An elementary reproof of Banchoff-Fabricius Bjerre formula for polygons by panorama views

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Abstract

Panorama views is a tool for shape detection. Using this and a new characteristic to generic height functions, we give an elementary reproof to a theorem by T. Banchoff on the shape of polygons, which is a PL-version of the well-known formula by Fabricius-Bjerre.

1 Introduction

For a smooth closed and regular plane curve C, the following formula in [Fb1] for C satisfying certain generic conditions is known as Fabricius-Bjerre formula:

$$\sharp\mathrm{Bt}_1 - \sharp\mathrm{Bt}_2 = \frac{1}{2}\,\sharp\mathrm{Ifx} + \sharp\mathrm{D},$$

where

$\sharp\mathrm{Bt}_1$	denotes the number of bi-tangent lines to C of type I (monoside bi-tangent),
$\sharp\mathrm{Bt}_2$	denotes the number of bi-tangent lines to C of type II (biside bi-tangent),
$\sharp\mathrm{Ifx}$	denotes the number of inflexions, and
$\sharp\mathrm{D}$	denotes the number of double points of C.

Here a straight line L in $\mathbb{R}^2$ is called a *bi-tangent line* if it is tangent to C exactly at distinct two points and is called *type* I if C locates in the same side of L near the two tangent points, while it is called *type* II otherwise. We note that C is assumed to have no non-simple points other than a finite number of normal crossings and to have no straight line tangent to C at more than two points. In Figure 1, we put two examples of this formula.

In [Ba] Banchoff proved the same formula for a piecewise linear closed plane curve C (polygon, in short) when it satisfies some generic conditions. Namely let C be a polygon with k vertices v_i $(i = 1, \cdots, k)$ whose edges $v_i v_{i+1}$ are denoted by

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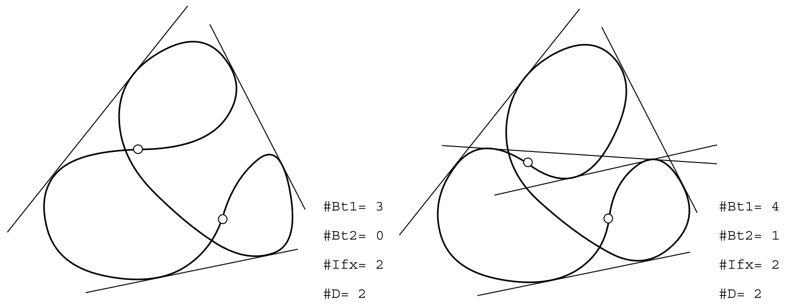

Figure 1: Fabricius-Bjerre formula, two examples

l_i, where v_{k+1} stands for v_1. A straight line L is *tangent* to C at a point $x \in C$ if either;

(a) $x = v_i$ and one component of $\mathbb{R}^2 \setminus L$ is disjoint from $l_{i-1} \cup l_i$ ($l_k \cup l_1$, in case $i = 1$), or

(b) x lies in $\mathrm{Int} l_i$ for some l_i.

We say that C is *non-degenerate* if no three vertices are placed on a straight line and further no three edges are extended to straight lines meeting at a single point. The non-degenerate polygons are generic and we consider only non-degenerate polygon C in the following. Any tangent line to C of the (a) type is tangent to C at most two distinct vertices and that of the (b) type contains no other vertices of C than v_{i-1} and v_i. The bi-tangent line to C and their types (type I or type II) are defined in the same way as in the smooth case and they are of a finite number. An edge l_i of C is called an *inflexion edge* if the two adjacent vertices v_{i-1} and v_{i+2} lie in the different sides of the straight line containing l_i. Any non-simple point of C is a transverse intersection by certain two edges l_i and l_j. Then the same formula as Fabricius-Bjerre holds for a generic polygon C, where $\sharp$Ifx in this case denotes the number of inflexion edges.

The purpose of this paper is to show the formula for non-degenerate polygons in an elementary way by introducing a characteristic to a generic height function of C. Before that we introduce a tool for shape analysis of a polygon named Panorama view. We note that the Fabricius-Bjerre formula for smooth closed curves are generalized by Fabricius-Bjerre [Fb2] and by others such as in [Ha], [Pi] and [Th], *etc.*.

2 Panorama views

We introduce the panorama views by following [KSY] (refer also to [KS] for smooth case), in a little simplified style that fits to our purpose to show Fabricius-Bjerre formula. For $\theta \in [0, \pi]$ let $h_\theta : \mathbb{R}^2 \to \mathbb{R}P^1$ be the map defined by $h_\theta(x, y) = x\cos\theta + y\sin\theta$, and let

$$H : \mathbb{R}^2 \times [0, \pi] \to \mathbb{R} \times [0, \pi]$$

be the map given by $H(x,y,\theta) = (h_\theta(x,y),\theta)$.

Note that h_θ is an orthogonal projection onto the line at 0 of direction $(\cos\theta, \sin\theta)$ and for the restriction $h_\theta|C$, we define its singular points as follows (refer to Figure 2):

$(x,y) \in C$ is a *singular point* of $h_\theta|C$ if and only if the line $h_\theta^{-1}(a)$ for $a = h_\theta(x,y)$ is tangent to C at (x,y).

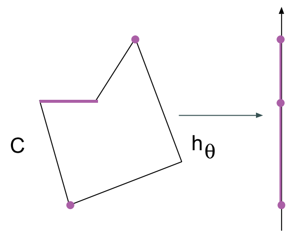

Figure 2: Singular points (bold) of $h_\theta|C$

For the restriction $H_C = H|C \times [0,\pi]$, set

$$S_{H_C} = \{(x,y,\theta) \in C \times [0,\pi]; (x,y) \text{ is a singular point of } h_\theta|C\}$$

and we define

$$\mathrm{PV} = H(C \times [0,\pi]) \quad \text{and} \quad \mathrm{PV}_0 = H(S_{H_C}),$$

which are referred to as the *Panorama view image* and the *Panorama view locus* of C, respectively. A sample construction is given in Figure 3.

Each edge l_i of C has a special $\theta = \theta_i \in [0,\pi]$ which gives the normal direction to l_i that l_i is projected by h_θ onto a single point, say $a = h_\theta(l_i)$. Then the H_C image of $l_i \times \{\theta_i\}$ is a single point $(a,\theta_i) \in \mathbb{R} \times [0,\pi]$, which we denote by (l_i). By the same reason, for a straight line L in $\mathbb{R}^2$ and its normal direction θ_L, we denote the point $(a,\theta_L) \in \mathbb{R} \times [0,\pi]$ by (L), where $a = h_{\theta_L}(L)$. Using these notations, one can see the following correspondences between C and its panorama view locus PV_0.

Theorem 2.1 ([KSY])**.** *Let C be a non-degenerate polygon C with vertices $v_1, \cdots,$ v_k and edges $l_1, \cdots, l_k$.*

1. *PV_0 is a curve in $\mathbb{R} \times [0,\pi]$ broken into certain sine arcs $(v_1), \cdots, (v_k)$ at the points $(l_1), \cdots, (l_k)$.*
2. *The sine arc (v_i) in 1 is the locus (L_s) by a family of tangent lines L_s to C at v_i, for a certain parametrization s of the tangent lines to C.*
3. *A straight line L in $\mathbb{R}^2$ is bi-tangent to C at v_i and v_j if and only if the point $(L) \in \mathbb{R} \times [0,\pi]$ is a normal crossing by the arcs (v_i) and (v_j) in 1.*
4. *An edge l_i is an inflexion edge if and only if the point $(l_i) \in \mathbb{R} \times [0,\pi]$ is a cusp joint of the two arcs (v_i) and (v_{i+1}) in 1.*

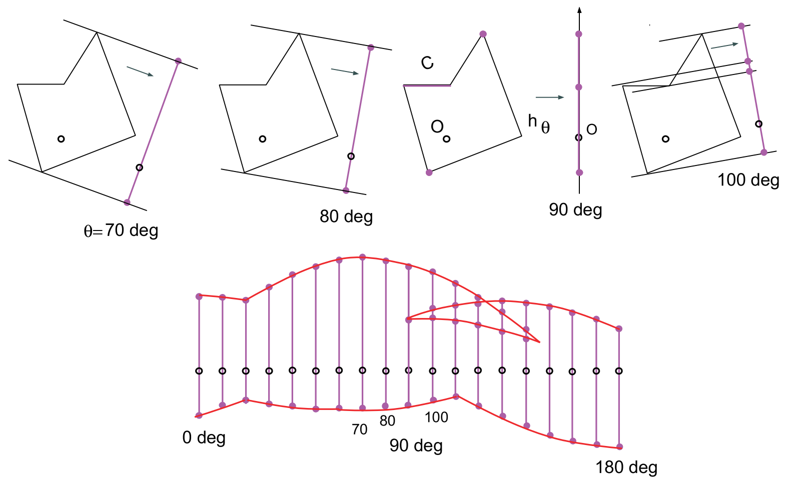

Figure 3: Panorama view for a polygon

Remark The cusp joint in 4 of Theorem 2.1 stands for the following point of PV_0. Let $\mu : \mathbb{R} \times [0, \pi] \to [0, \pi]$ be the canonical projection $\mu(u, \theta) = \theta$. By 1 of the theorem the two sine arcs (v_i) and (v_{i+1}) are jointed at (l_i). The point (l_i) is called a *regular joint* in case μ restricted to a neighbourhood of (l_i) in PV_0 is an injection and a *cusp joint* otherwise.

3 B-characteristic

Let C be a non-degenerate polygon. We say that a restriction $h : C \to \mathbb{R}$ of the orthogonal projection $\mathbb{R}^2 \to \mathbb{R}$ to the y-axis of $\mathbb{R}^2$ has the *Morse property* if the level set $h^{-1}(t)$ for any $t \in h(C)$ consists of a finite number of points and further, all the singular points of h (in the sense defined for $h_\theta|C$ in Section 2) have distinct values. For h with Morse property we define a number $B(h)$ as follows. Let p be a singular point of h. We count each regular point $q \in C$ on the critical level line $h^{-1}(c)$, $c = h(p)$ with weight by $+1$ or -1 given as below, by its relative position to p on the line $h^{-1}(c)$:

In the case where p is a local maximum,

$$\text{weight at } q \text{ is } \begin{cases} -1 & \text{if } q \text{ is right to } p \\ +1 & \text{if } q \text{ is left to } p, \end{cases}$$

and in the case where p is a local minimum,

$$\text{weight at } q \text{ is } \begin{cases} -1 & \text{if } q \text{ is left to } p \\ +1 & \text{if } q \text{ is right to } p. \end{cases}$$

In case q is a double point of C, we give double of the above weight. Then we define $B(h)$ to be the total of the above weighted counts of q's for all the singular points of h. Examples of $B(h)$'s are given below.

$B(h) = 2$ $\qquad$ $B(h) = -2$

Figure 4: Sample computations of $B(h)$

Let $h_\theta|C : C \to \mathbb{R}$ be the restriction of $h_\theta(x, y) = x\cos\theta + y\sin\theta$ in Section 2 for a non-degenerate polygon C. Considering $\mathbb{R}^2$ equipped with the first axis of direction $(\sin\theta, -\cos\theta)$ and the second axis of direction $(\cos\theta, \sin\theta)$ we regard $h_\theta|C$ as the orthogonal projection h above. Then it has the Morse property except for θ in a finite set $\{\eta_1, \cdots, \eta_s\}$ where $\mu^{-1}(\eta_i)$ contains a cusp joint, regular joint, or a normal crossing of PV_0, and hence one can consider $B(h_\theta|C)$ for θ not equal to any η_i. We may assume by rotation to C if necessary that η_i is neither 0 nor π.

Note that $B(h_\theta|C)$ is constant for θ in each component of $[0, \pi] - \{\eta_1, \cdots, \eta_s\}$ and the difference

$$\Delta B = B(h_b|C) - B(h_a|C)$$

at $\eta \in \{\eta_1, \cdots, \eta_s\}$, where $a = \eta - \varepsilon$ and $b = \eta + \varepsilon$ for a positive small constant ε, is well defined.

Recall that a cusp joint and a regular joint are represented by (l_i) for an edge l_i of C, by Thorem 2.1. We denote by D_i the number of edges of C which have intersections with the edge l_i. Then one can find the following.

Lemma 3.1. *The difference ΔB at $\eta \in \{\eta_1, \cdots, \eta_s\}$ is given as below:*

$$\Delta B = \begin{cases} -2 - 2D_i & \text{if } \mu^{-1}(\eta) \text{ contains a cusp joint } (l_i), \\ 4 & \text{if } \mu^{-1}(\eta) \text{ contains a normal crossing of type I}, \\ -4 & \text{if } \mu^{-1}(\eta) \text{ contains a normal crossing of type II, and} \\ -2D_i & \text{if } \mu^{-1}(\eta) \text{ contains a regular joint } (l_i). \end{cases}$$

The difference ΔB in Lemma 3.1 can be evaluated elementary. For example, a cusp joint on $\mu^{-1}(\eta)$ can be classified into two types; a *birth of a cusp joint* which means a cusp joint of PV_0 such that $h_b|C$ has exactly two more singular points than $h_a|C$, and a *death of a cusp joint* which means a cusp joint of PV_0 such that $h_a|C$ has exactly two more singular points than $h_b|C$. The difference ΔB caused by a birth of a cusp joint can be examined by Figure 5 upper as follows.

In Figure 5 upper, an edge l_i of C that induces this cusp joint (l_i), two connecting edges to l_i, and other three types of edges of C that meet the critical level line of

$h_\eta|C$ at the possible three relative positions to l_i are indicated. Comparing h_a in the left picture and h_b in the right picture, one can deduce that; two points P_1, P_2 weighted both by -1 are produced in the right figure for each edge of C that meets l_i, two points Q_1, Q_2 weighted both by -1 are produced by the connecting edges to l_i in the right figure, while the other two types of edges meeting the critical level line affect nothing to ΔB, by nature of the weights. This implies $\Delta B = -2 - 2D_i$, as required. Evaluation for a death of a cusp joint is similar.

In Figure 5 bottom, the difference ΔB caused by a normal crossing (L) of type I is indicated, where L is a bi-tangent line L to C and the normal crossing (L) is caused by the coincidence of two maximum values of $h_\theta|C$. Two points P_1, P_2 both weighted by -1 in the left figure disappear while two points Q_1, Q_2 both weighted by $+1$ are produced in the right figure, while edges meeting the line L affect nothing to ΔB, by nature of weights. This implies that $\Delta B = 4$, as required. Evaluation for the case where the normal crossing is caused by the coincidence of two minimum values of $h_\theta|C$ is similar.

Evaluations of ΔB in the other cases in Lemma 3.1 is similar. We put pictures for two more cases, as a reference in Figure 6.

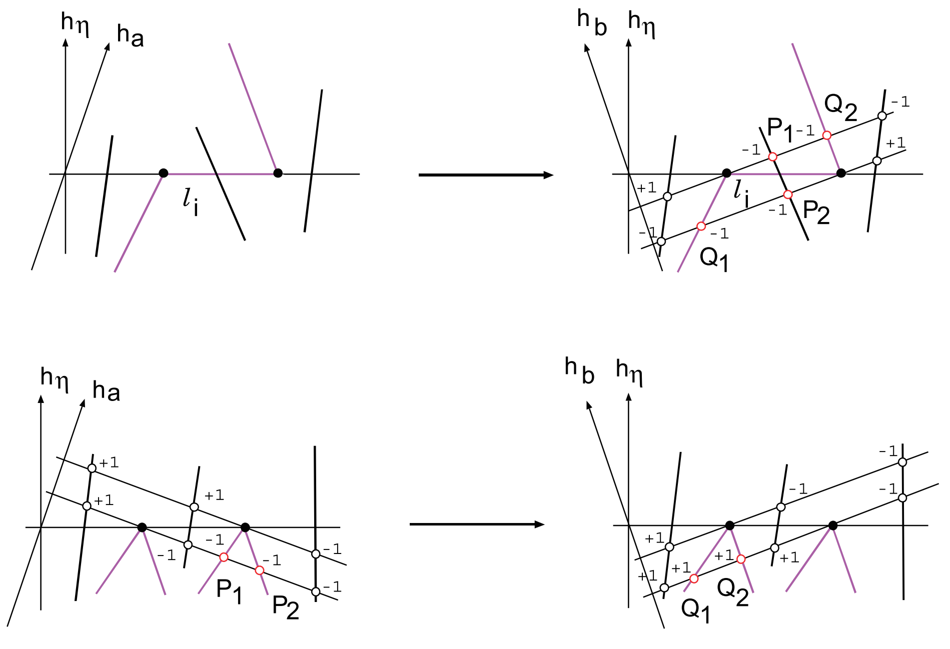

Figure 5: ΔB at a birth of a cusp joint (upper), at a normal crossing of type I (bottom)

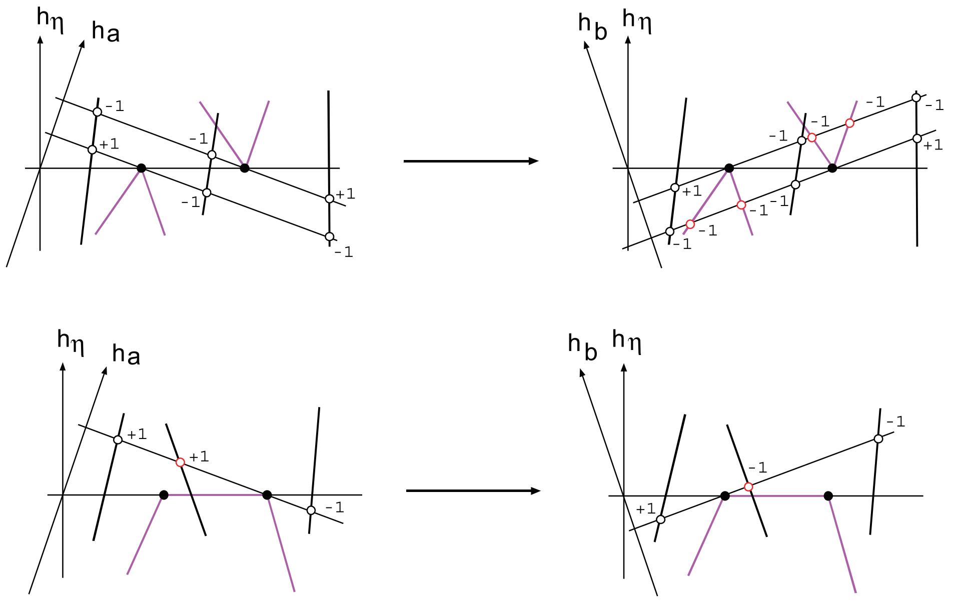

Figure 6: ΔB at a normal crossing of type II (upper), at a regular joint (bottom)

4 Proof of the Fabricius-Bjerre formula

We note that $B(h_0|C) = B(h_\pi|C)$ because in the identifications of $h_0|C$ and $h_\pi|C$ with the projection $h(x,y) = y$, both directions of x and y axes are reversed at $\theta = 0$ and at π, which implies both a singular point p being a maximum or a minimum points and a regular point q on the critical level set $h_\theta^{-1}(c)$, $c = h_\theta(p)$ being left or right of p exchange at $\theta = 0$ and at π.

It is clear by nature of ΔB that

$$B(h_\pi|C) = B(h_0|C) + \sum_\eta \Delta B$$

and hence

$$\sum_\eta \Delta B = 0,$$

where the sum is taken for $\eta \in \{\eta_1, \cdots, \eta_s\}$. Recalling that ΔB is caused by a cusp joint, a regular joint, a normal crossing of type I, and a normal crossing of type II, one can sum up ΔB by these types and use Lemma 3.1 to see that

$$\begin{aligned}\sum_\eta \Delta B = &- 2\ \text{(the number of cusp joints)} - 2\sum_i (D_i \text{ for a cusp joint } (l_i)) \\ &- 2\sum_i (D_i \text{ for a regular joint } (l_i)) \\ &+ 4\ \text{(the number of normal crossing of type I)} \\ &- 4\ \text{(the number of normal crossing of type II)}.\end{aligned}$$

By Theorem 2.1 and by $\sum_\eta \Delta B = 0$, this implies

$$0 = -2\ \sharp\mathrm{Ifx} - 2\sum_i (D_i \text{ for an edge } l_i) + 4\ \sharp\mathrm{Bt}_1 - 4\ \sharp\mathrm{Bt}_2.$$

By nature of D_i, the sum of D_i is twice of the double points of C. Hence we have

$$0 = -2\ \sharp\mathrm{Ifx} - 4\ \sharp\mathrm{D} + 4\ \sharp\mathrm{Bt}_1 - 4\ \sharp\mathrm{Bt}_2,$$

which implies the Fabricius-Bjerre formula

$$\sharp\mathrm{Bt}_1 - \sharp\mathrm{Bt}_2 = \frac{1}{2}\ \sharp\mathrm{Ifx} + \sharp\mathrm{D}.$$

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References

[Ba] T. F. Banchoff, *Global geometry of polygons. I: The theorem of Fabricius-Bjerre*, Proc. Amer. Math. Soc. **45** (1974), no2, 237–241

[Fb1] F. Fabricius-Bjerre, *On the double tangents of plane closed curves*, Math. Scand. **11** (1962), 113–116

[Fb2] F. Fabricius-Bjerre, *A relation between the numbers of singular points and singular lines of a plane closed curve*, Math. Scand. **40** (1977), 22–24

[Ha] B. Halpern, *Global theorems for closed plane curves*, Bull. Amer. Math. Soc. **76** (1970), 96–100

[KS] M. Kobayashi and T. Sano, *A short introduction to shape nalysis of apparent contours by "panorama views"*, Jour. of Math-for-Industry **2** (2010A-S), 85–91

[KSY] M. Kobayashi, T. Sano and M. Yamamoto, *Panorama views for closed piecewise linear plane curves*, Preprint (2023), 29 pages,

[Pi] R. Pignoni, *Integral relations for pointed curves in a real projective plane*, Geom. Dedicate **45** (1993), 263–287

[Th] A. Thompson, *Invariants of curves in* $\mathbb{R}P^2$ *and* $\mathbb{R}^2$, Algebraic and Geometric Topology **6** (2006), 2175–2187

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