On vertices of frontals in the Euclidean plane

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1 Introduction

The four vertex theorem is a classical result in differential geometry. It states that there are four vertices for simple closed regular convex plane curves (cf. [2]). In general, there are four vertices for simple closed regular plane curves (cf. [1]). There are counter examples for non-simple closed regular plane curves. See also [9].

We investigate vertices for plane curves with singular points. As smooth curves with singular points, we consider Legendre curves (respectively, Legendre immersions) in the unit tangent bundle over the Euclidean plane and frontals (respectively, fronts) in the Euclidean plane. We define a vertex using evolutes of frontals. After that we define a vertex of a frontal in the general case. We give conditions under which a frontal has a vertex and the four vertex theorem holds for closed frontals.

The content of this paper is based on joint research with Masatomo Takahashi (cf. [8]).

2 Preliminaries

We quickly review the theories of regular curves, Legendre curves on the unit tangent bundle over $\mathbb{R}^2$ and the evolutes of fronts.

Let I be an interval or $\mathbb{R}$, and let $\mathbb{R}^2$ be the Euclidean plane with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$, where $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$.

2.1 Regular plane curves

Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve, that is, $\dot{\gamma}(t) = (d\gamma/dt)(t) \neq 0$ for all $t \in I$. We have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$ and the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$, where J is the anti-clockwise rotation by $\pi/2$ on $\mathbb{R}^2$. Then we have the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where the curvature $\kappa : I \to \mathbb{R}$ is given by

$$\kappa(t) = \frac{\dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)}{|\dot{\gamma}(t)|} = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}.$$

Note that the curvature $\kappa(t)$ is independent of the choice of a parametrization up to sign.

We say that $t_0 \in I$ is an *inflection point* of γ if $\kappa(t_0) = 0$ and a *vertex* of γ if $\dot{\kappa}(t_0) = 0$. Then the classical four vertex theorem is as follows.

Theorem 2.1 (The four vertex theorem of regular curves, [1, 2, 9])**.** If $\gamma : I \to \mathbb{R}^2$ is a simple convex regular curve, then γ has at least four vertices. In general, if $\gamma : I \to \mathbb{R}^2$ is a simple closed regular curve, then γ has at least four vertices.

In particular, [9] has a detailed explanation of vertices and the four-vertex theorem in Japanese.

Suppose that γ does not have inflection points. We define an *evolute* $Ev(\gamma) : I \to \mathbb{R}^2$ of γ by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t).$$

Then t_0 is a singular point of $Ev(\gamma)$ if and only if t_0 is a vertex of γ.

2.2 Legendre curves in the Euclidean plane

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a smooth mapping, where S^1 is the unit circle. We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) *a Legendre immersion.* We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* (respectively, *a front* or *a wave front*) if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. We put on $\mu(t) = J(\nu(t))$ and call the pair $\{\nu(t), \mu(t)\}$ a *moving frame along the frontal* $\gamma(t)$ in $\mathbb{R}^2$. Then we have the Frenet type formula of the Legendre curve which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\mu}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix}, \ \dot{\gamma}(t) = \beta(t)\mu(t),$$

where $\ell(t) = \dot{\nu}(t) \cdot \mu(t), \beta(t) = \dot{\gamma}(t) \cdot \mu(t)$. The pair (ℓ, β) is an important invariant of Legendre curves. We call the pair (ℓ, β) *the (Legendre) curvature of the Legendre curve* (cf. [3]). Then $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion if and only if $(\ell(t), \beta(t)) \neq (0, 0)$ for all $t \in I$. We have the existence and the uniqueness for Legendre curves in the unit tangent bundle like as regular plane curves, see [3].

Theorem 2.2 (The Existence Theorem for Legendre curves [3])**.** Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).

Definition 2.3 (Congruent as Legendre curves, [3])**.** Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\boldsymbol{a}$ on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{a}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$.

Theorem 2.4 (The Uniqueness Theorem for Legendre curves, [3])**.** Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves with curvatures of Legendre curves (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are congruent as Legendre curves if and only if (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$ coincide.

For $n \in \mathbb{N} \cup \{0\}$, we say that a Legendre curve $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ is C^n-*closed* if $(\gamma^{(k)}(a), \nu^{(k)}(a)) = (\gamma^{(k)}(b), \nu^{(k)}(b))$ for all $k \in \{0, \cdots, n\}$, where $\gamma^{(k)}(a)$, $\nu^{(k)}(a)$, $\gamma^{(k)}(b)$ and $\nu^{(k)}(b)$ mean one-sided k-th differential. Similarly, we say that a Legendre curve $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ is C^∞-*closed* if $(\gamma^{(k)}(a), \nu^{(k)}(a)) = (\gamma^{(k)}(b), \nu^{(k)}(b))$ for all $k \in \mathbb{N} \cup \{0\}$. In this paper, we say that (γ, ν) is a *closed* Legendre curve, if the curve is at least C^1-closed (cf. [6]). When a and b are singular points of γ, we treat these singular points as one singular point. Moreover, a frontal $\gamma : [a, b] \to \mathbb{R}^2$ is *simple closed* if for $t_1 < t_2$, we have $\gamma(t_1) = \gamma(t_2)$ if and only if $t_1 = a$ and $t_2 = b$.

We define a convex frontal in the Euclidean plane. From now on, I is a closed interval. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. We denote the tangent line at t of γ by L_t, that is, $L_t = \{\lambda\mu(t) + \gamma(t) \mid \lambda \in \mathbb{R}\}$. Any tangent line L_t divides $\mathbb{R}^2$ into two half-planes H_+ and H_- such that $H_+ \cup H_- = \mathbb{R}^2$ and $H_+ \cap H_- = L_t$. By using ν, the half-planes H_+ and H_- are presented by $H_+ = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (\boldsymbol{x} - \gamma(t)) \cdot \nu(t) \geq 0\}$ and $H_- = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (\boldsymbol{x} - \gamma(t)) \cdot \nu(t) \leq 0\}$. For a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, we say that (γ, ν) is a *convex Legendre curve* (or, γ is a *convex frontal*) if $\gamma(I) \subset H_+$ for any tangent line of γ or $\gamma(I) \subset H_-$ for any tangent line of γ. Note that if γ is a regular curve, then $\mu(t)$ is equal to the unit tangent vector of γ at $\gamma(t)$ up to sign. Therefore, γ is a convex curve as a frontal if and only if γ is a convex curve as the usual mean when γ is a regular curve (cf. [7]).

Theorem 2.5 ([6])**.** Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a closed Legendre curve with curvature (ℓ, β) which the frontal γ is simple closed. Suppose that zeros of ℓ and of β are isolated points. Then the frontal γ is convex if and only if the curvature satisfy one of the following condition:
(i) Both of $\ell(t)$ and $\beta(t)$ are always non-negative,
(ii) $\ell(t)$ is always non-negative and $\beta(t)$ is always non-positive,
(iii) Both of $\ell(t)$ and $\beta(t)$ are always non-positive,
(iv) $\ell(t)$ is always non-positive and $\beta(t)$ is always non-negative.

2.3 Evolutes of fronts

In order to define a vertex of a front, we consider an evolute of the front, in detail see [4]. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β). If (γ, ν) does not have inflection points, namely, $\ell(t) \neq 0$ for all $t \in I$, then (γ, ν) is a Legendre immersion. In this subsection, we assume that (γ, ν) does not have inflection points.

Definition 2.6 (The evolute of the front, [4])**.** The *evolute* $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ of the front γ is given by

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t).$$

Proposition 2.7 ([4])**.** The evolute $\mathcal{E}v(\gamma)$ is also a front. More precisely, $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature

$$\left(\ell(t), \frac{d}{dt}\frac{\beta(t)}{\ell(t)}\right).$$

Definition 2.8 (The vertex of the front, [4]). For a Legendre immersion (γ, ν) with curvature (ℓ, β), t_0 is *a vertex of the front* γ (or *a Legendre immersion* (γ, ν)) if $(d/dt)\mathcal{E}v(t_0) = 0$, namely, $(d/dt)(\beta/\ell)(t_0) = 0$.

Proposition 2.9 ([4]). Let $(\gamma, \nu) : [a, b] \to \mathbb{R}^2 \times S^1$ be a closed Legendre immersion without inflection points.

(1) If γ has at least two singular points which degenerate more than 3/2 cusp, then γ has at least four vertices.

(2) If γ has at least four singular points, then γ has at least four vertices.

3 Evolutes of frontals and vertices

In order to define a vertex of a frontal, we consider an evolute of the frontal, in detail see [5]. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β).

Definition 3.1 (The evolute of the frontal, [5]). The *evolute* $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ *of the frontal* γ is given by

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \alpha(t)\nu(t),$$

if there exists a unique smooth function $\alpha : I \to \mathbb{R}$ such that $\beta(t) = \alpha(t)\ell(t)$. In this case, we say that *the evolute* $\mathcal{E}v(\gamma)$ *exists*.

Proposition 3.2 ([5]). If the evolute $\mathcal{E}v(\gamma)$ of the frontal exists and $\beta = \alpha\ell$, then the evolute $\mathcal{E}v(\gamma)$ is also a frontal. More precisely, $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre curve with the curvature $(\ell, \dot{\alpha})$.

Definition 3.3 (The vertex of the frontal, [5]). We say that t_0 is a *vertex of the frontal* γ (or, *of the Legendre curve* (γ, ν)) if the evolute $\mathcal{E}v(\gamma)$ of the frontal exists and $\dot{\mathcal{E}v}(\gamma)(t_0) = 0$.

Let $n, k \in \mathbb{N}$ and $m = n+k$. We consider a smooth map germ $(\gamma, \nu) : (\mathbb{R}, 0) \to \mathbb{R}^2 \times S^1$,

$$\begin{aligned}
\gamma(t) &= (\pm t^n, t^m f(t)),\\
\nu(t) &= \frac{1}{\sqrt{(mt^k f(t) + t^{k+1}\dot{f}(t))^2 + n^2}}(-mt^k f(t) - t^{k+1}\dot{f}(t), \pm n),
\end{aligned}$$

where $f : (\mathbb{R}, 0) \to \mathbb{R}$ is a smooth function germ with $f(0) \neq 0$. Note that 0 is a singular point of γ when $n > 1$. Then (γ, ν) is a Legendre curve with curvature

$$\begin{aligned}
\ell(t) &= \pm\frac{nt^{k-1}(mkf(t) + (m+k+1)t\dot{f}(t) + t^2\ddot{f}(t))}{(mt^k f(t) + t^{k+1}\dot{f}(t))^2 + n^2},\\
\beta(t) &= -t^{n-1}\sqrt{(mt^k f(t) + t^{k+1}\dot{f}(t))^2 + n^2}.
\end{aligned}$$

We say γ is of type (n, m). Suppose that a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ at $t_0 \in I$ is of type (n, m), that is, (γ, ν) at t_0 is $\mathcal{R}$-equivalent to $(\widetilde{\gamma}, \widetilde{\nu}) : (\mathbb{R}, 0) \to \mathbb{R}^2 \times S^1$,

$$\begin{aligned}
\widetilde{\gamma}(t) &= (\pm t^n, t^m f(t)),\\
\widetilde{\nu}(t) &= \frac{1}{\sqrt{(mt^k f(t) + t^{k+1}\dot{f}(t))^2 + n^2}}(-mt^k f(t) - t^{k+1}\dot{f}(t), \pm n),
\end{aligned}$$

where $f : (\mathbb{R}, 0) \to \mathbb{R}$ is a smooth function germ with $f(0) \neq 0$ and the curvature $(\widetilde{\ell}, \widetilde{\beta})$. Then we say that γ *has a point* t_0 *of type* (n, m).

If $n \geq k$, then there exists a unique function $\widetilde{\alpha} : (\mathbb{R}, 0) \to \mathbb{R}$,

$$\widetilde{\alpha}(t) = \mp \frac{t^{n-k}((mt^k f(t) + t^{k+1}\dot{f}(t))^2 + n^2)^{\frac{3}{2}}}{n(mkf(t) + (m+k+1)t\dot{f}(t) + t^2\ddot{f}(t))}$$

such that $\widetilde{\beta} = \widetilde{\alpha}\widetilde{\ell}$.

Proposition 3.4 ([8])**.** Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β) and $\beta = \alpha\ell$.

(1) In the case of $n = k$. If γ has four points of type $(n, m) = (n, 2n)$ with $\dot{f}(0) = 0$, then γ has at least four vertices.

(2) In the case of $n = k + 1$. If γ has four points of type $(n, m) = (n, 2n - 1)$, then γ has at least four vertices.

(3) In the case of $n \geq k + 2$. If γ has two points of type (n, m), then γ has at least four vertices.

Theorem 3.5 ([8])**.** Let $(\gamma, \nu) : I = [a, b] \to \mathbb{R}^2 \times S^1$ be a closed Legendre curve with curvature (ℓ, β) and $\beta = \alpha\ell$. Suppose that zeros of ℓ and of β are isolated points. If γ is a simple convex frontal, then γ has at least four vertices.

4 Vertices of frontals in the general case

In the previous section, we define the vertex using the evolute of the Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ with curvature (ℓ, β) under the condition $\beta = \alpha\ell$. We give a definition of a generalisation of the vertex of frontals.

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β). In this section, we assume that ℓ and β are linearly dependent on I, that is, there exists a non-zero smooth mapping $(k_1, k_2) : I \to \mathbb{R}^2 \setminus \{0\}$ such that $k_1(t)\ell(t) + k_2(t)\beta(t) = 0$ for all $t \in I$.

Definition 4.1 (Vertex of the frontal, [8])**.** Under the above assumption, we say that $t_0 \in I$ is a *vertex* of the frontal γ (or, *of the Legendre curve* (γ, ν)) if $\dot{k}_1(t_0)k_2(t_0) - k_1(t_0)\dot{k}_2(t_0) = 0$.

If $\beta(t) \neq 0$ for all $t \in I$, then γ is a regular curve and we can take $k_1(t) = 1, k_2(t) = -\ell(t)/\beta(t)$. It follows that t_0 is a vertex of γ as a regular curve if and only if $t_0 \in I$ is a vertex of γ as the frontal. Moreover, if $\ell(t) \neq 0$ for all $t \in I$, then (γ, ν) is a Legendre immersion and we can take $k_1(t) = -\beta(t)/\ell(t), k_2(t) = 1$. It follows that t_0 is a vertex of the front γ if and only if $t_0 \in I$ is a vertex of γ as the frontal. Furthermore, if $\beta(t) = \alpha(t)\ell(t)$ for all $t \in I$, we can take $k_1(t) = -\alpha(t), k_2(t) = 1$. It follows that t_0 is a vertex of the frontal γ in the sense of the previous section if and only if $t_0 \in I$ is a vertex of γ as the frontal.

Proposition 4.2 ([8])**.** Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β) and $\ell = \alpha\beta$, where $\alpha : I \to \mathbb{R}$ is a smooth function.

(1) In the case of $n + 1 = k$. If γ has four points of type $(n, m) = (n, 2n + 1)$, then γ has at least four vertices.

(2) In the case of $n + 2 \leq k$. If γ has two points of type (n, m), then γ has at least four vertices.

In general case, we have the four vertex theorem of frontals by the Propositions 3.4 and 4.2.

Theorem 4.3 ([8])**.** Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with curvature (ℓ, β). Suppose that there exists a non-zero smooth mapping $(k_1, k_2) : I \to \mathbb{R}^2 \setminus \{0\}$ such that $k_1(t)\ell(t) + k_2(t)\beta(t) = 0$ for all $t \in I$. If there exist four points of the following types (n, m):

$$\text{(i) } n = k \text{ and } \dot{f}(0) = 0, \quad \text{(ii) } n \geq k + 2, \quad \text{(iii) } n + 2 \leq k,$$

then γ has at least four vertices.

Theorem 4.4 ([8])**.** Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a closed Legendre immersion with curvature (ℓ, β). If γ is a simple convex front with at least two singular and two inflection points, then γ has at least four vertices.

Example 4.5. Let $(\gamma, \nu) : [0, 2\pi] \to \mathbb{R}^2 \times S^1$ be

$$\gamma(t) = \left(\frac{1}{5}\cos^5 t, \frac{1}{5}\sin^5 t\right), \ \nu(t) = \frac{1}{\sqrt{\cos^6 t + \sin^6 t}}\left(-\sin^3 t, -\cos^3 t\right).$$

By a direct calculation, (γ, ν) is a closed Legendre curve with the curvature

$$\ell(t) = -\frac{3\cos^2 t \sin^2 t}{\cos^6 t + \sin^6 t}, \ \beta(t) = -\cos t \sin t \sqrt{\cos^6 t + \sin^6 t}.$$

If $k_1(t) = -(\cos^6 t + \sin^6 t)^{\frac{3}{2}}$, $k_2(t) = 3\cos t \sin t$, then (k_1, k_2) is a non-zero smooth function and $k_1(t)\ell(t) + k_2(t)\beta(t) = 0$ for all $t \in I$. By a direct calculation, we have

$$\dot{k}_1(t) = 9\cos 2t \cos t \sin t \sqrt{\cos^6 t + \sin^6 t}, \ \dot{k}_2(t) = 3\cos 2t.$$

Since

$$\dot{k}_1(t)k_2(t) - k_1(t)\dot{k}_2(t) = 3\cos 2t\sqrt{\cos^6 t + \sin^6 t}(9\cos^2 t \sin^2 t + \cos^6 t + \sin^6 t),$$

$\dot{k}_1(t)k_2(t) - k_1(t)\dot{k}_2(t) = 0$ if and only if $\cos 2t = 0$. It follows that $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ are vertices of γ. Therefore, γ has four vertices. Note that the singular points of γ are 5/2-cusps at $t = 0, \pi/2, \pi, 3\pi/2$. See Figure 1 left.

Example 4.6. Let $(\gamma, \nu) : [0, 2\pi] \to \mathbb{R}^2 \times S^1$ be

$$\gamma(t) = \left(\cos t, \frac{1}{3}\sin^3 t\right), \ \nu(t) = -\frac{1}{\sqrt{\cos^2 t \sin^2 t + 1}}\left(\cos t \sin t, 1\right).$$

By a direct calculation, (γ, ν) is a closed Legendre curve with the curvature

$$\ell(t) = -\frac{\cos 2t}{\cos^2 t \sin^2 t + 1}, \ \beta(t) = -\sin t\sqrt{\cos^2 t \sin^2 t + 1}.$$

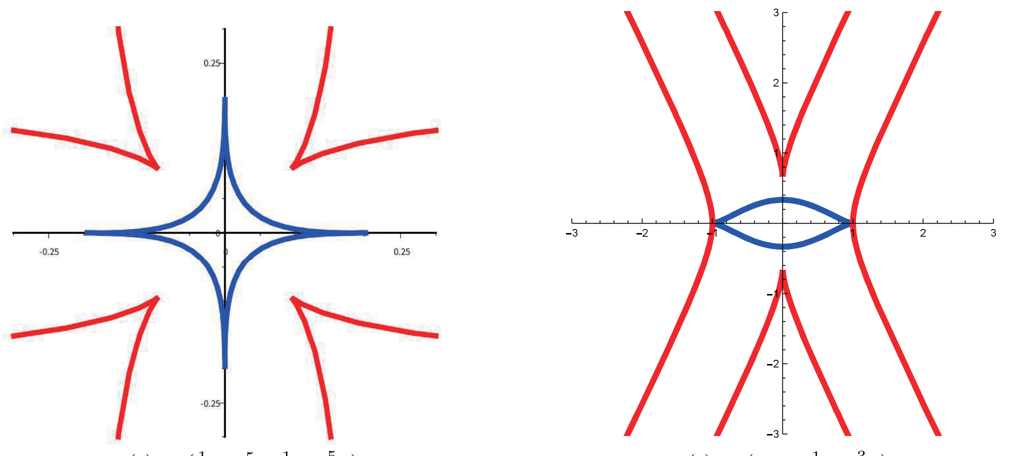
Note that (γ, ν) is a closed Legendre immersion, since $(\ell(t), \beta(t)) \neq (0, 0)$ for all $t \in [0, 2\pi]$. By Theorem 2.5, γ is not a convex front. Moreover, $t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ are inflection points of γ. If $k_1(t) = -\sin t(\cos^2 t \sin^2 t + 1)^{\frac{3}{2}}$, $k_2(t) = \cos 2t$, then (k_1, k_2) is a non-zero smooth function and $k_1(t)\ell(t) + k_2(t)\beta(t) = 0$ for all $t \in I$. By a direct calculation, we have

$$\dot{k}_1(t) = -\cos t(\cos^2 t \sin^2 t + 1 + 3\sin^2 t \cos 2t)\sqrt{\cos^2 t \sin^2 t + 1}, \dot{k}_2(t) = -2\sin 2t.$$

Since

$$\begin{aligned} &\dot{k}_1(t)k_2(t) - k_1(t)\dot{k}_2(t) \\ &= -\cos t\sqrt{\cos^2 t\sin^2 t + 1}(\cos 2t(\cos^2 t \sin^2 t + 1 + 3\sin^2 t\cos 2t) + 4\sin^2 t(\cos^2 t\sin^2 t + 1)), \end{aligned}$$

$\dot{k}_1(t)k_2(t) - k_1(t)\dot{k}_2(t) = 0$ if and only if $\cos t = 0$. It follows that γ has only two vertices. Therefore, it does not satisfy on the four vertex theorem. Note that the singular points of γ are 3/2-cusps at $t = 0, \pi$. See Figure 1 right.



$\gamma(t) = (\frac{1}{5}\cos^5 t, \frac{1}{5}\sin^5 t)$ $\qquad$ $\gamma(t) = (\cos t, \frac{1}{3}\sin^3 t)$

Figure 1: Closed Legendre curves

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