Deformations of S_1 singularities

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Abstract

For a one parameter deformation of the the S_1 singularity, we give a form using only isometric map of the target. Using this form, we study differential geometric properties of S_1 singularities and the Whitney umbrellas appearing in the deformation.

This note is an addition to the author's talk given in the RIMS workshop "Extension of the Singularity theory" which is held from November 27th to 29th, 2023.

1 Introduction

In recent decades, differential geometric studies of curves and surfaces with singularities have been studied by many authors. The main differential geometric invariants of regular surfaces are the Gaussian curvature and the mean curvature, and which plays central role of differential geometric studies of surfaces. However, they usually cannot be defined if the surface has singularities. As an alternative, we will take the method of normal form to study the S_1 singularities. Singularities are classified by the codimension. The codimension one singularities correspond to the birth/death of generic singularities. Therefore, it is natural to include the deformation when we study such singularities. In this paper, we give a normal form for the deformations of the $S_1^\pm$ singularities using only isometric maps of the target. Furthermore, using this form, we investigate the differential geometric properties of the deformations of the $S_1^\pm$ singularities.

The generic singularities on a surface in the three space is the Whitney umbrella, which is a map-germ $\mathcal{A}$-equivalent to the map-germ defined by

$$f(u,v) = (u, uv, v^2)$$

at the origin. See Figure 1.

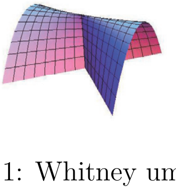

Figure 1: Whitney umbrella

The following theorem is known.

Theorem 1.1. [1] *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a Whitney umbrella. Then there exist a diffeomorphism-germ $\varphi : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ and $T \in SO(3)$ such that*

$$T \circ f \circ \varphi(u, v) = \left(u, uv + O(3), \frac{1}{2}(a_{20}u^2 + 2a_{11}uv + a_{02}v^2) + O(3)\right)$$

with $a_{20}, a_{11}, a_{02} \in \boldsymbol{R}$. Here, $O(n)$ stands for the terms whose degrees are equal to or greater than n.

The coefficients a_{20}, a_{11}, a_{02} are the differential geometric invariants of the Whitney umbrella, and these invariants can be used to reveal the differential geometric properties of the Whitney umbrella instead of the Gaussian curvature and the mean curvature, which have lost their meanings at the Whitney umbrella.

Next we deal with the $S_1^{\pm}$ singularities, which are map-germs $\mathcal{A}$-equivalent to the map-germ defined by

$$f(u, v) = (u, v^2, v(u^2 \pm v^2))$$

at the origin. See Figure 2.

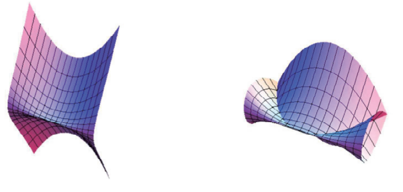

Figure 2: S_1^+ singularity (left), S_1^- singularity (right)

2 Deformations of $S_1^{\pm}$ singularities and their normal forms

The main objective of this paper is to study geometry of deformations of singularities.

Definition 2.1. A map germ $f : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ is a *deformation of* $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$*, or an $\mathcal{A}$-class* $[g]$ if it is smooth and $f(u, v, 0)$ at $(u, v) = (0, 0)$ is $\mathcal{A}$-equivalent to g.

In this definition, $\boldsymbol{R}$ added to the source space is called the *deformation parameter*. We define an equivalence relation between two deformations preserving the deformation parameters.

Definition 2.2. Let $f, g : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be deformations of $S_1^{\pm}$ singularities. Then f and g are *equivalent as deformations* if there exist orientation preserving diffeomorphism germs $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R}, 0)$ with the form $\varphi(u, v, s) = (\varphi_1(u, v, s), \varphi_2(u, v, s), \varphi_3(s))$ and $\psi : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ such that $\psi \circ f \circ \varphi^{-1}(u, v, s) = g(u, v, s)$ holds.

The above form on φ, namely φ_3 is defined depending only on s, implies that we allow change of parameter of deformation itself and prevent affecting the other parameters to the parameter of deformation. Since the third component of the source space is the deformation parameter, therefore this definition means an $\mathcal{A}$-equivalence that preserves the deformation parameters is defined.

Example 2.3. Let ${f_s}^{\pm}$ be a deformation of $S_1^{\pm}$ singularities defined by

$$ {f_s}^{\pm} : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \ni (u, v, s) \mapsto (u, v^2, v(u^2 \pm v^2) + sv) \in (\boldsymbol{R}^3, 0) $$

We show the deformation of ${f_s}^{+}$ in Figure 3 and ${f_s}^{-}$ Figure 4.

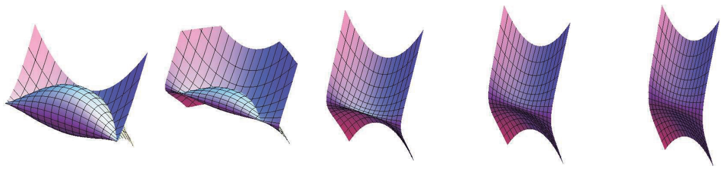

Figure 3: Deformation of S_1^+ singularity (from left to right $f_{-1}^+, f_{-1/2}^+, f_0^+, f_{1/2}^+$ and f_1^+)

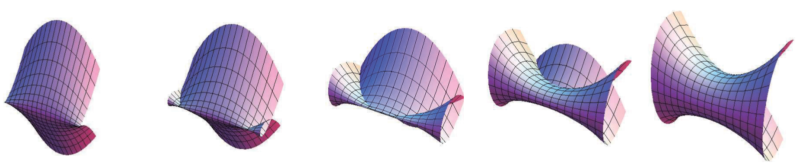

Figure 4: Deformation of S_1^- singularity (from left to right $f_{-1}^-, f_{-1/2}^-, f_0^-, f_{1/2}^-$ and f_1^-)

Theorem 2.4. *Let $f : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be a deformation of an $S_1^{\pm}$ singularity. Then there exist an orientation preserving diffeomorphism germ $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R}, 0)$ with the form of Definition 2.2, $T \in SO(3)$ and the functions $f_{21}, f_{31} \in C^\infty(1,1)$, $f_{24}, f_{33}, f_{34} \in C^\infty(2,1)$, $f_{32} \in C^\infty(3,1)$ such that*

$$ \begin{aligned} {f_n}^s &= T \circ f \circ \varphi(u, v, s) \\ &= (u, u^2 f_{21}(u) + v^2 + usf_{24}(u, s), \\ &\qquad u^2 f_{31}(u) + v^2 f_{32}(u, v, s) + vf_{33}(u, s) + usf_{34}(u, s)), \end{aligned} \tag{2.1} $$

where $(f_{32})_v(0,0,0) \neq 0$, $(f_{33})_{uu}(0,0) \neq 0$, $f_{32}(0,0,0) = f_{33}(0,0) = (f_{33})_u(0,0) = 0$. If $(df_{33}/ds)(0,0) \neq 0$, then one can further reduce $f_{33}(0,s) = s$.

The form ${f_n}^s$ is called the *normal form of the deformations of an $S_1^{\pm}$ singularity.* If $(df_{33}/ds)(0,0) \neq 0$, then it is a generic deformation. Thus we assume $(df_{33}/ds)(0,0) \neq 0$, and from now on, we assume $f_{33}(0,s) = s$ in the form ${f_n}^s$.

We see the set of singular points $S({f_n}^s)$ of ${f_n}^s$ is

$$ S({f_n}^s) = \{(u, v) \mid v = 0, f_{33}(u, s) = 0\}. $$

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In this Theorem 2.4, the given f and ${f_n}^s$ are equivalent as deformations. Furthermore, they have the same differential geometric properties. The uniqueness of the normal form holds as in the following sense.

Proposition 2.5. *We assume that $f_{24}(0,0) \neq 0$. Let us set $u = \tilde{u}$ and $\varphi(u,v,s) = (\tilde{u},\tilde{v},\tilde{s})$, where*

$$\tilde{v} = \sum_{i+j+k=1}^{r} \frac{a_{ijk}}{(i+j+k)!} u^i v^j s^k \quad (a_{ijk} \in \boldsymbol{R}),$$
$$\tilde{s} = \sum_{i=1}^{r} \frac{b_i}{i!} s^i \quad (b_i \in \boldsymbol{R}),$$

and let $T \in SO(3)$. If $T \circ f \circ \varphi(u,v,s) = f(u,v,s)$, then

$$a_{100} = a_{001} = a_{200} = a_{020} = a_{002} = a_{110} = a_{011} = a_{101} = \cdots = 0, a_{010} = 1,$$

and

$$b_1 = 1, b_2 = b_3 = \cdots = 0,$$

hold.

Example 2.6. Let ${f_n}^s$ be a deformation of S_1^+ singularity defined by

$$f(u,v,s) = (u, u^2 + v^2 - us, u^2 - uv^2 + v^3 + vs + u^2 v + us).$$

Then we can observe the Whitney umbrellas appear in the deformation. See Figure 5.

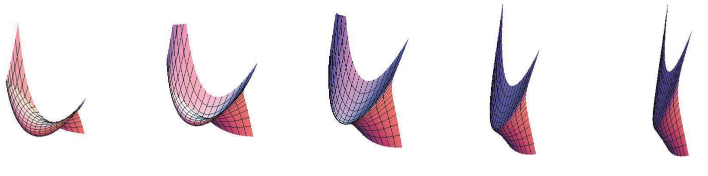

Figure 5: The surfaces in Example 2.6 (from left to right $s = -1, -1/2, 0, 1/2, 1$)

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Example 2.7. Let ${f_n}^s$ be a deformation of S_1^+ singularity defined by

$$f(u,v,s) = (u, u^2 + v^2 - us, u^2 - uv^2 + v^2 s + v^3 + uvs + u^2 v + vs^2 - us).$$

This map is not a generic deformation because of $(df_{33}/ds)(0,0) = 0$, and the Whitney umbrella does not appear. See Figure 6.

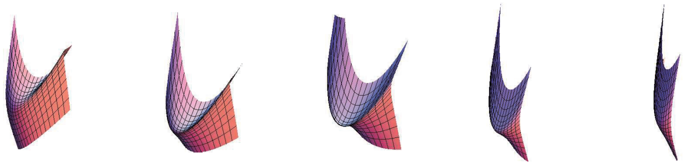

Figure 6: The surfaces in Example 2.7 (from left to right $s = -1, -1/2, 0, 1/2, 1$)
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3 Geometry on deformations of S_1^+ singularity

In this section, we study differential geometric properties of deformations of $S_1^{\pm}$ singularities.

We set $f = f_n{}^s$ (see (2.1)). The functions appearing in f can be expressed as

$$f_{32}(u,v,s) = uf_{321}(u,v,s) + vf_{322}(u,v,s) + sf_{323}(u,v,s),$$
$$f_{33}(u,s) = f_{330}(s) + uf_{331}(s) + u^2 f_{332}(s) + u^3 f_{333}(s) + \cdots .$$

Assuming $f_{332}(0) > 0$, we set $f_{332}(0) = f_{3320}^2$. If $f_{332}(0) < 0$, the same calculation can be done by setting $f_{332}(0) = -f_{3320}^2$. Furthermore, since $S(f) \neq \emptyset$ is equivalent to $s \leq 0$, we set $s = -\tilde{s}^2$.

Theorem 3.1. *If $(u,v) \in S(f)$, then $v = 0$ holds, and u can be expanded as a function of $\tilde{s}^2$, say $u(\tilde{s})$, as follows:*

$$u(\tilde{s}) = \frac{1}{f_{3320}}\tilde{s} + \frac{1}{2f_{3320}^2}\Big(f_{331}(0) - \frac{f_{333}(0)}{f_{3320}^2}\Big)\tilde{s}^2$$
$$-\frac{1}{4f_{3320}^3}\Big(f_{331}(0) - \frac{f_{333}(0)}{f_{3320}}\Big)\Big(f_{331}(0) + \frac{f_{333}(0)}{f_{3320}} + \frac{f_{332}(0)}{2f_{3320}}(f_{331}(0) - \frac{f_{333}(0)}{f_{3320}^2})\Big)\tilde{s}^3 + O(4).$$

Moreover, if $u \neq 0$, then f at $(u,0)$ are both Whitney umbrellas.

Since Whitney umbrellas appear on the deformations of the $S_1^{\pm}$ singularities, we introduce a formula for invariants of a Whitney umbrella.

Theorem 3.2. [3] *Let $f : (\boldsymbol{R}^2, p) \to (\boldsymbol{R}^3, 0)$ be a Whitney umbrella. For coordinate system (u,v) satisfying* $\mathrm{Ker} df_0 = \langle \partial v \rangle$ *and $|f_u, f_{uv}, f_{vv}| > 0$, the invariants a_{02}, a_{20}, a_{11} in Theorem* 1.1 can be written as:

$$a_{02} = \frac{|f_u||f_u \times f_{vv}|^3}{|f_u, f_{uv}, f_{vv}|^2}(p),$$
$$a_{20} = \frac{|f_u \times f_{vv}|}{4|f_u|^3|f_u, f_{uv}, f_{vv}|^2}(|f_u, f_{uu}, f_{vv}|^2 + 4|f_u, f_{uv}, f_{vv}||f_u, f_{uv}, f_{uu}|)(p),$$
$$a_{11} = \frac{1}{2|f_u||f_u, f_{uv}, f_{vv}|^2}\Big(2|f_u, f_{uv}, f_{vv}|\begin{vmatrix} f_u \cdot f_u & f_u \cdot f_{uv} \\ f_{vv} \cdot f_u & f_{vv} \cdot f_{uv} \end{vmatrix} - |f_u \times f_{vv}|^2 |f_u, f_{uu}, f_{vv}|\Big)(p).$$

As we saw in the above, the singular point of $f = f_n{}^s$ is $(u(\tilde{s}), 0)$. Considering the above invariants a_{02}, a_{20}, a_{11} at $(u(\tilde{s}), 0)$, one can regard the invariants a_{02}, a_{20}, a_{11} as functions of $\tilde{s}$. Let $a_{20}(\tilde{s})$, $a_{11}(\tilde{s})$ $a_{02}(\tilde{s})$ denote these functions. We remark that

$$\lim_{\tilde{s} \to 0} |f_u, f_{uv}, f_{vv}|(u(\tilde{s}), 0) = 0.$$

Thus these functions generally diverge at the $S_1^{\pm}$ singularities. More precisely, we obtain the following.

Theorem 3.3. *The functions $a_{20}(\tilde{s}), a_{11}(\tilde{s})$ and $a_{02}(\tilde{s})$ can be expanded as follows:*

$$\begin{aligned}
a_{02}(\tilde{s}) &= \frac{1}{\tilde{s}^2}\Big(\frac{1}{2f_{3320}^2} + O(1)\Big),\\
a_{20}(\tilde{s}) &= \frac{1}{\tilde{s}^2}\Big(\frac{f_{31}^2(0)}{2f_{3320}^2} - \frac{f_{31}(0)(c_1 + 2f_{3320}c_2)}{64}\tilde{s} + O(2)\Big),\\
a_{11}(\tilde{s}) &= \frac{1}{\tilde{s}^2}\Big(\frac{4}{f_{3320}}(f_{21}(0)f_{321}(0,0,0) - 3f_{31_u}(0))\tilde{s} + O(2)\Big),
\end{aligned}$$

where $c_1, c_2 \in \boldsymbol{R}$.

We have the following corollary.

Corollary 3.4. *Taking the limit $\tilde{s} \to 0$, then the following hold. The invariant $a_{02}(\tilde{s})$ diverges to positive infinity. If $f_{31}(0) \neq 0$, then $a_{20}(\tilde{s})$ diverges to positive infinity, and if $f_{31}(0) = 0$, then $a_{20}(\tilde{s})$ is bounded. If $f_{21}(0)f_{321}(0,0,0) - 3(f_{31})_u(0) \neq 0$, then $a_{11}(\tilde{s})$ diverges to infinity, and if $f_{21}(0)f_{321}(0,0,0) - 3(f_{31})_u(0) = 0$, then $a_{11}(\tilde{s})$ is bounded*

Definition 3.5. [2] Let $f : (\boldsymbol{R}^2, (u_0, v_0)) \to \boldsymbol{R}^3$ be a map germ with rank $df_{(u_0,v_0)} = 1$. Then the *focal set* of f at (u_0, v_0) is defined by

$$\{x \in \boldsymbol{R}^3 | D_u^x(u_0, v_0) = D_v^x(u_0, v_0) = 0, D_{uu}^x(u_0, v_0)D_{vv}^x(u_0, v_0) - D_{uv}^x(u_0, v_0)^2 = 0\}$$

where D^x is the distance squared function $D^x(u, v) = \frac{1}{2}|x - f|^2$.

The focal set is located on the normal plane of the image of $df_{(u_0,v_0)}$. Since the focal set is a conic for the Whitney umbrella, the focal set is called the *focal conic*.

Proposition 3.6. [2] *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a map germ with a Whitney umbrella. Then the focal conic of f is*

(1) *an ellipse if and only if $a_{02}a_{20} < 0$,*

(2) *a hyperbola if and only if $a_{02}a_{20} > 0$, and*

(3) *a parabola if and only if $a_{20} = 0$.*

By Theorem 3.3 if $f_{31}(0) \neq 0$, then $a_{02}a_{20} \to \infty (\tilde{s} \to 0)$. Therefore, by Proposition 3.6, we obtain the following corollary.

Corollary 3.7. *In the small deformations of ${S_1}^{\pm}$ singularities with $f_{31}(0) \neq 0$, all the focal conics for sufficiently small $\tilde{s}$ are hyperbolas.*

Example 3.8. Let f be a deformation of S_1^+ singularity defined by

$$f(u, v, \tilde{s}) = (u, v^2, -u^2 + v^3 + v\tilde{s}^2 + u^2 v),$$

Then the focal conic of f is a hyperbola for small $\tilde{s} \neq 0$ and two transversal lines when $\tilde{s} = 0$. See Figure 7 and Figure 8.

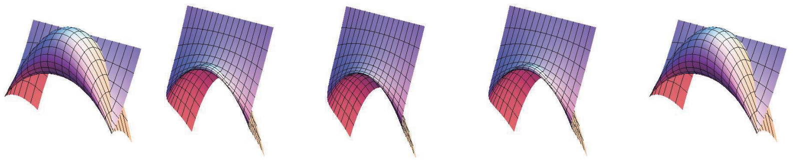

Figure 7: The surfaces in Example 3.8 (from left to right $\tilde{s} = -1, -1/2, 0, 1/2, 1$)

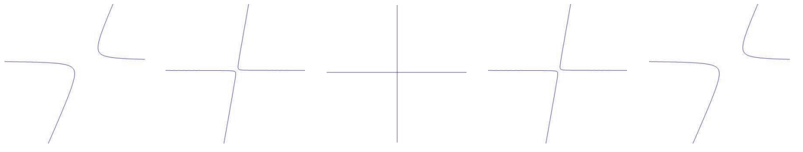

Figure 8: The focal conics of the singularities in Example 3.8 (from left to right $\tilde{s} = -1, -1/2, 0, 1/2, 1$)

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Next we also give an example of the case $f_{31}(0) \neq 0$ whose focal conics are ellipses when $\tilde{s}$ is not close to 0 and they are hyperbolas when $\tilde{s}$ is close to 0.

Example 3.9. Let f be a deformation of S_1^+ singularity defined by

$$f(u, v, \tilde{s}) = (u, -u^2 + v^2, u^2 + v^3 - v\tilde{s}^2 + u^2 v),$$

This case $f_{31}(0) \neq 0$ holds. The focal conic of f is an ellipse when $\tilde{s} < -1/2, 1/2 < \tilde{s}$, a parabola when $\tilde{s} = \pm 1/2$, a hyperbola when $0 < |\tilde{s}| < 1/2$ and two transversal lines when $\tilde{s} = 0$. See Figure 9 and Figure 10.

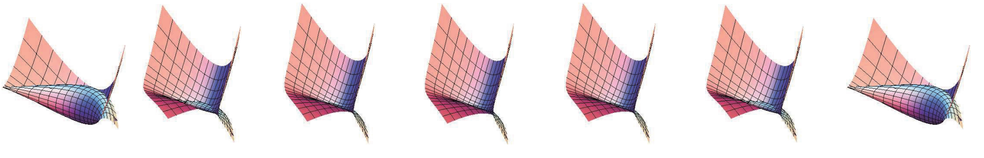

Figure 9: The surfaces in Example 3.9 (from left to right $\tilde{s} = -1, -1/2, -1/5, 0, 1/5, 1/2, 1$)

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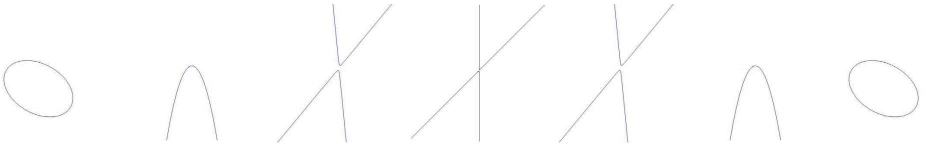

Figure 10: The focal conics of the singularities in Example 3.9 (from left to right $\tilde{s} = -1, -1/2, -1/5, 0, 1/5, 1/2, 1$)

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Finally we give an example in the case of $f_{31}(0) = 0$. We can observe the focal conic is a parabola even if $\tilde{s}$ is close to 0.

Example 3.10. Let f be a deformation of S_1^+ singularity defined by

$$f(u, v, \tilde{s}) = (u, v^2, v^3 + v\tilde{s}^2 + u^2 v).$$

This case, $f_{31}(0) = 0$ holds. The focal conic of f is a parabola when $\tilde{s} \neq 0$ and a line when $\tilde{s} = 0$. See Figure 11 and Figure 12.

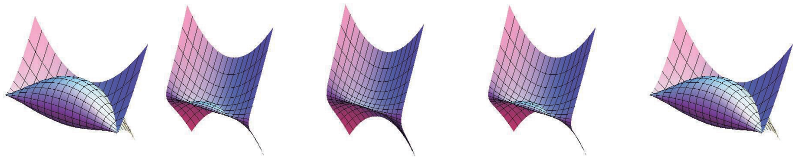

Figure 11: The surfaces in Example 3.10 (from left to right $\tilde{s} = -1, -1/2, 0, 1/2.1$)

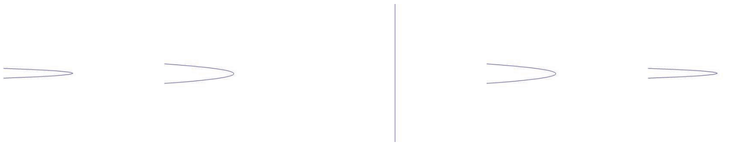

Figure 12: The focal conics of the singularities in Example 3.10 (from left to right $\tilde{s} = -1, -1/2, 0, 1/2, 1$)

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We close this note by giving a geometric meaning of the lowest order coefficients $f_{24}(0,0)$, $f_{34}(0,0)$ including the deformation parameters. The trajectory of the singular points $S({f_n}^{-\tilde{s}^2})$ for the deformation of the $S_1^\pm$ singularities $f = {f_n}^s$ is a space curve passing through the origin. It is parameterized by

$$\gamma(\tilde{s}) := {f_n}^{-\tilde{s}^2}(u(\tilde{s}), 0),$$

where $u(\tilde{s})$ is given in Theorem 3.1. If $f_{21}^2(0) + f_{31}^2(0) \neq 0$, then

$$f_{24}(0,0) = \frac{2\tau f_{31}(0) + \dfrac{\kappa' f_{3320} f_{21}(0)}{\sqrt{f_{21}^2(0) + f_{31}^2(0)}} + 6\dfrac{df_{21}}{du}(0)}{6 f_{3320}^2}$$

and

$$f_{34}(0,0) = \frac{-2\tau f_{21}(0) + \dfrac{\kappa' f_{3320} f_{31}(0)}{\sqrt{f_{21}^2(0) + f_{31}^2(0)}} + 6\dfrac{df_{31}}{du}(0)}{6 f_{3320}^2}$$

hold, where κ and τ are the curvature and the torsion of γ respectively, and $\kappa' = d\kappa/d\tilde{s}$.

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