some non-simply-connected finite complexes, which covers the previous results on non-simply-connected spaces. So far, all explicit computations are only for finite complexes, and there is no result on the components of mapping spaces. In this paper, we consider:

Problem 1.1 Does the self-closeness number distinguish the rational homotopy types of $\mathrm{Map}(M, S^n; 0)$ and $\mathrm{Map}(M, S^n; 1)$ for n even?

A space X is called a *rational Poincaré complex* of dimension n if it is a finite complex of dimension n such that $H^n(X; \mathbb{Z}) \cong \mathbb{Z}$ and the map

$$H^i(X; \mathbb{Z}) \to H^{n-i}(X; \mathbb{Z}), \quad x \mapsto w \frown x$$

is an isomorphism after tensoring with $\mathbb{Q}$, where w is a generator of $H^n(X; \mathbb{Z})$. Clearly, a closed connected oriented manifold of dimension n is a rational Poincaré complex of dimension n. Moreover, we can consider Problem 1.1 for a rational Poincaré complex of dimension n, instead of a manifold M.

To state the main result, we set notations. For a graded algebra A, let QA^i denote its module of indecomposables of degree i. Let X be a rational Poincaré complex of dimension $2n$. We say that X is *primitive* if $QH^i(X;\mathbb{Q}) = H^i(X;\mathbb{Q})$ for $i < 2n$, as $H_i(X;\mathbb{Q})$ consists of primitive homology classes for $i < 2n$ with respect to the comultiplication of $H_*(X;\mathbb{Q})$ induced by the diagonal map of X. We define $d(X)$ to be the least integer d such that $H^d(X;\mathbb{Q}) \neq 0$ and $d \geq n$. Let $Y_{(0)}$ denote the rationalization of a nilpotent space Y in the sense of [7]. Now we state the main theorem.

Theorem 1.2 *Let X be a rational Poincaré complex of dimension $2n$ with finite π_1. Then we have*

$$N\mathcal{E}(\mathrm{Map}(X, S^{2n}; 1)_{(0)}) = 4n - 1$$

and

$$N\mathcal{E}(\mathrm{Map}(X, S^{2n}; 0)_{(0)}) = \begin{cases} 2n & X \text{ is primitive}, \\ d(X) & X \text{ is not primitive}. \end{cases}$$

We immediately get the following corollary, which gives a positive answer to Problem 1.1.

Corollary 1.3 *Let X be a rational Poincaré complex of dimension $2n$ with finite π_1. The self-closeness number distinguishes the rational homotopy types of* $\mathrm{Map}(X, S^{2n}; k)$ *for* $k = 0, 1$.

This paper is organized as follows. In Sect. 2, we introduce the self-closeness number of a minimal Sullivan algebra, and prove that the self-closeness number of a simply-connected rational space coincides with that of its minimal model. In Sect. 3, we recall the Brown–Szczarba models for $\mathrm{Map}(X, S^{2n})$ and its components, where X is a rational Poincaré complex of dimension $2n$ with finite π_1. In Sects. 4 and 5, we compute the self-closeness numbers of the minimal models for $\mathrm{Map}(X, S^{2n}; k)_{(0)}$ for $k = 0, 1$.

2 Algebraic self-closeness number

In this section, we define the self-closeness number of a minimal Sullivan algebra, and prove that it coincides with the self-closeness number of a corresponding rational space. Hereafter, we will assume that all algebras and vector spaces will be over the field $\mathbb{Q}$.

We say that a dga A is an algebraic model for a space X if there is a zig-zag of quasi-isomorphisms

$$A \xrightarrow{\simeq} A_1 \xleftarrow{\simeq} A_2 \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} A_n \xrightarrow{\simeq} A_{\mathrm{PL}}(X)$$

where $A_{\mathrm{PL}}(X)$ denotes the dga of piecewise linear forms on X. If X is a nilpotent space, then the rationalization $X \to X_{(0)}$ is a rational homotopy equivalence, implying that a dga A is an algebraic model for X if and only if so is for $X_{(0)}$. We recall a minimal model for a space. Let V be a positively graded vector space, and let ΛV denote the free commutative graded algebra generated by V. We say that a dga is a minimal Sullivan algebra if it is of the form $(\Lambda V, d)$ such that

$$d(V_n) \subset \Lambda(V_{\leq n-1})$$

for all $n \geq 1$, where V_n and $V_{\leq n-1}$ are the degree n and degree $\leq n-1$ parts of V, respectively. If an algebraic model for a space X is a minimal Sullivan algebra, then we call it a minimal model for X. If $(\Lambda V, d)$ is a minimal model for a space X, then a zig-zag of quasi-isomorphisms patch together to yield a quasi-isomorphism

$$(\Lambda V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X).$$

We state the fundamental theorem of rational homotopy theory.

Theorem 2.1 *For every simply-connected rational space X of finite rational type, there is a minimal model*

$$\alpha\colon (\Lambda V, d) \xrightarrow{\simeq} A_{\mathrm{PL}}(X)$$

satisfying the following properties.

(1) $(\Lambda V, d)$ *is unique, up to isomorphism.*
(2) *The quasi-isomorphism* α *is natural, up to homotopy.*
(3) *There is a natural isomorphism*

$$V \cong \mathrm{Hom}(\pi_*(X), \mathbb{Q}).$$

If X is a nilpotent space of finite type such that $X_{(0)}$ is simply-connected, then we often say that a minimal model for $X_{(0)}$ is a minimal model for X. For a map $f\colon \Lambda V \to \Lambda W$ between free commutative graded algebras, we define its linear part $f_0\colon V \to W$ by the composite

$$V \xrightarrow{\text{incl}} \Lambda V \xrightarrow{f} \Lambda W \xrightarrow{\text{proj}} \Lambda W/(\Lambda^+ W)^2 \cong W$$

where $\Lambda^+ W$ denotes the ideal of ΛW generated by elements of positive degrees. We define the self-closeness number of a minimal Sullivan algebra.

Definition 2.2 The self-closeness number of a minimal Sullivan algebra $(\Lambda V, d)$, denoted by $N\mathcal{E}(\Lambda V, d)$, is the least integer n such that any dga map $(\Lambda V, d) \to (\Lambda V, d)$ is an isomorphism whenever its linear part is an isomorphism in degree $\leq n$.

We will use the following lemma to compute the self-closeness number of a rational space.

Proposition 2.3 *Let X be a simply-connected rational CW complex of finite rational type, and let $(\Lambda V, d)$ be its minimal model. Then we have*

$$N\mathcal{E}(X) = N\mathcal{E}(\Lambda V, d).$$

Proof Let $f\colon X \to X$ be a map, and let $g\colon (\Lambda V, d) \to (\Lambda V, d)$ be a dga map corresponding to a map f, which exists, up to homotopy, by Theorem 2.1. Since there is a natural isomorphism $V \cong \mathrm{Hom}(\pi_*(X), \mathbb{Q})$ as in Theorem 2.1, there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "g_0"] \arrow[d, "\cong"'] & V \arrow[d, "\cong"] \\ \mathrm{Hom}(\pi_*(X), \mathbb{Q}) \arrow[r, "(f_*)^*"] & \mathrm{Hom}(\pi_*(X), \mathbb{Q}). \end{tikzcd}$$

Then g_0 is an isomorphism in degree k if and only if $f_*\colon \pi_k(X) \to \pi_k(X)$ is an isomorphism. Thus if $N\mathcal{E}(X) = m$, then g is an isomorphism whenever g_0 is an isomorphism in degree $\leq m$, that is, $N\mathcal{E}(\Lambda V, d) \leq m$. On the other hand, if $N\mathcal{E}(\Lambda V, d) = n$, then f is an isomorphism in π_* whenever it is an isomorphism in π_* for $* \leq n$, implying $N\mathcal{E}(X) \leq n$ by the J.H.C. Whitehead theorem. Thus the proof is finished. □

For the rest of this section, let X denote a rational Poincaré complex of dimension $2n$ with finite π_1. Then by [7, Theorem 2.5], the mapping space $\mathrm{Map}(X, S^{2n}; k)$ is a nilpotent complex of finite type. Then to apply Proposition 2.3 to $\mathrm{Map}(X, S^{2n}; k)_{(0)}$, we need the following lemma.

Lemma 2.4 *For any integer k, $\mathrm{Map}(X, S^{2n}; k)_{(0)}$ is simply-connected.*

Proof By [7, Theorem 3.11], there is a homotopy equivalence

$$\mathrm{Map}(X, S^{2n}; k)_{(0)} \simeq \mathrm{Map}(X, S^{2n}_{(0)}; r \circ k)$$

where $r\colon S^{2n} \to S^{2n}_{(0)}$ denotes the rationalization. Since there is a $(4n-1)$-equivalence $i\colon S^{2n}_{(0)} \to K(\mathbb{Q}, 2n)$, we get a $(2n-1)$-equivalence

$$i_*\colon \mathrm{Map}(X, S^{2n}_{(0)}; r \circ k) \to \mathrm{Map}(X, K(\mathbb{Q}, 2n); i \circ r \circ k)$$

and since $K(\mathbb{Q}, 2n)$ is an H-group, we have

$$\mathrm{Map}(X, K(\mathbb{Q}, 2n); i \circ r \circ k) \simeq \mathrm{Map}(X, K(\mathbb{Q}, 2n); 0).$$

On the other hand, by the theorem of Thom [22], there is a homotopy equivalence

$$\mathrm{Map}(X, K(\mathbb{Q}, 2n); 0) \simeq \prod_{k=0}^{2n-1} K(H^k(X; \mathbb{Q}), 2n-k).$$

Since $\pi_1(X)$ is finite, we have $H^{2n-1}(X;\mathbb{Q}) \cong H^1(X;\mathbb{Q}) = 0$ by the definition of a rational Poincaré complex. Thus the proof is finished. □

3 Brown–Szczarba model

In this section, we give an algebraic model for $\mathrm{Map}(X, S^{2n})$ of Brown and Szczarba [1] for a general space X, and specialize it to the case when X is a rational Poincaré complex of dimension $2n$. We also give algebraic models for the components of $\mathrm{Map}(X, S^{2n})$. We will write the rational homology and cohomology of X simply by $H_*(X)$ and $H^*(X)$, respectively.

Brown and Szczarba [1, Theorem 5.3] proved that for a CW complex X and a nilpotent complex Y of finite type, there is an algebraic model for $\mathrm{Map}(X, Y)$ of the form

$$(\Lambda(V \otimes H_*(X)), d)$$

where $(\Lambda V, d)$ is a minimal model for Y and we set

$$|x \otimes y| = |x| - |y|$$

for $x \in V$ and $y \in H_*(X)$, where $|a|$ denotes the degree of an element a in a graded vector space. The differential is defined in terms of the differential of the minimal model $(\Lambda V, d)$ and the comultiplication of the chain complex of X, instead of the comultiplication of $H_*(X)$ in general. We specialize this algebraic model to $\mathrm{Map}(X, S^{2n})$. Recall that S^{2n} has a minimal model given by

$$(\Lambda(u, v), d), \quad du = 0, \quad dv = u^2$$

where $|u| = 2n$ and $|v| = 4n - 1$. Let V be a graded vector space spanned by u, v. Then there is an algebraic model for $\mathrm{Map}(X, S^{2n})$ of the form

$$(\Lambda(V \otimes H_*(X)), d)$$

where the differential is defined in terms of the comultiplication of the chain complex of X. However, the proof of [1, Lemma 5.1] implies that in our special case, the differential is actually defined in terms of the comultiplication of $H_*(X)$ as follows.

Theorem 3.1 *There is an algebraic model for* $\mathrm{Map}(X, S^{2n})$ *of the form*

$$(\Lambda(V \otimes H_*(X)), d)$$

such that

$$d(u \otimes x) = 0 \quad \textit{and} \quad d(v \otimes x) = \sum_i (u \otimes y_i)(u \otimes z_i)$$

where $\Delta(x) = \sum_i y_i \otimes z_i$ *for the comultiplication* Δ *of* $H_*(X)$.

Hereafter, let X denote a rational Poincaré complex of dimension $2n$ with finite π_1. We specialize the algebraic model in Theorem 3.1. First, for $x, y, z \in H_*(X)$, we define a rational number $\epsilon(x, y, z)$ by

$$\epsilon(x, y, z) = \langle y^* \smile x^*, z \rangle$$

where $x^*, y^* \in H^*(X)$ denote the dual cohomology classes of x, y and $\langle -, - \rangle$ denotes the pairing of cohomology and homology classes. Note that

$$|x| + |y| = |z| \quad \text{whenever} \quad \epsilon(x, y, z) \neq 0. \tag{3.1}$$

Next, we choose a basis of $H_*(X)$ as follows. Hereafter, we fix a generator w of $H_{2n}(X) \cong \mathbb{Q}$. Let

$$\mathcal{B}_0 = \{1\} \subset H_0(X) \quad \text{and} \quad \mathcal{B}_{2n} = \{w\} \subset H_{2n}(X).$$

We choose any basis $\mathcal{B}_i$ of $H_i(X)$ for $i = 2, 3, \ldots, n-1$, where $H_1(X) = 0$ since $\pi_1(X)$ is finite. By definition, the cup product

$$H^i(X) \otimes H^{2n-i}(X) \to H^{2n}(X) \cong \mathbb{Q}, \quad x \otimes y \mapsto x \smile y$$

is nondegenerate. Then for each $x \in \mathcal{B}_i$ with $i = 2, 3, \ldots, n-1$, there is unique $\mathrm{PD}(x) \in H_{2n-i}(X)$ satisfying that for $y \in \mathcal{B}_i$,

$$\epsilon(y, \mathrm{PD}(x), w) = \begin{cases} 1 & y = x, \\ 0 & y \neq x. \end{cases}$$

We set $\mathcal{B}_{2n-i} = \{\mathrm{PD}(x) \mid x \in \mathcal{B}_i\}$ for $i = 2, 3, \ldots, n-1$. Then $\mathcal{B}_{2n-i}$ is a basis of $H_{2n-i}(X)$ for $i = 2, 3, \ldots, n-1$. Since $H_{2n-1}(X) \cong H_1(X) = 0$, it remains to choose a basis for $H_n(X)$. Since the cup product

$$H^n(X) \otimes H^n(X) \to H^{2n}(X) \cong \mathbb{Q}$$

is a nondegenerate symmetric bilinear form on $H^n(X)$ for n even and a nondegenerate anti-symmetric bilinear form on $H^n(X)$ for n odd, we can choose a basis $\mathcal{B}_n = \{x_1, \ldots, x_{b_n}\}$ of $H_n(X)$ such that the matrix $(\epsilon(x_i, x_j, w))_{1 \leq i, j \leq b_n}$ is a regular diagonal matrix for n even and of the form

$$\begin{pmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_{\frac{b_n}{2}} \\ & & & \lambda_{\frac{b_n}{2}} & 0 \end{pmatrix}$$

for n odd, where $\lambda_1 \cdots \lambda_{\frac{b_n}{2}} \neq 0$ and $b_i = \dim H_i(X)$. We set

$$\mathcal{B} = \mathcal{B}_0 \sqcup \cdots \sqcup \mathcal{B}_{2n}.$$

Then $\mathcal{B}$ is a basis of $H_*(X)$. By definition, we have

$$\Delta(x) = \sum_{x_1, x_2 \in \mathcal{B}} \epsilon(x_1, x_2, x) x_1 \otimes x_2.$$

for $x \in \mathcal{B}$, where $\epsilon(x_1, x_2, x) = 0$ unless $|x_1| + |x_2| = |x|$ by (3.1). By Theorem 3.1, we get:

Theorem 3.2 *There is an algebraic model for* $\mathrm{Map}(X, S^{2n})$ *of the form*

$$(\Lambda(V \otimes H_*(X)), d)$$

such that for $x \in \mathcal{B}$,

$$\begin{aligned} d(u \otimes x) &= 0, \\ d(v \otimes x) &= \sum_{x_1, x_2 \in \mathcal{B}} \epsilon(x_1, x_2, x)(u \otimes x_1)(u \otimes x_2). \end{aligned}$$

Now we give an algebraic model for the component $\mathrm{Map}(X, S^{2n}; k)$. Note that the degree zero part of $\Lambda(V \otimes H_*(X))$ is spanned by $u \otimes w$ and the unit 1. Let I_k be the ideal of $\Lambda(V \otimes H_*(X))$ generated by the element $u \otimes w - k$ of degree zero. Since $d(u \otimes w - k) = 0$, I_k is an ideal of dga. In particular, we can define the quotient dga $(\Lambda(V \otimes H_*(X))/I_k, d)$. By [1, Theorem 6.1], we get:

Theorem 3.3 *The quotient dga* $(\Lambda(V \otimes H_*(X))/I_k, d)$ *is an algebraic model for* $\mathrm{Map}(X, S^{2n}; k)$.

Remark 3.4 Let Y be a connected CW complex of dimension at most $4n - 2$. Then there is a one-to-one correspondence between $H^{2n}(Y; \mathbb{Z})$ and path-components of $\mathrm{Map}(Y, S^{2n})$. For each $\lambda \in H^{2n}(Y; \mathbb{Z})$, Møller and Raussen [15, Propositions 2.3, 2.4] constructed a minimal model for $\mathrm{Map}(Y, S^{2n}; \lambda)$ by using the rational Postnikov tower of S^{2n}. We will construct minimal models for $\mathrm{Map}(X, S^{2n}; k)$ with $k = 0, 1$ in Corollary 4.1 and Proposition 5.2, which are the special cases of Møller and Raussen [15, Propositions 2.3, 2.4] by Theorem 2.1. However, our minimal models are more explicit so that we can determine the self-closeness numbers of $\mathrm{Map}(X, S^{2n}; k)_{(0)}$ from them.

We show some properties of $\epsilon(x, y, z)$ that we will use later.

Lemma 3.5 *For* $x, y, z \in H_*(X)$, *we have:*

$$\epsilon(x, 1, x) = \epsilon(1, x, x) = 1, \quad \epsilon(x, y, z) = (-1)^{|x||y|} \epsilon(y, x, z).$$

Proof The first identity follows from

$$\epsilon(x,1,x)=\langle 1\smile x^*,x\rangle=1=\langle x^*\smile 1,x\rangle=\epsilon(1,x,x).$$

The second identity follows from

$$\epsilon(x,y,z)=\langle y^*\smile x^*,z\rangle=(-1)^{|x||y|}\langle x^*\smile y^*,z\rangle=\epsilon(y,x,z)$$

completing the proof. □

Lemma 3.6 *For $x_1,x_2,x_3,x_4\in H_*(X)$, we have:*

$$\sum_{y\in\mathcal{B}}\epsilon(x_1,x_2,y)\epsilon(y,x_3,x_4)=\sum_{z\in\mathcal{B}}\epsilon(x_2,x_3,z)\epsilon(x_1,z,x_4).$$

Proof By definition, we have

$$x_2^*\smile x_1^*=\sum_{y\in\mathcal{B}}\epsilon(x_1,x_2,y)y^*.$$

Then we get

$$\langle x_3^*\smile x_2^*\smile x_1^*,x_4\rangle=\sum_{y\in\mathcal{B}}\epsilon(x_1,x_2,y)\langle x_3^*\smile y^*,x_4\rangle=\sum_{y\in\mathcal{B}}\epsilon(x_1,x_2,y)\epsilon(y,x_3,x_4).$$

Quite similarly, we can get

$$\langle x_3^*\smile x_2^*\smile x_1^*,x_4\rangle=\sum_{z\in\mathcal{B}}\epsilon(x_2,x_3,z)\epsilon(x_1,z,x_4).$$

Thus the statement is proved. □

4 Degree zero component

In this section, we compute the self-closeness number of $\mathrm{Map}(X,S^{2n};0)_{(0)}$ by using the algebraic model in Theorem 3.3. Let

$$\widehat{\mathcal{B}}=\{x\in\mathcal{B}\mid 0<|x|<2n\}.$$

For a graded set S, let $\langle S\rangle$ denote the graded vector space spanned by S. We define a graded vector space by

$$W=\langle u\otimes x,v\otimes x,v\otimes w\mid x\in\mathcal{B}_0\sqcup\widehat{\mathcal{B}}\rangle \tag{4.1}$$

and a dga $(\Lambda W, d)$ by

$$\begin{aligned} d(u \otimes x) &= 0, \\ d(v \otimes x) &= \sum_{x_1, x_2 \in \mathcal{B}} \epsilon(x_1, x_2, x)(u \otimes x_1)(u \otimes x_2), \\ d(v \otimes w) &= \sum_{x_1, x_2 \in \widehat{\mathcal{B}}} \epsilon(x_1, x_2, w)(u \otimes x_1)(u \otimes x_2) \end{aligned}$$

for $x \in \mathcal{B}_0 \sqcup \widehat{\mathcal{B}}$.

Corollary 4.1 *The dga $(\Lambda W, d)$ is the minimal model for* $\mathrm{Map}(X, S^{2n}; 0)$.

Proof By definition, the term $u \otimes w$ is not included in any differential of the quotient dga $(\Lambda(V \otimes H_*(X))/I_0, d)$ in Theorem 3.3, and so $(\Lambda(V \otimes H_*(X))/I_0, d)$ is isomorphic with $(\Lambda W, d)$. Clearly, $(\Lambda W, d)$ is minimal, and the statement is proved. □

By Lemmas 2.3 and 2.4, we aim to compute the self-closeness number of $(\Lambda W, d)$ for determining that of $\mathrm{Map}(X, S^{2n}; 0)_{(0)}$. Let $f : (\Lambda W, d) \to (\Lambda W, d)$ be a dga map. We give a matrix representation of f_0. For $0 \le k < 2n$, we equip $\mathcal{B}_k$ with any total order such that

$$\mathcal{B}_k = \{x_1^k < \cdots < x_{b_k}^k\}.$$

Let W_k denote the degree k part of W. Then we have

$$W = \left(\bigoplus_{k=0}^{2n-2} W_{2n-k} \right) \oplus \left(\bigoplus_{k=0}^{2n} W_{4n-k-1} \right).$$

For $0 \le k \le 2n-2$, we have

$$W_{2n-k} = \langle u \otimes x \mid x \in \mathcal{B}_k \rangle \quad \text{and} \quad W_{4n-k-1} = \langle v \otimes x \mid x \in \mathcal{B}_k \rangle.$$

Moreover, we have $W_{2n-1} = \langle v \otimes w \rangle$. For $0 \le k \le 2n-2$, let $A_k(f)$ denote the matrix representation of $f_0 : W_{2n-k} \to W_{2n-k}$ with respect to the ordered basis $\mathcal{B}_{2n-k}$. For $0 \le k \le 2n$, let $B_k(f)$ denote the matrix representation of $f_0 : W_{4n-k-1} \to W_{4n-k-1}$ with respect to the ordered basis $\mathcal{B}_{4n-k-1}$. Let C_{ij} denote the (i, j)-entry of a matrix C. Then by definition, we have

$$f_0(u \otimes x_j^k) = \sum_{i=1}^{b_k} A_k(f)_{ij} u \otimes x_i^k \quad \text{and} \quad f_0(v \otimes x_j^k) = \sum_{i=1}^{b_k} B_k(f)_{ij} v \otimes x_i^k$$

for $1 \le j \le b_k$. Obviously, a dga map f is an isomorphism if and only if its linear part f_0 is an isomorphism. Then the following lemma is immediate from the definition of $A_k(f)$ and $B_k(f)$.

Lemma 4.2 *The following are equivalent:*

(1) $N\mathcal{E}(\Lambda W, d) \leq m$;
(2) *For any dga map* $f\colon (\Lambda W, d) \to (\Lambda W, d)$, *if* $A_k(f)$ *and* $B_l(f)$ *are regular for* $2n - m \leq k \leq 2n - 2$ *and* $4n - m - 1 \leq l \leq 2n$, *then* $A_k(f)$ *and* $B_l(f)$ *are regular for all* $0 \leq k \leq 2n - 2$ *and* $0 \leq l \leq 2n$.

For $x \in \mathcal{B}_i$ and $k \leq i$, we also define an $b_k \times b_{i-k}$ matrix $E_k(x)$ by

$$E_k(x)_{pq} = \epsilon(x_p^k, x_q^{i-k}, x).$$

By our choice of the basis $\mathcal{B}$ and the definition of a rational Poincaré complex, we have:

Lemma 4.3 *The matrix* $E_k(w)$ *is regular for* $0 \leq k \leq 2n$.

We prove relations among $A_k(f)$, $B_k(f)$ and $E_k(x)$.

Lemma 4.4 *Let* $f\colon (\Lambda W, d) \to (\Lambda W, d)$ *be a dga map. For each* $x_p^i \in \mathcal{B}_i$, *we have*

$$A_k(f)E_k(x_p^i)A_{i-k}(f)^T = \sum_{a=1}^{b_i} B_i(f)_{ap}E_k(x_a^i)$$

where $0 \leq k \leq i$ *for* $i < 2n$ *and* $0 < k < 2n$ *for* $i = 2n$.

Proof We only prove the $i < 2n$ case, and the $i = 2n$ case can be proved verbatim, where the only difference is the range of k. Let $i < 2n$. The quadratic part of $f(d(v \otimes x_p^i))$ is

$$\begin{aligned}
&\sum_{k=0}^{i}\sum_{r=1}^{b_k}\sum_{t=1}^{b_{i-k}} \epsilon(x_r^k, x_t^{i-k}, x_p^i) f_0(u \otimes x_r^k) f_0(u \otimes x_t^{i-k}) \\
&\quad = \sum_{k=0}^{i}\sum_{q,r=1}^{b_k}\sum_{s,t=1}^{b_{i-k}} \epsilon(x_r^k, x_t^{i-k}, x_p^i) A_k(f)_{qr} A_{i-k}(f)_{st}(u \otimes x_q^k)(u \otimes x_s^{i-k})
\end{aligned}$$

and the quadratic part of $d(f(v \otimes x_p^i))$ is

$$\begin{aligned}
d(f_0(v \otimes x_p^i)) &= d\left(\sum_{a=1}^{b_i} B_i(f)_{ap} v \otimes x_a^i\right) \\
&= \sum_{a=1}^{b_i}\sum_{k=0}^{i}\sum_{q=1}^{b_k}\sum_{s=1}^{b_{i-k}} \epsilon(x_q^k, x_s^{i-k}, x_a) B_i(f)_{ap}(u \otimes x_q^k)(u \otimes x_s^{i-k}).
\end{aligned}$$

Then since $f(d(v \otimes z)) = d(f(v \otimes z))$ for $z \in \mathcal{B}$, we get

$$\sum_{r=1}^{b_k}\sum_{t=1}^{b_{i-k}} \epsilon(x_r^k, x_t^{i-k}, x_p^i) A_k(f)_{qr} A_{i-k}(f)_{st} = \sum_{a=1}^{b_i} \epsilon(x_q^k, x_s^{i-k}, x_a) B_i(f)_{ap}$$

for fixed $q = 1, \dots, b_k$, $s = 1, \dots, b_{i-k}$ and $k = 0, \dots, i$. Thus the statement is proved. □

Corollary 4.5 *Let $f\colon (\Lambda W, d) \to (\Lambda W, d)$ be a dga map. For $0 \le i < 2n$, we have*

$$B_i(f) = A_0(f) A_i(f)$$

Proof Clearly, the matrix $(E_i(x_1^i), \dots, E_i(x_{b_i}^i))$ is the $b_i \times b_i$ identity matrix, and so by Lemma 4.4, we have

$$A_i(f) A_0(f)^{\mathrm{T}} = A_i(f)(E_i(x_1^i), \dots, E_i(x_{b_i}^i)) A_0(f)^{\mathrm{T}} = B_i(f).$$

Since $A_0(f)$ is a 1×1 matrix, the proof is finished. □

We will use the following property of $d(X)$.

Lemma 4.6 *For $x \in \mathcal{B}$, if $2n - |x| < d(X)$, then $|x| > n$.*

Proof If $d(X) = n$, we have $|x| > 2n - d(X) = n$. If $d(X) > n$, we have $|x| \ge d(X) > n$ since by definition, $\mathcal{B}_i = \varnothing$ for $2n - d(X) < i < d(X)$. In either case we have $|x| > n$, completing the proof. □

Now we are ready to compute the self-closeness number of $(\Lambda W, d)$.

Proposition 4.7 *If X is not primitive, then*

$$N\mathcal{E}(\Lambda W, d) = d(X).$$

Proof First, we prove $N\mathcal{E}(\Lambda W, d) \le d(X)$. Let $f\colon (\Lambda W, d) \to (\Lambda W, d)$ be a dga map such that f_0 is an isomorphism in degrees $\le d(X)$. Then $A_k(f)$ is regular for $2n - k \le d(X)$. By Lemma 4.2, it is sufficient to show that $A_k(f)$ and $B_l(f)$ are regular for $0 \le k < 2n$ and $0 \le l \le 2n$. To this end, we take two steps.
Step 1. By Lemma 4.4, for $x_p^i = w$, we have

$$A_k(f) E_k(w) A_{2n-k}(f)^{\mathrm{T}} = B_{2n}(f) E_k(w) \tag{4.2}$$

for $0 < k < 2n$, where $B_{2n}(f)$ is a 1×1 matrix. By definition, $b_{d(X)} = b_{2n-d(X)} \neq 0$, implying $W_{d(X)}$ and $W_{2n-d(X)}$ are non-trivial. By assumption, $A_{2n-d(X)}(f)$ is regular. Moreover, since $2n - d(X) \le d(X)$, $A_{d(X)}(f)$ is regular too. Then the 1×1 matrix $B_{2n}(f)$ is regular by Lemma 4.3 and (4.2) for $k = d(X)$. This implies that $A_k(f)$ is regular for $0 < k < 2n$ too by Lemma 4.3 and (4.2).

Step 2. By assumption, there is $0 < i < 2n$ such that $QH^i(X) \neq H^i(X)$, implying $E_j(x_p^i)$ is non-trivial for some $x_p^i \in \mathcal{B}_j$ and $0 < j < i$. This also implies that W_{2n-i}, W_{2n-j} and W_{2n-i+j} are non-trivial. On the other hand, by Lemma 4.4, we have

$$A_j(f)E_j(x_p^i)A_{i-j}(f)^{\mathrm{T}} = \sum_{a=1}^{b_i} B_i(f)_{ap}E_j(x_a^i).$$

We have seen in Step 1 that both $A_j(f)$ and $A_{i-j}(f)$ are regular since $0 < j, i-j < 2n$. Then $A_j(f)E_j(x_p^i)A_{i-j}(f)^{\mathrm{T}}$ is non-trivial, implying that $B_i(f)$ is non-trivial too. Then since $B_i(f)$ is non-trivial and $A_0(f)$ is a 1×1 matrix, it follows from Corollary 4.5 that $A_0(f)$ is regular. Thus we have obtained that $A_k(f)$ is regular for $0 \leq k < 2n$ by Step 1. Moreover, by Corollary 4.5, $B_l(f)$ is regular for $0 \leq l < 2n$ too, and so we obtained that $B_l(f)$ is regular for $0 \leq l \leq 2n$. Thus we obtain $N\mathcal{E}(\Lambda W, d) \leq d(X)$.

Next, we prove $N\mathcal{E}(\Lambda W, d) \geq d(X)$. Consider a self-map of a commutative graded algebra $g\colon \Lambda W \to \Lambda W$ such that $g(v \otimes w) = 0$, $g(v \otimes x) = 0$ and

$$g(u \otimes x) = \begin{cases} u \otimes x & 2n - |x| < d(X), \\ 0 & 2n - |x| \geq d(X) \end{cases}$$

for $x \in \mathcal{B}_0 \sqcup \widehat{\mathcal{B}}$. Then g_0 is an isomorphism in degrees $< d(X)$ and trivial in degree $d(X)$, where $W_{d(X)}$ is non-trivial. By definition, we have

$$dg(u \otimes x) = 0 = g(d(u \otimes x))$$

for $x \in \mathcal{B}_0 \sqcup \widehat{\mathcal{B}}$. For $x \in \mathcal{B}$, $d(v \otimes x)$ is a linear combination of $(u \otimes x_1)(u \otimes x_2)$ such that $|x_1| + |x_2| = |x|$. If $2n - |x_1| < d(X)$ and $2n - |x_2| < d(X)$, we have $|x| = |x_1| + |x_2| > 2n$ by Lemma 4.6, which is impossible. Thus either $2n - |x_1| \geq d(X)$ or $2n - |x_2| \geq d(X)$, implying that

$$dg(v \otimes x) = 0 = g(d(v \otimes x))$$

for $x \in \mathcal{B}$. Thus g is a dga map, implying $N\mathcal{E}(\Lambda W, d) \geq d(X)$. Therefore the proof is finished. □

Proposition 4.8 *If X is primitive, then*

$$N\mathcal{E}(\Lambda W, d) = 2n.$$

Proof Let $f\colon (\Lambda W, d) \to (\Lambda W, d)$ be a dga map. Suppose that f_0 is an isomorphism in degrees $\leq 2n$. Then $A_k(f)$ and $B_{2n}(f)$ are regular for $0 \leq k < 2n$. So by Corollary 4.5, $B_k(f)$ is regular for $0 \leq k < 2n$ too. Thus by Lemma 4.2, we obtain $N\mathcal{E}((\Lambda W, d)) \leq 2n$.

Consider a self-map of a commutative graded algebra $g\colon \Lambda W \to \Lambda W$ given by

$$g(u\otimes x)=\begin{cases}0 & |x|=0,\\ u\otimes x & 0<|x|<2n\end{cases} \quad \text{and} \quad g(v\otimes x)=\begin{cases}0 & 0\le |x|<2n,\\ v\otimes x & |x|=2n.\end{cases}$$

Then we have

$$dg(u\otimes x)=0=g(d(u\otimes x))$$

for $0\le |x|<2n$. Since $QH^i(X)=H^i(X)$ for $i<2n$, we have $\epsilon(x_1,x_2,x)=0$ for $0\le |x|<2n$ unless $x_1=1$ or $x_2=1$. Then we have

$$d(v\otimes x)=\begin{cases}(u\otimes 1)^2 & |x|=0,\\ 2(u\otimes 1)(u\otimes x) & 0<|x|<2n,\end{cases}$$

implying $d(g(v\otimes x))=0=g(d(v\otimes x))$ for $0\le |x|<2n$. By definition, $d(v\otimes w)$ is a linear combination of $(u\otimes x_1)(u\otimes x_2)$ for $0<|x_1|, |x_2|<2n$. Then we have

$$dg(v\otimes w)=d(v\otimes w)=g(d(v\otimes w)).$$

Thus g is a dga map. Clearly, g_0 is an isomorphism in degrees $<2n$ and trivial in degree $2n$, where $W_{2n}=\langle u\otimes 1\rangle$ is non-trivial. Thus we get $N\mathcal{E}(\Lambda W,d)\ge 2n$, completing the proof. □

5 Degree one component

In this section, we determine the self-closeness number of $\mathrm{Map}(X,S^{2n};1)_{(0)}$. To this end, we take two steps. In the first step, we construct the minimal model $(\Lambda\overline{W},d)$ for $\mathrm{Map}(X,S^{2n};1)_{(0)}$ by using its algebraic model in Theorem 3.3, where $\overline{W}$ concentrates in degrees $\le 4n-1$. If there is an isomorphism of dgas

$$(\Lambda U,d)\otimes(\Lambda(s),0)\xrightarrow{\cong}(\Lambda\overline{W},d)$$

where U is the degree $<4n-1$ part of $\overline{W}$ and $|s|=4n-1$, then the self-closeness number of $(\Lambda\overline{W},d)$ turns out to be $4n-1$. To prove the existence of such an isomorphism, it is sufficient to show that a certain element in $(\Lambda U,d)$ of degree $4n$ is a coboundary. In the second step, we show that the sum of the above element of degree $4n$ and a certain coboundary belongs to a vector subspace of ΛU having a direct sum decomposition. Then we show that the above sum is trivial in each direct summand, implying that the above element of degree $4n$ turns out to be a coboundary.

5.1 Minimal model

The algebraic model for $\mathrm{Map}(X, S^{2n}; 1)$ in Theorem 3.3 is not minimal, unlikely to the degree zero component in the previous section. Then we construct a minimal model for $\mathrm{Map}(X, S^{2n}; 1)_{(0)}$ from it. Let W be the graded vector space as in (4.1). We define a dga $(\Lambda W, d)$ by

$$d(u \otimes x) = 0, \tag{5.1}$$

$$d(v \otimes x) = \sum_{x_1, x_2 \in \mathcal{B}} \epsilon(x_1, x_2, x)(u \otimes x_1)(u \otimes x_2), \tag{5.2}$$

$$d(v \otimes w) = 2(u \otimes 1) + \sum_{x_1, x_2 \in \widehat{\mathcal{B}}} \epsilon(x_1, x_2, w)(u \otimes x_1)(u \otimes x_2) \tag{5.3}$$

for $x \in \mathcal{B}_0 \sqcup \widehat{\mathcal{B}}$, which is different from the one in the previous section. Quite similarly to Corollary 4.1, we can see that the dga $(\Lambda W, d)$ is isomorphic with the quotient dga $(\Lambda(V \otimes H_*(X))/I_1, d)$ in Theorem 3.3, and so we get:

Lemma 5.1 *The dga $(\Lambda W, d)$ is an algebraic model for* $\mathrm{Map}(X, S^{2n}; 1)_{(0)}$.

By definition, the dga $(\Lambda W, d)$ is not minimal, and so we construct a minimal model for $\mathrm{Map}(X, S^{2n}; 1)_{(0)}$ from it. Consider an element

$$\eta = d(v \otimes w) - 2(u \otimes 1)$$

of ΛW. Then we have $d\eta = 0$. We define

$$v \odot x = (v \otimes w)(u \otimes x) - v \otimes x \quad \text{and} \quad v \odot 1 = v \otimes 1 - \frac{1}{4}(v \otimes w)(2(u \otimes 1) - \eta)$$

for $x \in \widehat{\mathcal{B}}$. Then by (5.1), (5.2) and (5.3), we have

$$d(v \odot x) = \eta(v \otimes x) - \sum_{x_1, x_2 \in \widehat{\mathcal{B}}} \epsilon(x_1, x_2, x)(u \otimes x_1)(u \otimes x_2), \tag{5.4}$$

$$d(v \odot 1) = \frac{1}{4}\eta^2 \tag{5.5}$$

for $x \in \widehat{\mathcal{B}}$. Consider a vector subspace

$$\overline{W} = \langle u \otimes x, v \odot x, v \odot 1 \mid x \in \widehat{\mathcal{B}} \rangle$$

of ΛW. Then since $\eta \in \Lambda \overline{W}$, it follows from (5.1), (5.4) and (5.5) that we get a subdga $(\Lambda \overline{W}, d)$ of $(\Lambda W, d)$.

Proposition 5.2 *The dga $(\Lambda \overline{W}, d)$ is a minimal model for* $\mathrm{Map}(X, S^{2n}; 1)_{(0)}$.

Proof By definition, $(\Lambda\overline{W}, d)$ is minimal, and so it remains to show that the inclusion $(\Lambda\overline{W}, d) \to (\Lambda W, d)$ is a quasi-isomorphism. Consider a vector subspace

$$\widehat{W} = \langle v \otimes w, 2(u \otimes 1) + \eta \rangle$$

of ΛW. Then since $d(v \otimes w) = 2(u \otimes 1) + \eta$, we get a contractible subdga $(\Lambda\widehat{W}, d)$ of $(\Lambda W, d)$, and so we get a dga map

$$f\colon (\Lambda\overline{W}, d) \otimes (\Lambda\widehat{W}, d) \to (\Lambda W, d), \quad x \otimes y \mapsto xy.$$

Clearly, f_0 is an isomorphism, hence so is f. Thus we may identify the inclusion $(\Lambda\overline{W}, d) \to (\Lambda W, d)$ with the inclusion of $(\Lambda\overline{W}, d)$ into the first factor of $(\Lambda\overline{W}, d) \otimes (\Lambda\widehat{W}, d)$, completing the proof. □

By Lemmas 2.3 and 2.4, we aim to compute the self-closeness number of $(\Lambda\overline{W}, d)$ for determining that of $\mathrm{Map}(X, S^{2n}; 1)_{(0)}$. Let U be the vector subspace of $\overline{W}$ spanned by elements of degree $\leq 4n - 2$. Then we have

$$\overline{W} = U \oplus \langle v \odot 1 \rangle.$$

By (5.1) and (5.4), we get a subdga $(\Lambda U, d)$ of $(\Lambda\overline{W}, d)$.

Proposition 5.3 *There is an isomorphism*

$$(\Lambda U, d) \otimes (\Lambda(s), 0) \xrightarrow{\cong} (\Lambda\overline{W}, d)$$

where $|s| = 4n - 1$.

Assuming Proposition 5.3, we determine $N\mathcal{E}(\Lambda\overline{W}, d)$.

Proposition 5.4 $N\mathcal{E}(\Lambda\overline{W}, d) = 4n - 1$.

Proof The largest degree of elements of $\overline{W}$ is $4n - 1$, and so $N\mathcal{E}(\Lambda\overline{W}, d) \leq 4n - 1$. By Proposition 5.3, it is easy to construct a dga map $f\colon (\Lambda\overline{W}, d) \to (\Lambda\overline{W}, d)$ such that $f_0|_U$ is an isomorphism but f_0 itself is not an isomorphism. Then $N\mathcal{E}(\Lambda\overline{W}, d) \geq 4n - 1$, completing the proof. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 Combine Propositions 4.7, 4.8 and 5.4. □

It remains to prove Proposition 5.3. Suppose that there is a decomposable element ζ of ΛU such that

$$\frac{1}{4}\eta^2 = d\zeta. \tag{5.6}$$

Then we can define a dga map

$$f\colon (\Lambda U, d)\otimes(\Lambda(s), 0)\to(\Lambda\overline{W}, d)$$

by $f(x\otimes 1)=x$ for $x\in\Lambda U$ and $f(1\otimes s)=v\odot 1-\zeta$. Indeed, $df(x\otimes 1)=dx=f(dx\otimes 1)$ since $dx\in\Lambda U$, and

$$ds=0=d(v\odot 1)-\frac{1}{4}\eta^2=d(v\odot 1-\zeta)$$

by (5.5). Since ζ is decomposable, f_0 is an isomorphism, and so f is an isomorphism too. Thus Proposition 5.3 is proved by the following lemma.

Lemma 5.5 *There is a decomposable element ζ of ΛU satisfying* (5.6).

5.2 The vector space $\mathcal{U}$

To prove Lemma 5.5, we define a vector subspace

$$\mathcal{U}=\langle(u\otimes x_1)(u\otimes x_2)(u\otimes x_3)\mid |x_1|+|x_2|+|x_3|=2n\rangle$$

of ΛU. First, we will find an element ξ of $\mathcal{U}$ and a decomposable element α of ΛU such that

$$\eta^2=\xi+d\alpha. \tag{5.7}$$

For each $x\in\mathcal{B}_i\subset\widehat{\mathcal{B}}$, there is a unique element $\hat{x}\in\mathcal{B}$ such that $\epsilon(x,\hat{x},w)\neq 0$, where $|x|+|\hat{x}|=2n$. We abbreviate $\epsilon(x,\hat{x},w)$ by $\epsilon(x)$. Since $\hat{\hat{x}}=x$, we have

$$\eta=\sum_{x\in\widehat{\mathcal{B}}}\epsilon(x)(u\otimes x)(u\otimes\hat{x}).$$

We define

$$\xi=\sum_{x_1,x_2,x_3\in\widehat{\mathcal{B}}}\epsilon(x_3)\epsilon(x_1,x_2,x_3)(u\otimes x_1)(u\otimes x_2)(u\otimes\hat{x}_3). \tag{5.8}$$

Then ξ is a decomposable element of ΛU.

Lemma 5.6 *There is a decomposable element $\alpha\in\Lambda U$ satisfying* (5.7).

Proof Let

$$\alpha=\sum_{x\in\widehat{\mathcal{B}}}\epsilon(x)(v\odot x)(u\otimes\hat{x})$$

Then α is a decomposable element of ΛU, and by (5.1) and (5.4) we have

$$\begin{aligned} d\alpha &= d\left(\sum_{x\in\widehat{\mathcal{B}}} \epsilon(x)(v\odot x)(u\otimes\hat{x})\right) \\ &= \sum_{x\in\widehat{\mathcal{B}}} \epsilon(x)(d(v\odot x))(u\otimes\hat{x}) \\ &= \sum_{x\in\widehat{\mathcal{B}}} \epsilon(x)\left(\eta(u\otimes x)(u\otimes\hat{x}) - \sum_{x_1,x_2\in\widehat{\mathcal{B}}} \epsilon(x_1,x_2,x)(u\otimes x_1)(u\otimes x_2)(u\otimes\hat{x})\right) \\ &= \eta^2 - \xi. \end{aligned}$$

So the statement is proved. □

Next we define a decomposable element $\mu \in \Lambda U$ such that $d\mu \in \mathcal{U}$. Let

$$\mathcal{B}_- = \{x \in \mathcal{B} \mid 0 < |x| < n\}.$$

For $x \in \mathcal{B}_-$, let

$$\lambda(x) = \frac{3(n-|x|)}{n}\epsilon(\hat{x}), \quad \mu(x) = (-1)^{|x||\hat{x}|}(v\odot x)(u\otimes\hat{x}) - (v\odot\hat{x})(u\otimes x)$$

and let

$$\mu = \sum_{x\in\mathcal{B}_-} \lambda(x)\mu(x).$$

By (5.1) and (5.4) we have

$$\begin{aligned} d\mu(x) &= d\left((-1)^{|x||\hat{x}|}(v\odot x)(u\otimes\hat{x}) - (v\odot\hat{x})(u\otimes x)\right) \\ &= (-1)^{|x||\hat{x}|}\left(d(v\odot x)\right)(u\otimes\hat{x}) - \left(d(v\odot\hat{x})\right)(u\otimes x) \\ &= \sum_{x_1,x_2\in\widehat{\mathcal{B}}} \epsilon(x_1,x_2,\hat{x})(u\otimes x_1)(u\otimes x_2)(u\otimes x) \\ &\quad - (-1)^{|x||\hat{x}|} \sum_{x_1,x_2\in\widehat{\mathcal{B}}} \epsilon(x_1,x_2,x)(u\otimes x_1)(u\otimes x_2)(u\otimes\hat{x}). \end{aligned} \tag{5.9}$$

Then it is easy to see that $d\mu(x)$ belongs to $\mathcal{U}$, hence so does $d\mu$ too.

Proposition 5.7 *There is an equality*

$$\xi = d\mu.$$

We prove Lemma 5.5 by assuming Proposition 5.7.

Proof of Lemma 5.5 Let $\zeta = \frac{1}{4}(\alpha + \mu)$. Then by Lemma 5.6 and Proposition 5.7, ζ is a decomposable element of ΛU and satisfies (5.7). □

We devote the rest of this paper to prove Proposition 5.7. Since both ξ and $d\mu$ belong to $\mathcal{U}$, we need to understand the structure of the vector space $\mathcal{U}$. We introduce a total order on $\widehat{\mathcal{B}}$. First, we equip $\mathcal{B}_i$ with any total order for each $0 < i \leq n$. Next, for each $n < i < 2n$ we equip $\mathcal{B}_i$ with a total order such that $x < y$ if and only if $\hat{x} < \hat{y}$. Finally, we extend to a total order on $\widehat{\mathcal{B}}$ such that $|x| < |y|$ implies $x < y$. We write

$$\mathcal{B}_- = \{\theta_1 > \cdots > \theta_m\}.$$

Then $\mathcal{B}_{n+1} \sqcup \cdots \sqcup \mathcal{B}_{2n-2} = \{\hat{\theta}_1, \ldots, \hat{\theta}_m\}$. Let $\mathcal{F}_0 = \mathcal{B}_n$, and for $1 \leq i \leq m$, let $\mathcal{F}_i = \mathcal{F}_{i-1} \sqcup \{\theta_i, \hat{\theta}_i\}$. Then we get a filtration

$$\mathcal{B}_n = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \widehat{\mathcal{B}}.$$

Since $\theta_m < \cdots < \theta_1 < x$ for all $x \in \mathcal{B}_n \sqcup \cdots \sqcup \mathcal{B}_{2n-2}$, the least element of $\mathcal{F}_i$ is θ_i for $1 \leq i \leq m$.

We introduce a filtration of $\mathcal{U}$ by using the above filtration of $\widehat{\mathcal{B}}$. For $0 \leq i \leq m$, we define a linear map $f_i : \mathcal{U} \to \mathcal{U}$ by

$$f_i((u \otimes x_1)(u \otimes x_2)(u \otimes x_3)) = \begin{cases} (u \otimes x_1)(u \otimes x_2)(u \otimes x_3) & x_1, x_2, x_3 \in \mathcal{F}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{U}_i = \operatorname{Im} f_i$ for $0 \leq i \leq m$.

Lemma 5.8 *The vector space $\mathcal{U}_0$ is trivial.*

Proof Let $(u \otimes x_1)(u \otimes x_2)(u \otimes x_3)$ be an element of $\mathcal{U}$. By definition, $f_0((u \otimes x_1)(u \otimes x_2)(u \otimes x_3)) \neq 0$ if and only if $x_1, x_2, x_3 \in \mathcal{F}_0 = \mathcal{B}_n$, implying $|x_1| + |x_2| + |x_3| = 3n$. This is impossible because $|x_1| + |x_2| + |x_3| = 2n$, and so the statement is proved. □

It is easy to see that f_m is the identity map on $\mathcal{U}$ since $\mathcal{F}_m = \widehat{\mathcal{B}}$. Then we get a filtration

$$0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_m = \mathcal{U}.$$

For $1 \leq i \leq m$, let

$$\mathcal{V}_i = \langle (u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i) \in \mathcal{U} \mid x_1, x_2 \in \mathcal{F}_i \rangle$$

be a vector subspace of $\mathcal{U}$.

Lemma 5.9 *There is an equality*

$$\mathcal{U} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m.$$

Proof It is sufficient to show $\mathcal{U}_i = \mathcal{U}_{i-1} \oplus \mathcal{V}_i$ for $1 \le i \le m$. By definition, the vector space $\mathcal{U}_i$ is the direct sum of $\mathcal{U}_{i-1}$ and a vector subspace of $\mathcal{U}$ spanned by $(u \otimes x_1)(u \otimes x_2)(u \otimes x_3)$ such that $x_1, x_2 \in \mathcal{F}_i$ and $x_3 = \theta_i$ or $\hat{\theta}_i$. If $x_3 = \hat{\theta}_i$, then

$$|x_1| + |x_2| = 2n - |x_3| = |\theta_i|$$

implying $x_1, x_2 < \theta_i$. Then we get $x_1, x_2 \notin \mathcal{F}_i$, which is impossible. Clearly, the $x_3 = \theta_i$ case is possible, completing the proof. □

5.3 The derivation ∂_i

For $1 \le i \le m$, let $p_i : \mathcal{U} \to \mathcal{V}_i$ denote the projection. Then by Lemma 5.9, in order to prove Proposition 5.7, it is sufficient to show

$$p_i(\xi - d\mu) = 0 \tag{5.10}$$

for $1 \le i \le m$. To this end, we introduce the derivation ∂_i. Let

$$\widehat{U} = \langle u \otimes x \mid x \in \widehat{\mathcal{B}} \rangle.$$

Then $\mathcal{U}$ is a vector subspace of $\Lambda \widehat{U}$. For $1 \le i \le m$, we define a derivation $\partial_i : \Lambda \widehat{U} \to \Lambda \widehat{U}$ by

$$\partial_i(u \otimes x) = \begin{cases} 1 & x = \theta_i, \\ 0 & x \ne \theta_i \end{cases}$$

for $x \in \widehat{\mathcal{B}}$ and the Leibuniz rule

$$\partial_i(ab) = \partial_i(a)b + (-1)^{|a|} a \partial_i(b)$$

for $a, b \in \Lambda \widehat{U}$. The following is immediate.

Lemma 5.10 *For a non-trivial element $(u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i) \in \mathcal{V}_i$, we have*

$$\partial_i((u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i)) = \begin{cases} (-1)^{|x_1|+|x_2|}(u \otimes x_1)(u \otimes x_2) & x_1, x_2 \ne \theta_i, \\ 2(u \otimes x_1)(u \otimes \theta_i) & x_1 \ne \theta_i,\ x_2 = \theta_i, \\ 3(u \otimes \theta_i)^2 & x_1, x_2 = \theta_i. \end{cases}$$

Proof Since $(u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i) \in \mathcal{V}_i$, we have $|x_1| + |x_2| + |\theta_i| = 2n$.
(1) For $x_1, x_2 \ne \theta_i$,

$$\begin{aligned} \partial_i((u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i)) &= (-1)^{|u \otimes x_1| + |u \otimes x_2|}(u \otimes x_1)(u \otimes x_2)\partial_i(u \otimes \theta_i) \\ &= (-1)^{|x_1|+|x_2|}(u \otimes x_1)(u \otimes x_2). \end{aligned}$$

(2) For $x \neq \theta_i$, $x_2 = \theta_i$, $|u \otimes \theta_i|$ is even since $(u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i) \neq 0$. This also implies that $|x_1|$ is even since $|x_1| = 2n - |x_2| - |\theta_i| = 2n - 2|\theta_i|$. Then we have

$$\partial_i((u\otimes x_1)(u\otimes x_2)(u\otimes\theta_i)) = (-1)^{|u\otimes x_1|}(u \otimes x_1)\partial_i\left((u \otimes \theta_i)^2\right) = 2(u \otimes x_1)(u\otimes\theta_i).$$

(3) For $x_1, x_2 = \theta_i$, similarly to the above, we have $|\theta_i|$ even. Then we get

$$\partial_i((u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i)) = 3(u \otimes \theta_i)^2.$$

Thus the proof is finished. □

The derivation ∂_i has the following pleasant property.

Lemma 5.11 *For $1 \leq i \leq m$, the derivation ∂_i is injective on $\mathcal{V}_i \subset \Lambda\widehat{U}$.*

Proof Note that $\mathfrak{B}_i = \{(u \otimes x_1)(u \otimes x_2)(u \otimes \theta_i) \in \mathcal{U} \mid x_1, x_2 \in \mathcal{F}_i\}$ is a basis of $\mathcal{V}_i$. Then by Lemma 5.10, $\partial_i a$ for $a \in \mathfrak{B}_i$ are linearly independent, proving the statement.

In order to show (5.10), it is sufficient to prove

$$\partial_i(p_i(\xi - d\mu)) = 0 \tag{5.11}$$

for $1 \leq i \leq m$ by Lemma 5.11. So we describe $\partial_i p_i(\xi)$ and $\partial_i p_i(d\mu)$ respectively. To this end, we need the following properties of $\epsilon(x, y, z)$.

Lemma 5.12 *For $x \in \mathcal{B}_-$, we have*

$$\epsilon(x) = 1 \quad \textit{and} \quad \epsilon(\hat{x}) = (-1)^{|x||\hat{x}|}.$$

Proof The first equality follows immediately from the definition. Since $\hat{\hat{x}} = x$, we have

$$\epsilon(\hat{x}) = \epsilon(\hat{x}, x, w) = (-1)^{|\hat{x}||x|}\epsilon(x, \hat{x}, w) = (-1)^{|\hat{x}||x|}\epsilon(x)$$

by Lemma 3.5. Then we get the second equality. □

Lemma 5.13 *For $x_1, x_2, x_3 \in \widehat{\mathcal{B}}$, we have the following:*

(1) $\epsilon(x_1, x_2, \hat{x}_3)(u \otimes x_1)(u \otimes x_2)(u \otimes x_3) = \epsilon(x_2, x_1, \hat{x}_3)(u \otimes x_2)(u \otimes x_1)(u \otimes x_3)$.
(2) *For $1 \leq i \leq m$, $p_i(\epsilon(x_1, x_2, \theta_i)(u \otimes x_1)(u \otimes x_2)(u \otimes \hat{\theta}_i)) = 0$.*
(3) $\epsilon(\hat{x}_3)\epsilon(x_1, x_2, \hat{x}_3) = \epsilon(x_1)\epsilon(x_2, x_3, \hat{x}_1)$.
(4) *If at least two of x_1, x_2, x_3 are of the same degree and*

$$\epsilon(x_1, x_2, \hat{x}_3)(u \otimes x_1)(u \otimes x_2)(u \otimes x_3) \neq 0,$$

then $|x_1|$, $|x_2|$, $|x_3|$ are even.

Proof (1) The equality follows at once from Lemma 3.5.
(2) By (3.1), $|x_1|+|x_2| = |\theta_i|$ whenever $\epsilon(x_1, x_2, \theta_i) \neq 0$, which implies that $x_1, x_2 \notin \mathcal{F}_i$. Thus $p_i(\epsilon(x_1, x_2, \theta_i)(u \otimes x_1)(u \otimes x_2)(u \otimes \hat{\theta}_i)) = 0$ by definition.
(3) By Lemma 3.6, we have

$$\begin{aligned}\epsilon(\hat{x}_3)\epsilon(x_1, x_2, \hat{x}_3) &= \epsilon(\hat{x}_3, x_3, w)\epsilon(x_1, x_2, \hat{x}_3)\\ &= \epsilon(x_1, \hat{x}_1, w)\epsilon(x_2, x_3, \hat{x}_1)\\ &= \epsilon(x_1)\epsilon(x_2, x_3, \hat{x}_1)\end{aligned}$$

where $\hat{\hat{x}}_3 = x_3$.
(4) We only prove the $|x_1| = |x_2|$ case because other cases are proved quite similarly. Since $\epsilon(x_1, x_2, \hat{x}_3) \neq 0$, we have $|x_1| + |x_2| = |\hat{x}_3| = 2n - |x_3|$ by (3.1). Since $(u \otimes x_1)(u \otimes x_2)(u \otimes x_3) = (u \otimes x_1)^2(u \otimes x_3) \neq 0$, $|u \otimes x_1| = 2n - |x_1|$ is even. Then we get that $|x_1|, |x_2|, |x_3|$ are even too. ◻

We prove a lemma which we are going to use.

Lemma 5.14 *For $x_2 \in \mathcal{F}_{i-1}$, we have*

$$\begin{aligned}&\partial_i p_i\left(\sum_{x_1, x_1' \in \widehat{\mathcal{B}}} \epsilon(x_1, x_1', \hat{x}_2)(u \otimes x_1)(u \otimes x_1')(u \otimes x_2)\right)\\ &\quad = 2\sum_{x_1 \in \mathcal{F}_i} \epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes x_1)(u \otimes x_2).\end{aligned}$$

Proof By definition, $p_i((u \otimes x_1)(u \otimes x_1')(u \otimes x_2)) \neq 0$ if and only if $x_1, x_1' \in \mathcal{F}_i$ and at least one of x_1, x_1' is θ_i. Then we have

$$\begin{aligned}&p_i\left(\sum_{x_1, x_1' \in \widehat{\mathcal{B}}} \epsilon(x_1, x_1', \hat{x}_2)(u \otimes x_1)(u \otimes x_1')(u \otimes x_2)\right)\\ &= \sum_{x_1, x_1' \in \mathcal{F}_i} \epsilon(x_1, x_1', \hat{x}_2)(u \otimes x_1)(u \otimes x_1')(u \otimes x_2)\\ &= \epsilon(\theta_i, \theta_i, \hat{x}_2)(u \otimes \theta_i)^2(u \otimes x_2)\\ &\quad + \sum_{x_1 \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(x_1, \theta_i, \hat{x}_2)(u \otimes x_1)(u \otimes \theta_i)(u \otimes x_2)\\ &\quad + \sum_{x_1' \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(\theta_i, x_1', \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1')(u \otimes x_2).\end{aligned}$$

where the last equality holds because $\mathcal{F}_i = \mathcal{F}_{i-1} \sqcup \{\theta_i, \hat{\theta}_i\}$. By Lemma 5.13 (1),

$$\begin{aligned}
&\sum_{x_1 \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(x_1, \theta_i, \hat{x}_2)(u \otimes x_1)(u \otimes \theta_i)(u \otimes x_2) \\
&\quad + \sum_{x_1' \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(\theta_i, x_1', \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1')(u \otimes x_2) \\
&= \sum_{x_1 \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1)(u \otimes x_2) \\
&\quad + \sum_{x_1' \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(\theta_i, x_1', \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1')(u \otimes x_2) \\
&= 2 \sum_{x_1 \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1)(u \otimes x_2).
\end{aligned}$$

Then since $x_2 \in \mathcal{F}_{i-1}$, we have

$$\begin{aligned}
&\partial_i p_i \left(\sum_{x_1, x_1' \in \widehat{\mathcal{B}}} \epsilon(x_1, x_1', \hat{x}_2)(u \otimes x_1)(u \otimes x_1')(u \otimes x_2) \right) \\
&= \partial_i \left(\epsilon(\theta_i, \theta_i, \hat{x}_2)(u \otimes \theta_i)^2 (u \otimes x_2) \right) \\
&\quad + 2\partial_i \left(\sum_{x_1 \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1)(u \otimes x_2) \right) \\
&= 2\epsilon(\theta_i, \theta_i, \hat{x}_2)(u \otimes \theta_i)(u \otimes x_2) \\
&\quad + 2 \sum_{x_1 \in \mathcal{F}_{i-1} \sqcup \{\hat{\theta}_i\}} \epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1)(u \otimes x_2) \\
&= 2 \sum_{x_1 \in \mathcal{F}_i} \epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes \theta_i)(u \otimes x_1)(u \otimes x_2)
\end{aligned}$$

where the second equality holds by Lemma 5.10 and the last equality holds because $\mathcal{F}_i = \mathcal{F}_{i-1} \sqcup \{\theta_i, \hat{\theta}_i\}$. Thus the proof is finished. $\square$

First, we describe $\partial_i p_i(\xi)$.

Lemma 5.15 *For $1 \leq i \leq m$, we have*

$$\partial_i p_i(\xi) = \epsilon(\hat{\theta}_i)\partial_i p_i(d\mu(\theta_i)) + 2 \sum_{\substack{x_1 \in \mathcal{F}_i \\ x_2 \in \mathcal{F}_{i-1}}} \epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes x_1)(u \otimes x_2).$$

Proof By (5.8) we have

$$\begin{aligned}\partial_i p_i(\xi) &= \partial_i p_i\left(\sum_{x_1,x_1',x_2\in\widehat{\mathcal{B}}}\epsilon(x_2)\epsilon(x_1,x_1',x_2)(u\otimes x_1)(u\otimes x_1')(u\otimes \hat{x}_2)\right)\\ &= \partial_i p_i\left(\sum_{x_1,x_1',x_2\in\widehat{\mathcal{B}}}\epsilon(\hat{x}_2)\epsilon(x_1,x_1',\hat{x}_2)(u\otimes x_1)(u\otimes x_1')(u\otimes x_2)\right)\\ &= \sum_{x_2\in\mathcal{F}_i}\left(\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}}\epsilon(\hat{x}_2)\epsilon(x_1,x_1',\hat{x}_2)(u\otimes x_1)(u\otimes x_1')(u\otimes x_2)\right)\right)\end{aligned}$$

where the second equality holds because $\hat{\hat{x}}_2 = x_2$ and the last equality holds by the definition of p_i. Since $\mathcal{F}_i = \mathcal{F}_{i-1}\sqcup\{\theta_i,\hat{\theta}_i\}$, we have

$$\begin{aligned}&\sum_{x_2\in\mathcal{F}_i}\left(\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}}\epsilon(\hat{x}_2)\epsilon(x_1,x_1',\hat{x}_2)(u\otimes x_1)(u\otimes x_1')(u\otimes x_2)\right)\right)\\ &= \sum_{x_2\in\mathcal{F}_{i-1}}\epsilon(\hat{x}_2)\left(\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}}\epsilon(x_1,x_1',\hat{x}_2)(u\otimes x_1)(u\otimes x_1')(u\otimes x_2)\right)\right)\\ &\quad+\epsilon(\hat{\theta}_i)\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}}\epsilon(x_1,x_1',\hat{\theta}_i)(u\otimes x_1)(u\otimes x_1')(u\otimes \theta_i)\right)\\ &\quad+\epsilon(\theta_i)\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}}\epsilon(x_1,x_1',\theta_i)(u\otimes x_1)(u\otimes x_1')(u\otimes \hat{\theta}_i)\right)\\ &= \sum_{x_2\in\mathcal{F}_{i-1}}\epsilon(\hat{x}_2)\left(\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}}\epsilon(x_1,x_1',\hat{x}_2)(u\otimes x_1)(u\otimes x_1')(u\otimes x_2)\right)\right)\\ &\quad+\epsilon(\hat{\theta}_i)\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}}\epsilon(x_1,x_1',\hat{\theta}_i)(u\otimes x_1)(u\otimes x_1')(u\otimes \theta_i)\right)\end{aligned}$$

where the last equality holds by Lemma 5.13 (2). By Lemma 5.14, we have

$$\sum_{x_2\in\mathcal{F}_{i-1}} \epsilon(\hat{x}_2)\left(\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}} \epsilon(x_1,x_1',\hat{x}_2)(u\otimes x_1)(u\otimes x_1')(u\otimes x_2)\right)\right)$$
$$= 2\sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}}} \epsilon(\hat{x}_2)\epsilon(\theta_i,x_1,\hat{x}_2)(u\otimes x_1)(u\otimes x_2).$$

By Lemmas 5.12 and 5.13 (3), we have

$$\epsilon(\hat{x}_2)\epsilon(\theta_i,x_1,\hat{x}_2)=\epsilon(\theta_i)\epsilon(x_1,x_2,\hat{\theta}_i)=\epsilon(x_1,x_2,\hat{\theta}_i)$$

for $x_1,x_2\in\widehat{\mathcal{B}}$. This implies that

$$\sum_{x_2\in\mathcal{F}_{i-1}} \epsilon(\hat{x}_2)\left(\partial_i p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}} \epsilon(x_1,x_1',\hat{x}_2)(u\otimes x_1)(u\otimes x_1')(u\otimes x_2)\right)\right)$$
$$= 2\sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}}} \epsilon(x_1,x_2,\hat{\theta}_i)(u\otimes x_1)(u\otimes x_2). \tag{5.12}$$

On the other hand, by Lemma 5.13 (2), we get

$$p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}} \epsilon(x_1,x_1',\hat{\theta}_i)(u\otimes x_1)(u\otimes x_1')(u\otimes\theta_i)\right)$$
$$= p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}} \epsilon(x_1,x_1',\hat{\theta}_i)(u\otimes x_1)(u\otimes x_1')(u\otimes\theta_i)\right)$$
$$-(-1)^{|\theta_i||\hat{\theta}_i|}p_i\left(\sum_{x_1,x_1'\in\widehat{\mathcal{B}}} \epsilon(x_1,x_1',\theta_i)(u\otimes x_1)(u\otimes x_1')(u\otimes\hat{\theta}_i)\right)$$
$$= p_i\,(d\mu(\theta_i)) \tag{5.13}$$

where the last equality holds by (5.9). Thus the proof is finished by combining (5.12) and (5.13). □

Hereafter, we write

$$\alpha(x_1,x_2)=\epsilon(x_1,x_2,\hat{\theta}_i)(u\otimes x_1)(u\otimes x_2)(u\otimes\theta_i). \tag{5.14}$$

Let $\partial_i p_i(d\mu(\theta_i))=\sum_{x_1\leq x_2\in\mathcal{F}_i}\hat{a}(x_1,x_2)(u\otimes x_1)(u\otimes x_2)$ for $\hat{a}(x_1,x_2)\in\mathbb{Q}$.

Lemma 5.16 *There are equalities*

$$\hat{a}(x_1, x_2) = \begin{cases} \epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i < x_1 = x_2 \in \mathcal{F}_i, \\ 4\epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i = x_1 < x_2 \in \mathcal{F}_i, \\ (-1)^{|x_1|+|x_2|} 2\epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i < x_1 < x_2 \in \mathcal{F}_i, \\ 3\epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i = x_1 = x_2 \in \mathcal{F}_i. \end{cases}$$

Proof We have

$$\begin{aligned} p_i(d\mu(\theta_i)) &= \sum_{x_1, x_2 \in \mathcal{F}_i} \alpha(x_1, x_2) \\ &= \sum_{\theta_i < x_1 = x_2 \in \mathcal{F}_i} \alpha(x_1, x_2) + \sum_{\theta_i < x_1 < x_2 \in \mathcal{F}_i} (\alpha(x_1, x_2) + \alpha(x_2, x_1)) \\ &\quad + \sum_{\theta_i = x_1 < x_2 \in \mathcal{F}_i} (\alpha(x_1, x_2) + \alpha(x_2, x_1)) + \sum_{\theta_i = x_1 = x_2 \in \mathcal{F}_i} \alpha(x_1, x_2). \end{aligned}$$

By Lemma 5.13 (1), we have $\alpha(x_1, x_2) = \alpha(x_2, x_1)$. By Lemma 5.10, we can make the following calculations, in which it suffices to assume $\alpha(x_1, x_2) \neq 0$.
(1) For $\theta_i < x_1 = x_2$, $|x_1|$, $|x_2|$, $|\theta_i|$ are even by Lemma 5.13 (4). Then we have

$$\begin{aligned} \partial_i \alpha(x_1, x_2) &= (-1)^{|x_1|+|x_2|} \epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes x_1)(u \otimes x_2) \\ &= \epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes x_1)(u \otimes x_2). \end{aligned}$$

(2) For $\theta_i = x_1 < x_2$, $|x_1|$, $|x_2|$, $|\theta_i|$ are even by Lemma 5.13 (4). Then we have

$$\begin{aligned} \partial_i (\alpha(x_1, x_2) + \alpha(x_2, x_1)) &= 2\partial_i (\alpha(x_1, x_2)) \\ &= 2\epsilon(x_1, x_2, \hat{\theta}_i) \partial_i \left((u \otimes \theta_i)^2 (u \otimes x_2) \right) \\ &= 4\epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes \theta_i)(u \otimes x_2) \\ &= 4\epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes x_1)(u \otimes x_2). \end{aligned}$$

(3) For $\theta_i < x_1 < x_2$,

$$\begin{aligned} \partial_i (\alpha(x_1, x_2) + \alpha(x_2, x_1)) &= 2\partial_i (\alpha(x_1, x_2)) \\ &= (-1)^{|x_1|+|x_2|} 2\epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes x_1)(u \otimes x_2). \end{aligned}$$

(4) For $\theta_i = x_1 = x_2$,

$$\partial_i \alpha(x_1, x_2) = 3\epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes x_1)(u \otimes x_2).$$

Thus the statement is proved. ◻

Note that we can write

$$\partial_i p_i(\xi) = \sum_{x_1 \le x_2 \in \mathcal{F}_i} a(x_1, x_2)(u \otimes x_1)(u \otimes x_2).$$

for $a(x_1, x_2) \in \mathbb{Q}$. We compute $a(x_1, x_2)$ explicitly in order to compare them with the coefficients of $(u \otimes x_1)(u \otimes x_2)$ in $\partial_i p_i(d\mu)$ later, for all $x_1 \le x_2 \in \mathcal{F}_i$.

Proposition 5.17 *For $x_1, x_2 \in \mathcal{F}_i$, we have*

$$a(x_1, x_2) = \begin{cases} 3\epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i < x_1 = x_2, \\ 6\epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i = x_1 < x_2, \\ 6\epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i < x_1 < x_2, \\ 3\epsilon(x_1, x_2, \hat{\theta}_i) & \theta_i = x_1 = x_2. \end{cases}$$

Proof By Lemmas 5.11, 5.15 and 5.16, $p_i(\xi)$ is a linear combination of $\alpha(x_1, x_2)$, where $\alpha(x_1, x_2)$ is as in (5.14). Thus it suffices to assume $\alpha(x_1, x_2) \neq 0$. By Lemma 5.10, we can make the following calculations.
(1) For $\theta_i < x_1 = x_2$, $|\theta_i|$ is even by Lemma 5.13 (4). Then since $\epsilon(\hat{\theta}_i) = 1$ by Lemma 5.12, we have

$$a(x_1, x_2) = \epsilon(\hat{\theta}_i)\hat{a}(x_1, x_2) + 2\epsilon(x_1, x_2, \hat{\theta}_i) = 3\epsilon(x_1, x_2, \hat{\theta}_i).$$

(2) For $\theta_i = x_1 < x_2$, similarly to the above, we get $\epsilon(\hat{\theta}_i) = 1$. Then we have

$$a(x_1, x_2) = \epsilon(\hat{\theta}_i)\hat{a}(x_1, x_2) + 2\epsilon(x_1, x_2, \hat{\theta}_i) = 6\epsilon(x_1, x_2, \hat{\theta}_i).$$

(3) For $\theta_i < x_1 < x_2$, we get $\epsilon(\hat{\theta}_i) = (-1)^{|\theta_i||\hat{\theta}_i|} = (-1)^{|\hat{\theta}_i|} = (-1)^{|x_1|+|x_2|}$ by Lemma 5.12 and (3.1). Then by Lemma 3.5, we have

$$\begin{aligned} a(x_1, x_2) &= \epsilon(\hat{\theta}_i)\hat{a}(x_1, x_2) + 2\epsilon(x_1, x_2, \hat{\theta}_i) + (-1)^{|u\otimes x_1||u\otimes x_2|}2\epsilon(x_2, x_1, \hat{\theta}_i) \\ &= (2((-1)^{|x_1|+|x_2|})^2 + 2 + 2)\epsilon(x_1, x_2, \hat{\theta}_i) \\ &= 6\epsilon(x_1, x_2, \hat{\theta}_i). \end{aligned}$$

(4) For $\theta_i = x_1 = x_2$, similarly to the $\theta_i < x_1 = x_2$ case, we get $\epsilon(\hat{\theta}_i) = 1$. Then we have

$$a(x_1, x_2) = \epsilon(\hat{\theta}_i)\hat{a}(x_1, x_2) = 3\epsilon(x_1, x_2, \hat{\theta}_i).$$

Thus the proof is finished. □

Next, we describe $\partial_i p_i(d\mu)$. We extend the definition of $\lambda(x)$ to $\widehat{\mathcal{B}} - \mathcal{B}_-$ as follows. Let $\lambda(x) = -\lambda(\hat{x})$ for $n < |x| < 2n$ and $\lambda(x) = 0$ for $|x| = n$.

Lemma 5.18 *For $1 \leq i \leq m$, we have*

$$\partial_i p_i(d\mu) = \lambda(\theta_i)\partial_i p_i(d\mu(\theta_i)) + 2 \sum_{\substack{x_1 \in \mathcal{F}_i \\ x_2 \in \mathcal{F}_{i-1}}} (-1)^{|x_2||\hat{x}_2|} \lambda(x_2)\epsilon(x_1, x_2, \hat{\theta}_i)(u \otimes x_1)(u \otimes x_2).$$

Proof By definition, we have

$$\begin{aligned}
\partial_i p_i(d\mu) &= \partial_i p_i \left(\sum_{x_2 \in \mathcal{B}_-} \lambda(x_2) d\mu(x_2) \right) \\
&= \sum_{x_2 \in \mathcal{F}_i \cap \mathcal{B}_-} \lambda(x_2)\partial_i p_i\, (d\mu(x_2)) \\
&= \lambda(\theta_i)\partial_i p_i(d\mu(\theta_i)) + \sum_{x_2 \in \mathcal{F}_{i-1} \cap \mathcal{B}_-} \lambda(x_2)\partial_i p_i(d\mu(x_2)).
\end{aligned}$$

where the second equality holds by the definition of p_i and the last equality holds because $\mathcal{F}_i \cap \mathcal{B}_- = (\mathcal{F}_{i-1} \cap \mathcal{B}_-) \sqcup \{\theta_i\}$. Let $x_2 \in \mathcal{F}_{i-1} \cap \mathcal{B}_-$, and let

$$\begin{aligned}
A(x_2) &= \sum_{x_1, x_1' \in \widehat{\mathcal{B}}} \epsilon(x_1, x_1', \hat{x}_2)(u \otimes x_1)(u \otimes x_1')(u \otimes x_2), \\
B(x_2) &= \sum_{x_1, x_1' \in \widehat{\mathcal{B}}} \epsilon(x_1, x_1', x_2)(u \otimes x_1)(u \otimes x_1')(u \otimes \hat{x}_2).
\end{aligned}$$

Then $d\mu(x_2) = A(x_2) - (-1)^{|x_2||\hat{x}_2|} B(x_2)$ by (5.9). Since $x_2, \hat{x}_2 \in \mathcal{F}_{i-1}$, by Lemma 5.14, we have

$$\begin{aligned}
\partial_i p_i(A(x_2)) &= 2 \sum_{x_1 \in \mathcal{F}_i} \epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes x_1)(u \otimes x_2), \\
\partial_i p_i(B(x_2)) &= 2 \sum_{x_1 \in \mathcal{F}_i} \epsilon(\theta_i, x_1, x_2)(u \otimes x_1)(u \otimes \hat{x}_2).
\end{aligned}$$

Then we have

$$\begin{aligned}
\partial_i p_i(d\mu(x_2)) &= \partial_i p_i \left(A(x_2) - (-1)^{|x_2||\hat{x}_2|} B(x_2) \right) \\
&= 2 \sum_{x_1 \in \mathcal{F}_i} \Big(\epsilon(\theta_i, x_1, \hat{x}_2)(u \otimes x_1)(u \otimes x_2) \\
&\qquad - (-1)^{|x_2||\hat{x}_2|} \epsilon(\theta_i, x_1, x_2)(u \otimes x_1)(u \otimes \hat{x}_2) \Big).
\end{aligned}$$

By Lemmas 5.12 and 5.13 (3), we have

$$\epsilon(\theta_i, x_1, \hat{x}_2) = \frac{\epsilon(\theta_i)}{\epsilon(\hat{x}_2)}\epsilon(x_1, x_2, \hat{\theta}_i) = (-1)^{|x_2||\hat{x}_2|}\epsilon(x_1, x_2, \hat{\theta}_i),$$

$$\epsilon(\theta_i, x_1, x_2) = \frac{\epsilon(\theta_i)}{\epsilon(x_2)}\epsilon(x_1, \hat{x}_2, \hat{\theta}_i) = \epsilon(x_1, \hat{x}_2, \hat{\theta}_i).$$

Then we have

$$\begin{aligned}\partial_i p_i(d\mu(x_2)) = 2\sum_{x_1\in\mathcal{F}_i}(-1)^{|x_2||\hat{x}_2|}\Big(&\epsilon(x_1, x_2, \hat{\theta}_i)(u\otimes x_1)(u\otimes x_2)\\ &- \epsilon(x_1, \hat{x}_2, \hat{\theta}_i)(u\otimes x_1)(u\otimes \hat{x}_2)\Big).\end{aligned}$$

Write $\beta(x_1, x_2) = (-1)^{|x_2||\hat{x}_2|}\epsilon(x_1, x_2, \hat{\theta}_i)(u\otimes x_1)(u\otimes x_2)$. Then we get

$$\begin{aligned}&\sum_{x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}\lambda(x_2)\partial_i p_i(d\mu(x_2))\\ &\quad= 2\sum_{x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}\left(\lambda(x_2)\sum_{x_1\in\mathcal{F}_i}(-1)^{|x_2||\hat{x}_2|}\epsilon(x_1, x_2, \hat{\theta}_i)(u\otimes x_1)(u\otimes x_2)\right.\\ &\quad\left.- \lambda(x_2)\sum_{x_1\in\mathcal{F}_i}(-1)^{|\hat{x}_2||x_2|}\epsilon(x_1, \hat{x}_2, \hat{\theta}_i)(u\otimes x_1)(u\otimes \hat{x}_2)\right)\\ &= 2\sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}}\big(\lambda(x_2)\beta(x_1, x_2) - \lambda(x_2)\beta(x_1, \hat{x}_2)\big).\end{aligned} \tag{5.15}$$

Let

$$C = \sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}}\lambda(x_2)\beta(x_1, x_2),$$

$$D = \sum_{\substack{x_1\in\mathcal{F}_i\\ \hat{x}_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}}\lambda(x_2)\beta(x_1, x_2),$$

$$E = \sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{B}_n}}\lambda(x_2)\beta(x_1, x_2).$$

Since $\hat{\hat{x}}_2 = x_2$ and $\lambda(\hat{x}_2) = -\lambda(x_2)$,

$$
\begin{aligned}
-\sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}} \lambda(x_2)\beta(x_1,\hat{x}_2) &= \sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}} \lambda(\hat{x}_2)\beta(x_1,\hat{x}_2)\\
&= \sum_{\substack{x_1\in\mathcal{F}_i\\ \hat{x}_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}} \lambda(x_2)\beta(x_1,x_2)\\
&= D.
\end{aligned}
$$

Since $\lambda(x_2) = 0$ for $x_2 \in \mathcal{B}_n$, we have $E = 0$. it follows from (5.15) that

$$
\begin{aligned}
&= \sum_{x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-} \lambda(x_2)\partial_i p_i(d\mu(x_2))\\
&= 2\sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}} \lambda(x_2)\beta(x_1,x_2) + 2\sum_{\substack{x_1\in\mathcal{F}_i\\ \hat{x}_2\in\mathcal{F}_{i-1}\cap\mathcal{B}_-}} \lambda(x_2)\beta(x_1,x_2)\\
&\quad + 2\sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{B}_n}} \lambda(x_2)\beta(x_1,x_2)\\
&= 2C + 2D + 2E.
\end{aligned}
$$

On the other hand, since

$$
\mathcal{F}_{i-1} = (\mathcal{F}_{i-1}\cap\mathcal{B}_-) \sqcup \mathcal{B}_n \sqcup \{x_2 \in \widehat{\mathcal{B}} \mid \hat{x}_2 \in \mathcal{F}_{i-1}\cap\mathcal{B}_-\},
$$

we have

$$
\begin{aligned}
C + D + E &= \sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}}} \lambda(x_2)\beta(x_1,x_2)\\
&= \sum_{\substack{x_1\in\mathcal{F}_i\\ x_2\in\mathcal{F}_{i-1}}} (-1)^{|x_2||\hat{x}_2|}\lambda(x_2)\epsilon(x_1,x_2,\hat{\theta}_i)(u\otimes x_1)(u\otimes x_2).
\end{aligned}
$$

Thus the proof is finished. □

We show a property of $\lambda(x)$ that we are going to use.

Lemma 5.19 *If $x_1, x_2, x_3 \in \widehat{\mathcal{B}}$ satisfy $|x_1| + |x_2| + |x_3| = 2n$, then*

$$
\sum_{i=1}^{3} \epsilon(\hat{x}_i)\epsilon(x_i)\lambda(x_i) = 3.
$$

Proof We may assume $|x_1| \le |x_2| \le |x_3|$. Let $x \in \widehat{\mathcal{B}}$. By Lemma 5.12, we have that for $0 < |x| < n$,

$$\epsilon(\hat{x})\epsilon(x)\lambda(x) = \epsilon(\hat{x})^2\epsilon(x)\frac{3(n-|x|)}{n} = \frac{3(n-|x|)}{n}$$

and for $n < |x| < 2n$, we have

$$\epsilon(\hat{x})\epsilon(x)\lambda(x) = -\epsilon(\hat{x})\epsilon(x)^2\frac{3(n-|\hat{x}|)}{n} = -\frac{3(n-2n+|x|)}{n} = \frac{3(n-|x|)}{n}.$$

Then for $|x_1| \le |x_2| \le |x_3| < n$, we have

$$\sum_{i=1}^{3}\epsilon(\hat{x}_i)\epsilon(x_i)\lambda(x_i) = \frac{3}{n}(3n-|x_1|-|x_2|-|x_3|) = 3.$$

For $|x_1| \le |x_2| \le |x_3| = n$, we have $|x_1| + |x_2| = n$ and $\lambda(x_3) = 0$. Then

$$\sum_{i=1}^{3}\epsilon(\hat{x}_i)\epsilon(x_i)\lambda(x_i) = \frac{3}{n}(2n-|x_1|-|x_2|) = 3.$$

For $|x_1| \le |x_2| < n < |x_3|$, we have

$$\sum_{i=1}^{3}\epsilon(\hat{x}_i)\epsilon(x_i)\lambda(x_i) = \frac{3}{n}(3n-|x_1|-|x_2|-|x_3|) = 3.$$

Thus the proof is finished. □

Now we are ready to prove Proposition 5.7.

Proof of Proposition 5.7 As mentioned above, it is sufficient to prove (5.11) for $1 \le i \le m$. We can write

$$\partial_i p_i(d\mu) = \sum_{x_1 \le x_2 \in \mathcal{F}_i} b(x_1, x_2)(u \otimes x_1)(u \otimes x_2)$$

for $b(x_1, x_2) \in \mathbb{Q}$. Then we aim to show $a(x_1, x_2) = b(x_1, x_2)$ for all $x_1 \le x_2 \in \mathcal{F}_i$, where $a(x_1, x_2)$ is as in Proposition 5.17. By Lemmas 5.11, 5.16 and 5.18, $p_i(d\mu)$ is a linear combination of $\alpha(x_1, x_2)$ where $\alpha(x_1, x_2)$ is as in (5.14). Thus it suffices to assume $\alpha(x_1, x_2) \ne 0$. Since $|x_1| + |x_2| + |\theta_i| = 2n$ by (3.1), we have

$$\epsilon(x_1)\epsilon(\hat{x}_1)\lambda(x_1) + \epsilon(x_2)\epsilon(\hat{x}_2)\lambda(x_2) + \epsilon(\theta_i)\epsilon(\hat{\theta}_i)\lambda(\theta_i) = 3$$

by Lemma 5.19. Then by Lemma 5.10, we can make the following calculations.

(1) For $\theta_i < x_1 = x_2, x_1, x_2, \theta_i \notin \mathcal{B}_n$ since $|x_1|+|x_2|+|\theta_i| = 2n$. Since $|x_1|, |x_2|, |\theta_i|$ are even by Lemma 5.13 (4), $\epsilon(x)\epsilon(\hat{x})\lambda(x) = \lambda(x)$ for $x = x_1, x_2, \theta_i$ by Lemma 5.12. Then we get

$$\begin{aligned}
b(x_1, x_2) &= \lambda(\theta_i)\hat{a}(x_1, x_2) + 2\lambda(x_2)\epsilon(x_1, x_2, \hat{\theta}_i) \\
&= (\epsilon(\hat{x}_1)\epsilon(x_1)\lambda(x_1) + \epsilon(\hat{x}_2)\epsilon(x_2)\lambda(x_2) + \epsilon(\hat{\theta}_i)\epsilon(\theta_i)\lambda(\theta_i))\epsilon(x_1, x_2, \hat{\theta}_i) \\
&= 3\epsilon(x_1, x_2, \hat{\theta}_i).
\end{aligned}$$

(2) For $\theta_i = x_1 < x_2, \theta_i = x_1 \notin \mathcal{B}_n$ since $|x_1|+|x_2|+|\theta_i| = 2n$. If $x_2 \notin \mathcal{B}_n$, similarly to the above, we have $\epsilon(x)\epsilon(\hat{x})\lambda(x) = \lambda(x)$ for $x = x_1, x_2, \theta_i$. If $x_2 \in \mathcal{B}_n$, we also have $\epsilon(x)\epsilon(\hat{x})\lambda(x) = \lambda(x)$ for $x = x_1, x_2, \theta_i$ since $\lambda(x_2) = 0$. Then we get

$$\begin{aligned}
b(x_1, x_2) &= \lambda(\theta_i)\hat{a}(x_1, x_2) + 2\lambda(x_2)\epsilon(x_1, x_2, \hat{\theta}_i) \\
&= 2(\epsilon(\hat{x}_1)\epsilon(x_1)\lambda(x_1) + \epsilon(\hat{x}_2)\epsilon(x_2)\lambda(x_2) + \epsilon(\hat{\theta}_i)\epsilon(\theta_i)\lambda(\theta_i))\epsilon(x_1, x_2, \hat{\theta}_i) \\
&= 6\epsilon(x_1, x_2, \hat{\theta}_i).
\end{aligned}$$

(3) For $\theta_i < x_1 < x_2$, we have

$$\begin{aligned}
&(-1)^{|x_2||\hat{x}_2|}2\lambda(x_2)\epsilon(x_1, x_2, \hat{\theta}_i) + (-1)^{|u\otimes x_1||u\otimes x_2|+|x_1||\hat{x}_1|}2\lambda(x_1)\epsilon(x_2, x_1, \hat{\theta}_i) \\
&= (-1)^{|x_2||\hat{x}_2|}2\lambda(x_2)\epsilon(x_1, x_2, \hat{\theta}_i) + (-1)^{|x_1||\hat{x}_1|}2\lambda(x_1)\epsilon(x_1, x_2, \hat{\theta}_i) \\
&= 2(\epsilon(\hat{x}_1)\epsilon(x_1)\lambda(x_1) + \epsilon(\hat{x}_2)\epsilon(x_2)\lambda(x_2))\epsilon(x_1, x_2, \hat{\theta}_i)
\end{aligned}$$

by Lemmas 3.5 and 5.12. Then we get

$$\begin{aligned}
b(x_1, x_2) &= 2(\epsilon(\hat{x}_1)\epsilon(x_1)\lambda(x_1) + \epsilon(\hat{x}_2)\epsilon(x_2)\lambda(x_2))\epsilon(x_1, x_2, \hat{\theta}_i) + \lambda(\theta_i)\hat{a}(x_1, x_2) \\
&= 2(\epsilon(\hat{x}_1)\epsilon(x_1)\lambda(x_1) + \epsilon(\hat{x}_2)\epsilon(x_2)\lambda(x_2) + \epsilon(\hat{\theta}_i)\epsilon(\theta_i)\lambda(\theta_i))\epsilon(x_1, x_2, \hat{\theta}_i) \\
&= 6\epsilon(x_1, x_2, \hat{\theta}_i).
\end{aligned}$$

(4) For $\theta_i = x_1 = x_2$, $|\theta_i|$ is even and $3|\theta_i| = 2n$ by (3.1) and Lemma 5.13 (4) respectively. Then we get

$$\lambda(\theta_i) = \frac{3(n - \frac{2n}{3})}{n} = 1 \quad \text{and} \quad b(x_1, x_2) = \lambda(\theta_i)\hat{a}(x_1, x_2) = 3\epsilon(x_1, x_2, \hat{\theta}_i).$$

Thus by Proposition 5.17, we obtain $a(x_1, x_2) = b(x_1, x_2)$ for all $x_1 \leq x_2 \in \mathcal{F}_i$, completing the proof. $\square$

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