Curvatures on curves passing through Whitney umbrella

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We consider curves passing through Whitney umbrella, which is known as a singularity appearing on stable maps from a surface to the three space. By the study of curvature of such curves, we obtain relations between curves and the differential geometric properties of Whitney umbrella.

1 Preliminaries

Definition 1.1. Two map-germs $f, g : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ are said to be $\mathcal{A}$-*equivalent* if there exist diffeomorphism-germs $\phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\psi \circ f = g \circ \phi$.

Using $\mathcal{A}$-equivalence, we define Whitney umbrella. Generally, Whitney umbrella is a singular point of maps, but we define it for a map-germ.

Definition 1.2. A map-grem $X : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *Whitney umbrella* (or a *cross cap*) if X is $\mathcal{A}$-equivalent to the map-germ

$$X_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0), (u, v) \mapsto (u, uv, v^2)$$

at the origin.

For a Whitney umbrella, the following normal form is known.

Theorem 1.3 ([2]). *Let $W : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a Whitney umbrella. Then for any $k \geq 3$, there exist an orientation preserving diffeomorphism $\psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $T \in SO(3)$ such that*

$$T \circ W \circ \psi(u, v) = \begin{pmatrix} u \\ uv + \frac{b_3}{3!}v^3 + \cdots + O(u, v)^{k+1} \\ \frac{a_{20}}{2}u^2 + a_{11}uv + \frac{a_{02}}{2}v^2 + \cdots + O(u, v)^{k+1} \end{pmatrix},$$

where $b_j \in \mathbb{R} (j \geq 3), a_{jk} \in \mathbb{R} (j + k \geq 2)$, and $a_{02} \neq 0$. Here, $O(u, v)^{k+1}$ is the terms whose dgrees are greater than or equal to $k + 1$.

In particular, the coefficients b_j, a_{jk} are all differential geometric invariants. On the other hand, the unit normal vector can not be smoothly extended at the singular point of the Whitney umbrella. A map that can smoothly extend the unit normal vector at the singular point is called frontal, and it is defined as follows.

Definition 1.4. A map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *frontal* if there exists a C^∞ map-germ $\nu : (\mathbb{R}^2, 0) \to S^2$ such that for any $p \in ((\mathbb{R}^2; u, v), 0)$,

$$\nu(p) \cdot f_u(p) = \nu(p) \cdot f_v(p) = 0$$

holds, where S^2 is the unit sphere in $\mathbb{R}^3$.

The property that a map-germ is a frontal does not depend on the $\mathcal{A}$-equivalence. Since $X_0(u, v) = (u, uv, v^2)$ satisfies

$$\frac{(X_0)_u \times (X_0)_v}{|(X_0)_u \times (X_0)_v|}(u, v) = \frac{(2v^2, -2v, u)}{\sqrt{4v^4 + 4v^2 + u^2}},$$

Whitney umbrella is not a frontal. On the other hand, the following property is known that for Whitney umbrella there is a well-defined unit normal vector after the bowing-up operation as follows ([1]).

We set $\tilde{\pi} : \mathbb{R} \times S^1 \to \mathbb{R}^2; (r, \theta) \to (r\cos\theta, r\sin\theta)$, and set $\mathcal{M}$ which is the quotient set $\mathbb{R} \times S^1/\sim$ of the relation $(r, \theta) \sim (-r, \theta + \pi)$. Considering

$$\pi : \mathcal{M} \to \mathbb{R}^2, [(r, \theta)] \mapsto (r\cos\theta, r\sin\theta),$$

one can see

$$\frac{((X_0)_u \times (X_0)_v) \circ \pi}{|((X_0)_u \times (X_0)_v) \circ \pi|}(0, \theta) = \frac{1}{\sqrt{4\sin^2\theta + \cos^2\theta}} \begin{pmatrix} 0 \\ -2\sin\theta \\ \cos\theta \end{pmatrix}.$$

This implies that by composing π, the unit normal vector field is well-defined near a Whitney umbrella on $\mathcal{M}$. This property strongly suggests that if we fix the angle of incidence of a Whitney umbrella, a smooth unit normal vector is well-defined. We consider a curve passing through $(0, 0)$ which satisfies the following general enough condition.

Definition 1.5. A C^∞ curve-germ $\mathbf{c} : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is said to be of *finite multiplicity* if there exist a positive integer m and a C^∞-germ $\bar{\mathbf{c}} : (\mathbb{R}, 0) \to \mathbb{R}^2$ such that

$$\mathbf{c}(x) = x^m \bar{\mathbf{c}}(x), \bar{\mathbf{c}}(0) \neq 0.$$

We consider curves of finite multiplicity passing through a Whitney umbrella. Then, we obtain the following lemma.

Lemma 1.6. *Let* $\mathbf{c} : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *be a curve of finite multiplicity, and let* $W : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a Whitney umbrella. Then the unit normal vector field of* W *along* $W \circ \mathbf{c}$ *can be smoothly extended across the Whitney umbrella.*

Similarly, we obtain the following lemma.

Lemma 1.7. *Let* $\mathbf{c} : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *be a curve of finite multiplicity, and let* $W : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *be a Whitney umbrella. Consider the composition* $W \circ \mathbf{c}$*. Then the unit tangent vector field of* $W \circ \mathbf{c}$ *can be smoothly extended across the Whitney umbrella.*

Example 1.8. Let us set $Y_0(x) = (x, x)$ and $X_0(u, v) = (u, uv, -v^2)$. We consider the composition $X_0 \circ Y_0 : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$. Then we can take a unit normal vector field $\mathbf{n}_0$ and a unit tangent vector field $\mathbf{e}_0$ as

$$\mathbf{n}_0(x) = \frac{1}{\sqrt{4x^2+5}} \begin{pmatrix} -2x \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{e}_0(x) = \frac{1}{\sqrt{8x^2+1}} \begin{pmatrix} 1 \\ 2x \\ -2x \end{pmatrix}$$

along $X_0 \circ Y_0$.

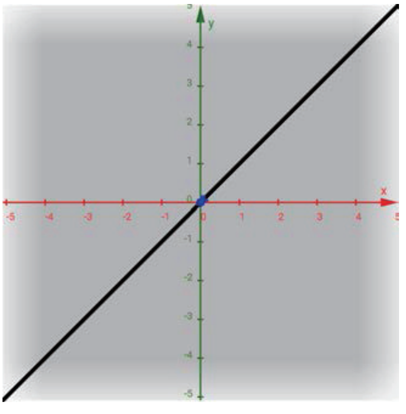

Figure 1.1: $Y_0(x)$

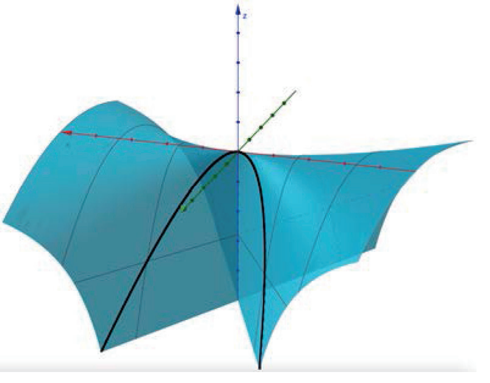

Figure 1.2: $Y_0 \circ X_0(x)$

2 Construction of frame

By the previous section, we have the following lemma.

Lemma 2.1. *The vectors*

$$\frac{(W \circ \mathbf{c})'}{|(W \circ \mathbf{c})'|}(x), \frac{(W_u \times W_v) \circ \mathbf{c}}{|(W_u \times W_v) \circ \mathbf{c}|}(x)$$

are well-defined across the singular point $(x = 0)$ *by taking an appropriate reduction. Setting* $\mathbf{b}(x) = \mathbf{n}(x) \times \mathbf{e}(x)$*, one obtain a Darboux frame* $\{\mathbf{e}(x), \mathbf{b}(x), \mathbf{n}(x)\}$ *along* $W \circ \mathbf{c}$*.*

We assume the condition

$$\mathrm{rank}(\mathbf{c}(0), \mathbf{c}'(0), \mathbf{c}''(0), \cdots) = 2$$

for a curve $\mathbf{c}(x)$. Then we can set

$$\mathbf{c}(x) = \begin{pmatrix} d_1(x)x^{m_1} \\ d_2(x)x^{m_2} \end{pmatrix}, \quad d_1(0) \neq 0,\ d_2(0) \neq 0,$$

where $m_1,\ m_2 \in \mathbb{Z}_{>0}, d_1, d_2 : (\mathbb{R}, 0) \to \mathbb{R}$. The following is an example of $\{\mathbf{e}(x), \mathbf{b}(x), \mathbf{n}(x)\}$.

Example 2.2. We set

$$m_1 = 2m_2,\ d_2 = 1,\ d_1(x) = \sum_{i=0} c_i x^i,$$

then $(W \circ \mathbf{c})'(x)$ and $(W_u \times W_v) \circ \mathbf{c}(x)$ satisfy

$$(W \circ \mathbf{c})'(x) = \begin{pmatrix} \{2m_2c_0 + O(x)\} \\ \{3m_2c_0 + \frac{1}{2}m_2b_3 + O(x)\}x^{m_2} \\ \{m_2a_{02} + O(x)\} \end{pmatrix} x^{2m_2-1} = \mathcal{A}(x)x^{2m_2-1},$$

$$(W_u \times W_v) \circ \mathbf{c}(x) = \begin{pmatrix} \{a_{02} + O(x)\}x^{m_2} \\ \{-a_{02} + O(x)\} \\ \{c_0 + \frac{b_3}{2} + O(x)\}x^{m_2} \end{pmatrix} x^{m_2} = \mathcal{B}(x)x^{m_2},$$

where $|\mathcal{A}(0)| \neq 0, |\mathcal{B}(0)| \neq 0$. Therefore, we have

$$\mathbf{e}(x) = \frac{\mathcal{A}}{|\mathcal{A}|} = \frac{1}{|\mathcal{A}|}\begin{pmatrix} e_1(x) \\ e_2(x)x^{m_2} \\ e_3(x) \end{pmatrix}, \begin{cases} e_1(0) = 2m_2c_0, \\ e_2(0) = m_2(3c_0 + \frac{1}{2}b_3), \\ e_3(0) = m_2a_{02}, \end{cases}$$

$$\mathbf{b}(x) = \frac{1}{|\mathcal{A}||\mathcal{B}|}\begin{pmatrix} b_1(x) \\ b_2(x)x^{m_2} \\ b_3(x) \end{pmatrix}, \begin{cases} b_1(0) = -m_2{a_{02}}^2, \\ b_2(0) = m_2(2{c_0}^2 + b_3c_0 - {a_{02}}^2), \\ b_3(0) = 2m_2a_{02}c_0, \end{cases}$$

$$\mathbf{n}(x) = \frac{\mathcal{B}}{|\mathcal{B}|} = \frac{1}{|\mathcal{B}|}\begin{pmatrix} n_1(x)x^{m_2} \\ n_2(x) \\ n_3(x)x^{m_2} \end{pmatrix}, \begin{cases} n_1(0) = a_{02}, \\ n_2(0) = -a_{02}, \\ n_3(0) = c_0 + \frac{1}{2}b_3. \end{cases}$$

3 Curvature

We define functions $\kappa_1, \kappa_2, \kappa_3$ by

$$\frac{d}{dx}\begin{pmatrix} \mathbf{e}(x) \\ \mathbf{b}(x) \\ \mathbf{n}(x) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(x) & \kappa_2(x) \\ -\kappa_1(x) & 0 & \kappa_3(x) \\ -\kappa_2(x) & -\kappa_3(x) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}(x) \\ \mathbf{b}(x) \\ \mathbf{n}(x) \end{pmatrix}.$$

We see the relations between $\kappa_1, \kappa_2, \kappa_3$ and the geodesic curvature κ_g, the normal curvature κ_ν, and the geodesic torsion κ_t which are defined on a set of regular points.

Lemma 3.1. *We set*

$$\gamma'(x) = \mathcal{A}(x)x^{\alpha}, \mathcal{A}(0) \neq 0,$$
$$(W_u \times W_v) \circ \mathbf{c}(x) = \mathcal{B}(x)x^{\beta}, \mathcal{B}(0) \neq 0.$$

Then it holds that

$$\begin{aligned}
\kappa_1(x) &= \operatorname{sgn}(x^{\alpha+\beta})|\mathcal{A}(x)|x^{\alpha}\kappa_g(x),\\
\kappa_2(x) &= \operatorname{sgn}(x^{\beta})|\mathcal{A}(x)|x^{\alpha}\kappa_\nu(x),\\
\kappa_3(x) &= |\mathcal{A}(x)|x^{\alpha}\kappa_t(x).
\end{aligned}$$

By lemma 3.1, we can see a boundedness of $\kappa_g, \kappa_\nu, \kappa_t$ if we give degrees and top terms of $\kappa_1, \kappa_2, \kappa_3$. We consider a curve $\mathbf{c}(x)$ such that

$$\mathbf{c}'(0) \in \operatorname{Ker} dW.$$

Then $\mathbf{c}(x)$ satisfies

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{m_1} \\ x^{m_2} \end{pmatrix}, c(0) \neq 0, m_1 > m_2.$$

Therefore, we set

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{mp+q} \\ x^{m} \end{pmatrix}, 1 \leq p, 1 \leq q < m, \tag{3.1}$$

or

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{mp} \\ x^{m} \end{pmatrix}, 2 \leq p \tag{3.2}$$

where $c(x) = c_0 + c_1x + c_2x^2 + \cdots,\ c_0 \neq 0$.

Theorem 3.2. *We assume that* $\mathbf{c}(x)$ *satisfies* (3.1). *Then it holds that*

$$\kappa_1(x) = \begin{cases} \tilde{\kappa}_1(x)x^{m-q-1} & (p = 1),\\ \tilde{\kappa}_1(x)x^{m(p-2)+q-1} & (2 \leq p < 4),\\ \tilde{\kappa}_1(x)x^{2m-1} & (4 \leq p), \end{cases}$$

$$\kappa_2(x) = \tilde{\kappa}_2(x)x^{m-1}$$

$$\kappa_3(x) = \begin{cases} \tilde{\kappa}_3(x)x^{q-1} & (p = 1),\\ \tilde{\kappa}_3(x)x^{m-1} & (2 \leq p). \end{cases}$$

Theorem 3.3. *We assume that* $\mathbf{c}(x)$ *satisfies* (3.2)*. Then it holds that*

$$\kappa_1(x) = \begin{cases} \tilde{\kappa}_1(x)x^0 & (p=2), \\ \tilde{\kappa}_1(x)x^{m-1} & (p=3), \\ \tilde{\kappa}_1(x)x^{2m-1} & (4 \le p), \end{cases}$$

$$\kappa_2(x) = \tilde{\kappa}_2(x)x^{m-1},$$
$$\kappa_3(x) = \tilde{\kappa}_3(x)x^{m-1}.$$

In particular, if

$$\mathbf{c}(x) = \begin{pmatrix} c(x)x^{2m} \\ x^m \end{pmatrix},$$

the top terms of $\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3$ satisfy

$$\tilde{\kappa}_1(x) = \begin{cases} \frac{1}{|\mathcal{A}|^2|\mathcal{B}|}a_{02}(6a_{11}c_0^2 + a_{03}c_0 - 3a_{02}c_1) + O(x)(m_2 = 1), \\ -\frac{1}{|\mathcal{A}|^2|\mathcal{B}|}m_2(2m_2+1)a_{02}^2c_1 + O(x)(2 \le m_2), \end{cases} \tag{3.3}$$

$$\tilde{\kappa}_2(x) = -\frac{1}{|\mathcal{A}||\mathcal{B}|}m_2^2a_{02}\left(3c_0 + \frac{b_3}{2}\right) + O(x), \tag{3.4}$$

$$\tilde{\kappa}_3(x) = -\frac{1}{|\mathcal{A}||\mathcal{B}|^2}m_2^2a_{02}(2c_0^2 + b_3c_0 - a_{02}^2) + O(x). \tag{3.5}$$

If we assume these top terms vanish, then we have the following relations in (3.3), (3.4), and (3.5),

$$6a_{11}c_0^2 + a_{03}c_0 - 3a_{02}c_1 = 0, \tag{3.6}$$

$$3c_0 + \frac{b_3}{2} = 0, \tag{3.7}$$

$$2c_0^2 + b_3c_0 - a_{02}^2 = 0, \tag{3.8}$$

where c_0, c_1 are differential geometric invariants of curve $\mathbf{c}$, $a_{02}, a_{11}, a_{03}, b_3$ are differential geometric invariants of Whitney umbrella W. When a Whitney umbrella and a curve satisfy these relations, the degrees of $\kappa_1, \kappa_2, \kappa_3$ are at least greater than 1 for the non-zero cases. Thus we can consider that these relations have a geometric meaning.

Example 3.4. Let us set

$$\mathbf{c}^1(x) = \begin{pmatrix} x^2 \\ x \end{pmatrix}, \tag{3.9}$$

$$W_1(u,v) = \begin{pmatrix} u \\ uv + \frac{b_3}{6}v^3 \\ a_{11}uv + \frac{a_{02}}{2}v^2 + \frac{a_{03}}{6}v^3 \end{pmatrix}. \tag{3.10}$$

We consider the composition $W_1 \circ \mathbf{c}^1$. We give examples of the cases

(1) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) \neq 0,$

(2) $\tilde{\kappa}_1(0) = 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) \neq 0,$

(3) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) = 0,\ \tilde{\kappa}_3(0) \neq 0,$

(4) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) = 0,$

by

(1) $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -1),$

(2) $(b_3, a_{11}, a_{02}, a_{03}) = (1, 1/6, -2, -1),$

(3) $(b_3, a_{11}, a_{02}, a_{03}) = (-6, -1, -4, -1),$

(4) $(b_3, a_{11}, a_{02}, a_{03}) = (2, -1, -2, -1).$

The figures of W_1 with $W_1 \circ \mathbf{c}^1$ of the cases (1) - (4) are given in Figure 3.1 - 3.4.

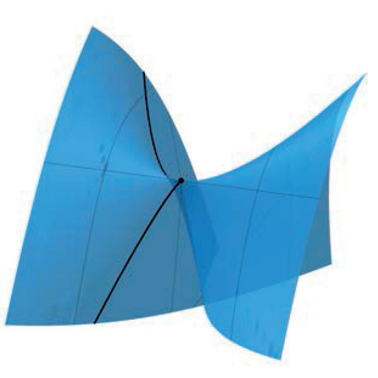

Figure 3.1: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

Figure 3.2: $\tilde{\kappa}_1(0) = 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

Figure 3.3: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) = 0, \tilde{\kappa}_3(0) \neq 0$

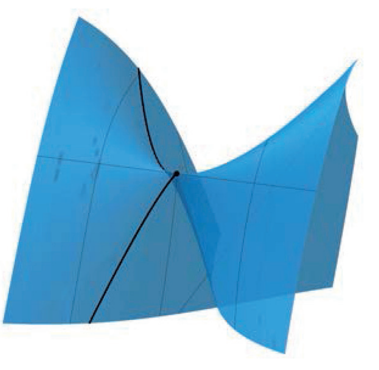

Figure 3.4: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) = 0$

Example 3.5. Let us set

$$\mathbf{c}^2(x) = \begin{pmatrix} x^4 + x^5 \\ x^2 \end{pmatrix}, \tag{3.11}$$

We consider the composition $W_1 \circ \mathbf{c}^2$. We give examples of the cases

(1) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) \neq 0,$

(2) $\tilde{\kappa}_1(0) = 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) \neq 0,$

(3) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) = 0,\ \tilde{\kappa}_3(0) \neq 0,$

(4) $\tilde{\kappa}_1(0) \neq 0,\ \tilde{\kappa}_2(0) \neq 0,\ \tilde{\kappa}_3(0) = 0,$

by

(1) $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -1),$

(2) $(b_3, a_{11}, a_{02}, a_{03}) = (1, -1, -2, -0),$

(3) $(b_3, a_{11}, a_{02}, a_{03}) = (-6, -1, -4, -1),$

(4) $(b_3, a_{11}, a_{02}, a_{03}) = (2, -1, -2, -1).$

The figures of W_1 with $W_1 \circ \mathbf{c}^2$ of the cases (1) - (4) are given in Figure 3.5 - 3.8.

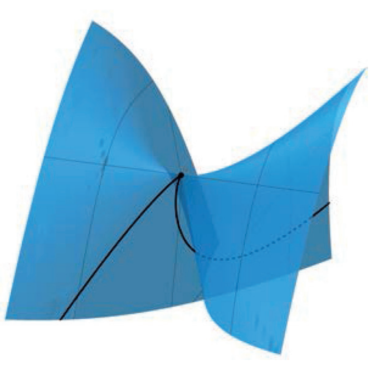

Figure 3.5: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

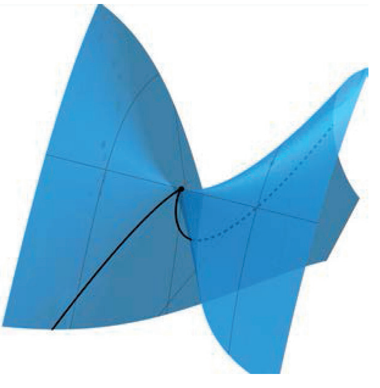

Figure 3.6: $\tilde{\kappa}_1(0) = 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) \neq 0$

Figure 3.7: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) = 0, \tilde{\kappa}_3(0) \neq 0$

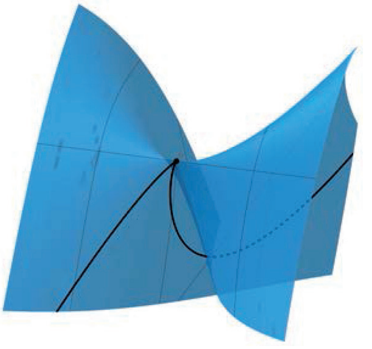

Figure 3.8: $\tilde{\kappa}_1(0) \neq 0, \tilde{\kappa}_2(0) \neq 0, \tilde{\kappa}_3(0) = 0$

This note is an addition to the author's talk given in the RIMS workshop "Extension of the Singularity theory" which is held from November 27th to 29th, 2023. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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