On the centers of polygons for Blaschke-like maps

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Abstract

Let E be the ellipse which is the boundary of the domain of definition of a Blaschke-like map B_φ. We show that the locus of the mean center of the preimages of λ under the map B_φ is an ellipse similar to E as λ ranges over E. This is an analog of the result for Poncelet polygons first found experimentally by Shestakov in 1814 and proved by Schwartz and Tabachnikov in 2016.

1 Introduction

1.1 Poncelet polygons

Let $E_2 \subset E_1$ be a pair of nested ellipses and P an n-sided polygon inscribed in E_1 and circumscribed about E_2. Poncelet's theorem guarantees that for any point of E_1, there exists an n-sided polygon inscribed in E_1 and circumscribed about E_2, which has this point as one of its vertices. Such an n-sided polygon is called an *n-sided Poncelet polygon.* For each n-sided Poncelet polygon P, the following "centers" can be considered.

(1) The mean center $C_v(P)$ (the center of mass of the vertices).

(2) The center of mass of the edges, $C_e(P)$.

(3) The centroid $C_l(P)$ (the center of mass of P, where P is considered as a homogeneous lamina).

Each center is obtained as follows.

Lemma 1 (cf. [ST16, Lemma 0.1])
Let P be an n-sided Poncelet polygon defined by connecting points $z_1, z_2, \cdots, z_n, (z_1)$ with line segments in this order. Then, each center is given as follows.

(1) The mean center $C_v(P)$ is the point that is equidistant from each vertex.

$$C_v(P) = \frac{z_1 + z_2 + \cdots + z_n}{n}.$$

(2) The center of mass of the edges $C_e(P)$ is given by

$$C_e(P) = \frac{l_1(z_1 + z_2) + l_2(z_2 + z_3) + \cdots + l_n(z_n + z_1)}{2(l_1 + \cdots + l_n)},$$

where $l_k = |z_{k+1} - z_k|$ $(k = 1, \cdots, n)$.

(3) The centroid $C_l(P)$ is given as follows.

If P is divided into triangles $T_1, \cdots, T_N$, and the area and the centroid of each T_k are A_k and c_k, respectively, then $C_l(P)$ is given by

$$C_l(P) = \frac{c_1 A_1 + \cdots c_N A_N}{A_1 + \cdots + A_N}.$$

Note that the centroid of each triangle T_k coincides with the mean center $C_v(T_k)$.

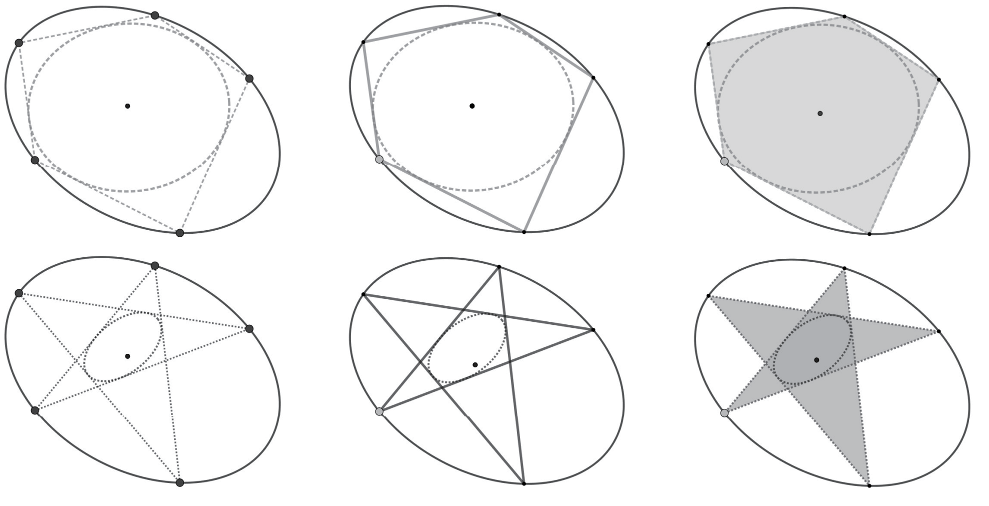

Figure 1: The three centers for a Poncelet pentagon (upper) and a pentagram (lower). The black dot in each ellipse indicates, from left to right, C_v, C_e, and C_l.

The following theorem is an interpretation of Shestakov's theorem.

Theorem 2 ([ST16, Theorem 1])
Let $E_2 \subset E_1$ be a pair of nested ellipses that admit a 1-parameter family of n-sided Poncelet polygons P_t. Then both loci $C_v(P_t)$ and $C_l(P_t)$ are ellipses similar to E_1 or single points.

Remark 1
In a letter to his brother in 1814, Shestakov had wrote that he obtained this theorem experimentally. According to the letter, he met and was inspired by Poncelet, who was a prisoner of war in Russia during the Napoleonic Wars. See [ST16] for more details.

1.2 Blaschke products

First, consider the locus of the "center" for the polygon constructed by the preimages of Blaschke products.

Let B be a Blaschke product of degree d,

$$B(z) = e^{i\theta} \frac{z - a_1}{1 - \overline{a_1} z} \cdots \frac{z - a_d}{1 - \overline{a_d} z}, \quad (a_k \in \mathbb{D},\ \theta \in \mathbb{R}).$$

In the case that $\theta = 0$ and $B(0) = 0$, B is called *canonical*.

By changing the coordinates using a Möbius transformation, we can see that it is sufficient to consider only the canonical Blaschke products to study the geometric properties of the preimages of the Blaschke products. A Blaschke product B is a holomorphic function on $\mathbb{D}$, is continuous on $\overline{\mathbb{D}}$, and maps $\mathbb{D}$ onto itself. Moreover, the derivative of B has no zeros on $\partial\mathbb{D}$ (see, for instance [Mas13]). So, there are d distinct preimages of $\lambda \in \partial\mathbb{D}$ by B.

Let $z_1, \cdots, z_d$ be the d distinct preimage of $\lambda \in \partial\mathbb{D}$ by B and l_λ the set of lines joining z_j and z_k with $j \neq k$. Then, the envelope I_B of the family of lines $\{l_\lambda\}_{\lambda \in \partial\mathbb{D}}$ is called the *interior curve associated with B*.

For a Blaschke product B of degree 3, the interior curve associated with B is an ellipse [DGM02], and the triangle formed by the preimages $B^{-1}(\lambda)$ with $\lambda \in \partial\mathbb{D}$ is exactly the Poncelet triangle [Fra04]. For $d > 3$, the interior curve associated with a Blaschke product of degree d is not always an ellipse (cf.

[Fuj13]). However, as λ ranges over the unit circle, the locus of the mean center for the preimages $B^{-1}(\lambda)$ forms a circle.

Theorem 3
Let $z_1, \cdots, z_d$ be the d distinct preimages of $\lambda \in \partial\mathbb{D}$ by B. As λ ranges over $\partial\mathbb{D}$, the center of mass $w = (z_1 + \cdots + z_d)/d$ of d-sided polygon with vertices $z_1, \cdots, z_d$, forms a circle or a single point.

We will show this result in Section 3. Moreover, for the Blaschke-like maps (see Section 3.2 for definition), we obtain the following result which is an analog of Theorem 2.

Corollary 4
Let $z_1, \cdots, z_d$ be the d distinct preimages of $\widetilde{\lambda} \in \partial\mathbb{E}_t$ by B_{φ_t}. As $\widetilde{\lambda}$ ranges over $\partial\mathbb{E}_t$, the center of mass $w = (z_1 + \cdots + z_d)/d$ of d-sided polygon with vertices $z_1, \cdots, z_d$, forms an ellipse which is similar to $\partial\mathbb{E}_t$ or a single point.

2 Computer experiments

Let P_λ be the d-sided polygon formed by the preimages of $\lambda \in \partial\mathbb{D}$ by Blaschke product B of degree d. Note that we will only consider convex or star polygons here.

For a Blaschke product B, let $C_v^B = \{C_v(P_\lambda)\,;\, \lambda \in \partial\mathbb{D}\}$, $C_e^B = \{C_e(P_\lambda)\,;\, \lambda \in \partial\mathbb{D}\}$, and $C_l^B = \{C_l(P_\lambda)\,;\, \lambda \in \partial\mathbb{D}\}$.

Figure 2 indicates examples of the loci C_v^B, C_e^B, and C_l^B for Blaschke products of degree 4. These images are drawn using GeoGebra [1)]. For a Blaschke product B of degree 4, the 4 vertices $z_1, \cdots, z_4 \in \partial\mathbb{D}$ of the polygon are determined by the preimages $B^{-1}(\lambda)$, and are given as the solution of the following equation

$$z(z-a_1)(z-a_2)(z-a_3) = \lambda(1-\overline{a_1}z)(1-\overline{a_2}z)(1-\overline{a_3}z), \quad \lambda \in \mathbb{D}.$$

So in general, a quadrilateral with vertices $z_1, \cdots, z_4$ connected by line segments in this order is not a convex. Therefore, we use the following procedure for drawing images by GeoGebra.

- For each $\lambda \in \partial\mathbb{D}$, calculate the preimages $z_1, \cdots, z_4$ using the "CSolutions" command in CAS.
  Note that each z_k can be written as $e^{i\theta_k}$ for some θ_k.
- These points $z_1, \cdots, z_4$ are not aligned clockwise or counter-clockwise on the unit circle.
- Use the `Sort(Arg(`z_1`),`$\cdots$`,Arg(`z_4`))` command to sort the arguments of $z_1, \cdots z_4$ in ascending order:
  $$\theta_1 < \theta_2 < \cdots < \theta_4 (< \theta_1 + 2\pi).$$
- Then, the points $e^{i\theta_1}, \cdots, e^{i\theta_4}$ are aligned counter-clockwise on the unit circle.

This procedure is also valid for Blaschke products of degree $d > 4$.

From Figures 2 and 3, it is natural to conjecture that the locus C_v is a circle. We can see that neither C_e^B nor C_l^B is a conic.

3 Results for Blaschke products and Blaschke-like maps

3.1 Blaschke polygons

Here, we will show Theorem 3 for the Blaschke product of degree $d \geq 3$.

Remark 2
The mean centers of a convex polygon and a star-shaped polygon defined by the same vertices are the same.

[1)] GeoGebra is an interactive mathematics software (https://www.geogebra.org/).

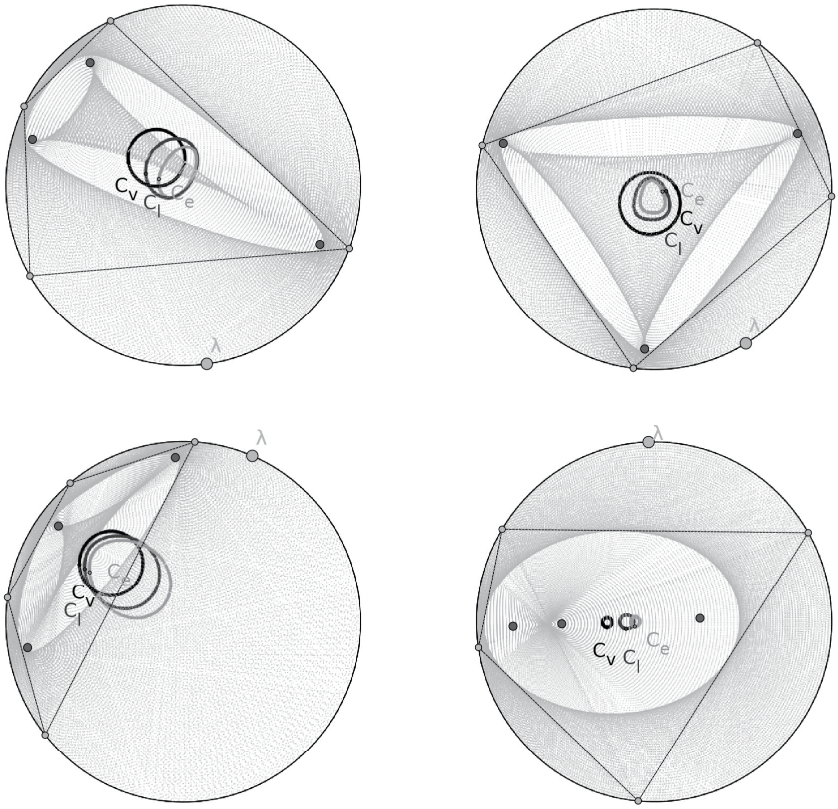

Figure 2: The case of Blaschke products B of degree 4: For each Blaschke product, the three loci C_v^B, C_e^B, C_l^B are drawn. Each disk is the unit disk, and the quadrilateral is obtained by connecting the preimages $B^{-1}(\lambda)$ of a point λ on the unit circle. The envelope coincides with the interior curve and the three black dots represent the zero points of B except the origin.

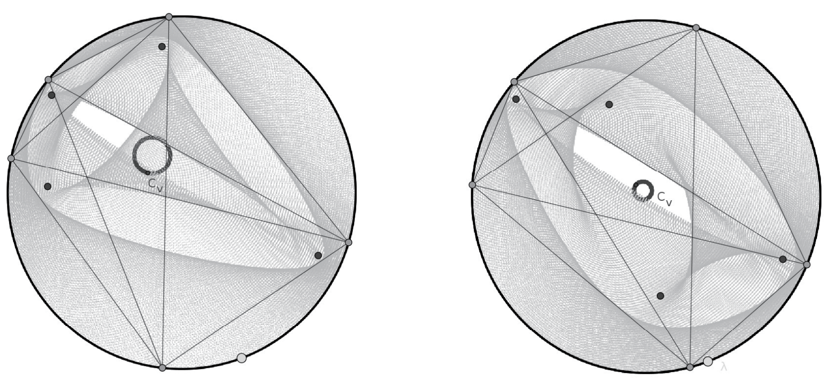

Figure 3: The case of Blaschke product B of degree 5: Each circle in the thick line indicates the locus C_v^B. Every pair of pentagon and pentagram is obtained by connecting the preimages $B^{-1}(\lambda)$ of a point λ on the unit circle. So, they share all vertices in common. Thus, their mean centers coincide.

Proof of Theorem 3. Let $B(w) = w\prod_{k=1}^{d-1}\frac{w-a_k}{1-\overline{a}_k w}$, and C_v^B the locus of the centers of mass for the vertices of each polygon given by $B^{-1}(\lambda)$ with $\lambda \in \partial\mathbb{D}$.

For $\lambda \in \partial\mathbb{D}$, let $w_1, \cdots, w_d \in \partial\mathbb{D}$ be the d distinct points mapped to λ under B. As the points $w_1, \cdots, w_d$ are the solution of $B(w) = \lambda$, we have

$$(w-w_1)(w-w_2)\cdots(w-w_d) = w(w-a_1)\cdots(w-a_{d-1}) - \lambda(1-\overline{a}_1 w)\cdots(1-\overline{a}_{d_1} w).$$

Set $\zeta = (w_1 + \cdots + w_d)/d$. Comparing each term of degree $d-1, 1$, and 0 for w in the above equation, we have

$$\zeta d = a_1 + \cdots + a_{d-1} + (-1)^{d-1}\overline{a}_1\cdots\overline{a}_{d-1}\lambda, \tag{1}$$

$$(-1)^{d-1}w_1\cdots w_d\Big(\frac{1}{w_1}+\cdots+\frac{1}{w_d}\Big) = (-1)^{d-1}a_1\cdots a_{d-1} + (\overline{a}_1+\cdots+\overline{a}_{d-1})\lambda, \tag{2}$$

$$(-1)^{d-1}w_1\cdots w_d = \lambda. \tag{3}$$

From (2) and (3),

$$\lambda(\overline{w_1}+\cdots+\overline{w_d}) = (-1)^{d-1}a_1\cdots a_{d-1} + (\overline{a}_1+\cdots+\overline{a}_{d-1})\lambda$$

holds. Dividing both sides of the above equation by λ, and substituting (1), we have

$$\overline{\zeta} d = (-1)^{d-1}a_1\cdots a_{d-1}\overline{\lambda} + (\overline{a}_1+\cdots+\overline{a}_{d-1}). \tag{4}$$

From $|\lambda|^2 = \lambda\overline{\lambda} = 1$, eliminating λ from (1) and (4), we have

$$\left|\zeta - \frac{a_1+\cdots+a_{d-1}}{d}\right| = \left|\frac{a_1\cdots a_{d-1}}{d}\right|, \tag{5}$$

and ζ is on the circle with center $(a_1+\cdots+a_{d-1})/d$ and radius $|a_1\cdots a_{d-1}|/d$. ∎

Remark 3
Note that C_v^B degenerates to a single point if and only if the Blaschke product B has multiple zeros at the origin.

3.2 Blaschke-like polygons

Let φ_t be the following Joukowski transformation

$$\varphi_t(w) = \frac{1}{1+t^2}\Big(t^2 w + \frac{1}{w}\Big), \quad 0 < t < 1.$$

φ_t conformally maps $\mathbb{D}$ onto the exterior of the elliptical disk $\mathbb{E}_t$, where

$$\mathbb{E}_t = \Big\{\Big|z - \frac{2t}{1+t^2}\Big| + \Big|z + \frac{2t}{1+t^2}\Big| < 2\Big\}.$$

For a canonical Blaschke product B of degree d, let $B_{\varphi_t} = \varphi_t \circ B \circ \varphi_t^{-1}$. The map B_{φ_t} maps the exterior of the elliptic disk $\widehat{\mathbb{C}} \setminus \overline{\mathbb{E}}_t$ onto itself. We call B_{φ_t} a *Blaschke-like map associated with B and φ_t*. See [FG23] for detail.

Here, we consider the locus $C_v^{B_{\varphi_t}}$ for the Blaschke-like map associated with B and φ_t. As a corollary of Theorem 3, we obtain a result that is analog to Theorem 2. We describe below the outline of the proof of Corollary 4.

Proof of Corollary 4. On $\partial\mathbb{D}$, φ_t is written as

$$z = \varphi_t(w) = \frac{t^2 w + \overline{w}}{1+t^2}, \quad w \in \partial\mathbb{D}.$$

That is,

$$\begin{array}{cccc} \varphi_t|_{\partial\mathbb{D}}: & \partial\mathbb{D} & \longrightarrow & \partial\mathbb{E}_t \\ & \Psi & & \Psi \\ & x+iy & \longmapsto & x + i\frac{t^2-1}{1+t^2}y \end{array}.$$

Therefore, the locus $C_v^{B_{\varphi_t}}$ is obtained by contracting the above circle (5) in the horizontal direction.

Since the outer unit circle is also horizontally contracted in the same ratio and transforms into an ellipse, these two ellipses are similar. ∎

Acknowledgements

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