SPHERICAL CONVEX BODY OF CONSTANT WIDTH AND ITS APPROXIMATION

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1. Basic definitions

Throughout this note, let $\mathbb{S}^n$ denote the unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For any given point P of $\mathbb{S}^n$, we denote by $H(P)$ the *hemisphere* whose center is P, namely,

$$H(P) = \{Q \in \mathbb{S}^n \mid P \cdot Q \geq 0\}.$$

Here the dot in the center stands for the scalar product of P, Q in $\mathbb{R}^{n+1}$. A non-empty subset W of $\mathbb{S}^n$ is *hemispherical* if there exists a point P of $\mathbb{S}^n$ such that the intersection set $W \cap H(P)$ is the empty set. A hemispherical set W of $\mathbb{S}^n$ is said to be *spherical convex* if the arc PQ between any two points $P, Q \in W$ lies in the W. Here

$$PQ = \left\{ \frac{tP + (1-t)Q}{|| \, tP + (1-t)Q \, ||} \in \mathbb{S}^n \mid 0 \leq t \leq 1 \right\}.$$

Denote the boundary of W is denoted by ∂W. A spherical convex set W of $\mathbb{S}^n$ is said to be *spherical convex body* if W has an interior point and closed. For any subset W of $\mathbb{S}^n$, the *spherical polar set* of W, denoted by W°, is the following set,

$$\bigcap_{P \in W} H(P).$$

For any non-empty closed hemispherical subset $W \subset \mathbb{S}^n$, the equality s-conv$(W) =$ (s-conv$(W))^{\circ\circ}$ holds ([15]), where s-conv(W) is the *spherical convex hull of* W, that is, the smallest spherical convex body contains W. The *diameter* of a spherical convex body W is defined by

$$\max\{\mid PQ \mid \mid P, Q \in W\}.$$

A spherical convex body W is said to be *constant diameter* τ, if the diameter of K is τ, and for every point $P \in \partial W$ there exists a point Q of ∂W such that $|PQ| = \tau$ ([12]). We say a hemisphere $H(Q)$ *supports* W *at* P if W is a subset of $H(Q)$ and P is a point of $\partial W \cap \partial H(Q)$. The hemisphere $H(Q)$ as defined above is called a *supporting hemisphere* of W at P. For any two points P, Q of $\mathbb{S}^n$, the intersection

$$H(P) \cap H(Q)$$

is called a *lune*, where $P \neq -Q$. The *thickness* of lune $H(P) \cap H(Q)$ is the real number $\pi - \mid PQ \mid$, denoted by $\Delta(H(P) \cap H(Q))$. Namely,

$$\Delta(H(P) \cap H(Q)) = \pi - \mid PQ \mid .$$

2010 *Mathematics Subject Classification.* 52A30.

Key words and phrases. Constant width, approximation, spherical convex body, spherical polytope, Wulff shape, self-dual Wulff shapes.

It is clear that thickness of any lune is greater than 0 and less than π. Let $H(P)$ be a supporting hemisphere of a spherical convex body W. The *width* of W with respect to $H(P)$ is defined by ([9])

$$\mathrm{width}_{H(P)}(K) = \min\{\Delta(H(P) \cap H(Q)) \mid W \subset H(Q)\}.$$

The minimum width of W is called *thickness of* W, denoted by ΔW. Following [9], a spherical convex body W is said to be *of constant width*, if all widths of W with respect to any supporting hemispheres $H(P)$ are equal; and a convex body W of $\mathbb{S}^n$ is said to be *reduced* if $\Delta(X) < \Delta(W)$ for every convex body X properly contained in W.

2. Some results

2.1. Width of spherical convex bodies.

Lemma 2.1 ([15]). *The subset $\mathcal{P}$ is a spherical polytope if and only if $\mathcal{P}^\circ$ is a spherical polytope.*

Lemma 2.2 ([12]). *Every spherical convex body of constant width smaller than $\pi/2$ on S^n is strictly convex.*

Lemma 2.3 ([8]). *Let C be a spherical convex body in S^n, and $0 < \tau < \pi$. The following two statements are equivalent:*

1. *C is of constant width τ;*
2. *C° is of constant width $\pi - \tau$.*

An alternative proof of Lemma 2.3 in the case of $\mathbb{S}^2$ is given in [14].

Theorem 1 ([8, 13]). *Let C be a spherical convex body in S^n, and $0 < \tau < \pi$. The following two statements are equivalent:*

1. *C is of constant diameter τ;*
2. *C is of constant width τ.*

Then by Theorem 1 and Lemma 2.3 we have the following:

Corollary 2.1 ([8]). *Let C be a spherical convex body in $\mathbb{S}^n$, and $0 < \tau < \pi$. The following two propositions are equivalent:*

1. *C is of constant diameter τ;*
2. *C° is of constant diameter $\pi - \tau$.*

We say a spherical convex body C is *self-dual* if $C = C^\circ$. A characterization of self-dual spherical convex body is given as follows:

Theorem 2 ([6]). *Let C be a spherical convex body. Then the following two statements are equivalent:*

1. *C is self dual;*
2. *C is of constant width $\pi/2$.*

Let $\mathcal{P}$ be a spherical polytope. By Lemma 2.4, its dual $\mathcal{P}^\circ$ is a spherical polytope. By Lemma 2.3, if $\mathcal{P}$ is of constant width τ if and only if $\mathcal{P}^\circ$ is of constant width $\pi - \tau$ for any $0 < \tau < \pi$. Then by Lemma 2.2, we have the following fact:

Lemma 2.4 ([2]). *Let $\mathcal{P}$ be a hemispherical convex polytope of constant width τ. Then $\tau = \pi/2$.*

Then by Lemma 2.4, Theorem 2 and Theorem 1, we know that the condition $\pi/2$ of Theorem 2 is unnecessary if $\mathcal{P}$ is a convex polytope:

Theorem 3 ([2]). *Let Wulff shape $\mathcal{P}$ be a spherical convex polytope. Then the following there statements are equivalent:*

(1) *$\mathcal{P}$ is self-dual;*
(2) *$\mathcal{P}$ is of constant width spherical convex body;*
(3) *$\mathcal{P}$ is of constant width spherical convex body.*

A Problem 2.7 from [11]: "Do there exist reduced spherical n-dimensional polytopes (possibly some simplices?) on $\mathbb{S}^n$, where $n \geq 3$, different from the $1/2^n$ part of $\mathbb{S}^n$?".

An affirmative answer (in the case $n = 3$): Let Q be an interior point of the spherical convex hull of $\{P_1, P_2, P_3, -N\}$ of $\mathbb{S}^3$. Set

$$Q_i = \partial H(Q) \cap NP_i, \quad i = 1, 2, 3.$$

Then the spherical polytope

$$\mathcal{P} = \text{s-conv}\{Q_1, Q_2, Q_3, P_1, P_2, P_3, Q\}$$

is of constant width $\pi/2$ (see the proof in general case in [2]).

2.2. Approximation of spherical convex bodies of constant width.

Theorem 4 ([10]). *For any spherical convex body $C \subset S^2$ of constant width $\tau < \pi/2$, and for any $\varepsilon > 0$ there exists a body C_ε of constant width $\Delta(C) = \Delta(C_\varepsilon)$ whose boundary consists only of arcs of circles of radius $\Delta(C)$ such that*

$$h(C, C_\varepsilon) \leq \varepsilon,$$

where $h(C_1, C_2)$ means the Hausdorff distance between C_1 and C_2 .

In [4], applying the fact that spherical dual transform is an isometry ([7]) and the relationships between boundary of C and its dual obtains a counterpart result of Theorem 4 as follows:

Theorem 5 ([4]). *For any spherical convex body $\widetilde{C} \subset S^2$ of constant width $\tau > \pi/2$, and for any $\varepsilon > 0$ there exists a body $\widetilde{C}_\varepsilon$ of constant width τ whose boundary consists only of arcs of circles of radius $\tau - \frac{\pi}{2}$ and great circle arcs, such that*

$$h(\widetilde{C}, \widetilde{C}_\varepsilon) \leq \varepsilon,$$

where h is the Hausdorff distance.

Notice that the spherical convex body of C in Theorem 4 is of constant width $\tau < \pi/2$ and the spherical convex body of C in Theorem 5 is of constant width $\tau > \pi/2$. For the remaining case (the spherical convex body C of $\pi/2$), we have a conjecture as follows.

Conjecture: Any spherical convex body of constant width $\pi/2$ can be approximated by a sequence of spherical convex polytopes of constant width $\pi/2$ ([4]).

Since a polytope $\mathcal{P}$ is of constant width τ, then $\tau = \pi/2$ (Lemma 2.4). This means any spherical convex body of constant width $\tau \neq \pi/2$ can not be approximated by a sequence of spherical convex polytopes of constant width τ.

3. Applications to Wulff shapes

Let $\gamma : \mathbb{S}^n \to \mathbb{R}_+$ be a continuous function, where $\mathbb{R}_+$ is the set consisting of positive real numbers. Then the *Wulff shape* associated with the function γ, denoted by $\mathcal{W}_\gamma$, is defined by

$$\bigcap_{\theta\in\mathbb{S}^n} \Gamma_{\gamma,\theta}.$$

Here $\Gamma_{\gamma,\theta}$ is the half space determined by the given continuous function γ and $\theta \in \mathbb{S}^n$,

$$\Gamma_{\gamma,\theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\}.$$

By definition, Wulff shape is a convex body, namely, convex, compact and contains the origin of $\mathbb{R}^{n+1}$ as an interior point. Conversely, for any convex body W contains the origin of $\mathbb{R}^{n+1}$ as an interior point, there exits a continuous function $\gamma : \mathbb{S}^n \to \mathbb{R}_+$ such that $\mathcal{W}_\gamma = W$. For more details in Wulff shapes, see for instance [1, 3, 5]. Let $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$ be the mapping defined by

$$Id(x) = (x, 1).$$

Let $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+2}$ be the north pole of $\mathbb{S}^{n+1}$, and let $\mathbb{S}^{n+1}_{N,+}$ denote the north open hemisphere of $\mathbb{S}^{n+1}$,

$$\mathbb{S}^{n+1}_{N,+} = \mathbb{S}^{n+1}\backslash H(-N) = \{Q \in \mathbb{S}^{n+1} \mid N \cdot Q > 0\}.$$

Let $\alpha_{N} : \mathbb{S}^{n+1}_{N,+} \to \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to N, defined by

$$\alpha_{N} (P_1, \dots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \dots, \frac{P_{n+1}}{P_{n+2}}, 1\right).$$

We call the spherical convex body $\widetilde{W}_\gamma = \alpha^{-1}(Id(\mathcal{W}_\gamma))$ is *the spherical Wulff shape of* $\mathcal{W}_\gamma$. The Wulff shape

$$Id^{-1} \circ \alpha_N\big((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^\circ\big).$$

is called *dual Wulff shape of* $\mathcal{W}_\gamma$, denoted by $\mathcal{DW}_\gamma$. We call a Wulff shape $\mathcal{W}$ is a *self-dual* if $\mathcal{W} = \mathcal{DW}$, namely, $\mathcal{W}$ and its dual Wulff shape $\mathcal{DW}$ are exactly the same convex body. By Theorem 1, Lemma 2.3 and Corollary 2.1, we have the following.

Corollary 3.1 ([8]). *Let $\gamma : \mathbb{S}^n \to \mathbb{R}_+$ be a continuous function. Suppose that the spherical Wulff shape $\widetilde{W}_\gamma = \alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ of $\mathcal{W}_\gamma$ is of constant width. Then*

(1) $\Delta(\widetilde{W}_\gamma) + diam\,(\widetilde{W}^\circ_\gamma) = \pi$,
(2) $\Delta(\widetilde{W}_\gamma) + \Delta\,(\widetilde{W}^\circ_\gamma) = \pi$,
(3) $diam(\widetilde{W}_\gamma) + \Delta\,(\widetilde{W}^\circ_\gamma) = \pi$,
(4) $diam(\widetilde{W}_\gamma) + diam\,(\widetilde{W}^\circ_\gamma) = \pi$,

where $\Delta(C)$ and $diam(C)$ are the width and the diameter of spherical convex body C in S^n, respectively.

A characterization of self-dual Wulff shape is given as follows.

Proposition 3.1 ([6]). *Let $\gamma : \mathbb{S}^n \to \mathbb{R}_+$ be a continuous function. Then $\mathcal{W}_\gamma$ is a self-dual Wulff shape if and only if its spherical Wulff shape is of constant width $\pi/2$, namely, the spherical convex body $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is of constant width $\pi/2$.*

By Theorem 1, we have the following:

Corollary 3.2 ([8]). *Let $\gamma : \mathbb{S}^n \to \mathbb{R}_+$ be a continuous function. Then $\mathcal{W}_\gamma$ is a self-dual Wulff shape if and only if its spherical Wulff shape is of constant diameter $\pi/2$, namely, the spherical convex body $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is of constant diameter $\pi/2$.*

By Theorem 3, we have the following:

Theorem 6 ([2]). *Let Wulff shape W_γ be a convex polytope. Then the following statements are equivalent:*

(1) *the Wulff shape W_γ is self-dual;*
(2) *the spherical Wulff shape of W_γ is constant width;*
(3) *the spherical Wulff shape of W_γ is constant diameter.*

References

[1] F. Morgan, *The cone over the Clifford torus in $\mathbb{R}^4$ in Φ- minimizing*, Math. Ann., **289** (1991), 341-534.

[2] H. Han, *Self-dual polytope and self-dual smooth Wulff shape*, to appear in Results in Mathematics.

[3] H. Han, *Maximum and minimum of convex integrands*, Hokkaido Mathematical Journal, **52** (2023), 381–399.

[4] H. Han, *Behavior of convex integrand at apex of its Wulff shape*, arXiv:2310.09710 (2023).

[5] H. Han and T. Nishimura, *Strictly convex Wulff shapes and C^1 convex integrands*, Proc. Amer. Math. Soc., **145** (2017), 3997–4008.

[6] H. Han and T. Nishimura, *Self-dual Wulff shapes and spherical convex bodies of constant width $\pi/2$*, J. Math. Soc. Japan., **69** (2017), 1475–1484.

[7] H. Han and T. Nishimura, *The spherical dual transform is an isometry for spherical Wulff shapes*, Studia Math. **245**, (2019), 201–211.

[8] H. Han and D. Wu, *Constant diameter and constant width of spherical convex bodies*, preprint (available from arXiv:1905.09098v2.)

[9] M. Lassak, *Width of spherical convex bodies*, Aequationes Math., **89** (2015), 555–567.

[10] M. Lassak, *Approximation of spherical convex bodies of constant width and reduced bodies*, J. Convex Analysis, **29** (2022), 921–928.

[11] Lassak M.: Spherical Geometry - A survey on width and thickness of convex bodies, In: Papadopoulos, A. (eds) Surveys in Geometry I. Springer, Cham, (2022)

[12] M. Lassak and M. Musielak, *Spherical bodies of constant width*, Aequationes Math., **92** (2018), 627–640.

[13] M. Lassak, *When a spherical body of constant diameter is of constant width?*, Aequat. Math., **94** (2020), 393–400.

[14] M. Musielak, *Covering a reduced spherical body by a disk*, arXiv:1806.04246.

[15] T. Nishimura and Y. Sakemi, *Topological aspect of Wulff shapes*, J. Math. Soc. Japan, **66** (2014), 89–109.

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