Null type framed surfaces in Minkowski 3-space

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Abstract

In this paper, we define null type framed surfaces in Minkowski 3-space and give a classification of null type framed base surfaces. Null type framed base surfaces contain regular null surfaces, but they are allowed to have singularities. Then we define the Bertrand offsets of null tangent developables, whose striction curves are null, and give some properties of Bertrand offsets.

1 Introduction

A smooth regular surface in Minkowski 3-space is called spacelike, timelike or null if its tangent planes are spacelike, timelike or null at each point, respectively. As for null surfaces, their normal vectors lie in tangent planes. It is obvious that null planes and the open nullcone are both null surfaces. But they are trivial. Inoguchi and Lee researched null revolution surfaces in Minkowski 3-space. They found that all null revolution surfaces are congruent to null planes or the nullcone [1]. In [2], Carlsen and Clelland classified the null surfaces of constant type. A surface of constant type means that the surface has the second-order contact with its tangent plane either everywhere or nowhere on the plane. They claimed every null surface of constant type is made up of null lines. The regular null surface of constant type is contained in a null plane, a nullcone or a tangent developable of a null curve.

Combining surfaces and the singularity theory, Fukunaga and Takahashi defined framed surfaces [3]. A framed surface is a smooth surface with a moving frame and it is allowed to contain singularities. In Minkowski 3-space, when a surface contains both null points and non-null points, we call it a mixed type surface. To research this, the lightcone frame is a useful tool. Li, Pei and Takahashi defined lightcone framed surfaces in Minkowski 3-space [4]. If a singular surface is null at its regular parts, it is a special mixed type surface. It has null geometry properties. We call it null type framed base surface. Of course, we also think regular null surfaces are null type framed base surfaces. As an application of the null type framed surfaces, we define their Bertrand offsets with null striction curve.

This paper is arranged as follows. First, we define generalized null framed surfaces in Minkowski 3-space and give the classification of them. Then we define Bertrand offsets

Key words and phrases: null type framed surface, null tangent developable, Bertrand offset.

of null tangent developables, whose striction curves are null, and give some properties of Bertrand offsets. Finally, we give two examples to show the Bertrand offsets of null tangent developables.

All maps and manifolds considered in this paper are differentiable of class C^∞.

2 Preliminaries

Let $\mathbb{R}^3$ be a 3-dimensional real vector space. For any vectors $\boldsymbol{y} = (y_1, y_2, y_3), \boldsymbol{z} = (z_1, z_2, z_3) \in \mathbb{R}^3$, their pseudo inner product is defined by

$$\langle \boldsymbol{y}, \boldsymbol{z} \rangle = -y_1 z_1 + y_2 z_2 + y_3 z_3.$$

We call $(\mathbb{R}^3, \langle\ ,\ \rangle)$ Minkowski 3-space and denote it $\mathbb{R}^3_1$. The pseudo vector product between $\boldsymbol{y}$ and $\boldsymbol{z}$ is

$$\boldsymbol{y} \wedge \boldsymbol{z} = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis in $\mathbb{R}^3_1$.

For any non-zero vector $\boldsymbol{y} \in \mathbb{R}^3_1$, it is called spacelike, timelike or null if $\langle \boldsymbol{y}, \boldsymbol{y} \rangle$ is positive, negative or zero, respectively. We call both spacelike vectors and timelike vectors non-null vectors. For any vectors $\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \in \mathbb{R}^3_1$, we define $\det(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) = \langle \boldsymbol{y} \wedge \boldsymbol{z}, \boldsymbol{w} \rangle$. The norm of $\boldsymbol{y}$ is $||\boldsymbol{y}|| = \sqrt{|\langle \boldsymbol{y}, \boldsymbol{y} \rangle|}$.

There are three pseudo spheres in $\mathbb{R}^3_1$:

$$S^2_1 = \{\boldsymbol{y} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{y}, \boldsymbol{y} \rangle = 1\},$$

$$H^2_0 = \{\boldsymbol{y} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{y}, \boldsymbol{y} \rangle = -1\}$$

and

$$NC^* = \{\boldsymbol{y} \in \mathbb{R}^3_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{y}, \boldsymbol{y} \rangle = 0\}.$$

We call them the de Sitter 2-space, the hyperbolic 2-space and the nullcone, respectively. H^2_0 can be divided into two connected subsets, that is $H^2_0 = H^2_+ \cup H^2_-$, where

$$H^2_\epsilon = \{\boldsymbol{y} = (y_1, y_2, y_3) \in H^2_0 \mid \epsilon y_1 > 0\},\ \epsilon = + \text{ or } -.$$

We denote

$$\Delta_4 = \{(\boldsymbol{\nu_1}, \boldsymbol{\nu_2}) \in NC^* \times NC^* \mid \langle \boldsymbol{\nu_1}, \boldsymbol{\nu_2} \rangle = -2\}.$$

Let I be an interval. A curve $\gamma : I \to \mathbb{R}^3_1$ is called spacelike, timelike or null if $\gamma'(t)$ is spacelike, timelike or null for all $t \in I$, respectively.

Inspired by Cartan curves [5], framed curves [6] and the lightcone frame [7], we defined null type curves in [8]. A null type curve is a smooth curve with a null vector field along it. The curve is allowed to have singularities and its tangent vectors are null at regular parts.

Definition 2.1 ([8]) *We call $\gamma : I \to \mathbb{R}^3_1$ a null type curve if there exists a function $\alpha : I \to \mathbb{R}$ and a nullconical curve $\mathsf{T} : I \to NC^*$ such that $\gamma'(t) = \alpha(t)\mathsf{T}(t)$ for all $t \in I$.*

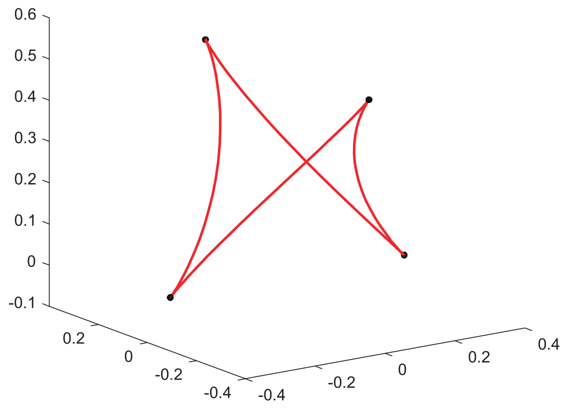

Figure 1: A null type curve with four singularities.

It is obvious that $t_0 \in I$ is a singularity of γ if and only if $\alpha(t_0) = 0$.

If $\langle \mathsf{T}'(t), \mathsf{T}'(t)\rangle \neq 0$ for any $t \in I$, we take $s = \int_{t_0}^{t} ||\mathsf{T}'(u)||\mathrm{d}u$. The parameter s is called the pseudo arc-length parameter of γ. Let

$$\mathsf{N}(s) = \mathsf{T}'(s),\ \mathsf{B}(s) = -\mathsf{T}''(s) - \frac{1}{2}\langle \mathsf{T}''(s), \mathsf{T}''(s)\rangle \mathsf{T}(s).$$

We get the Cartan-type frame $\{\mathsf{T}(s), \mathsf{N}(s), \mathsf{B}(s)\}$ of γ, where $\mathsf{T}(s)$ and $\mathsf{B}(s)$ are null and $\langle \mathsf{T}(s), \mathsf{B}(s)\rangle = 1$. $\mathsf{N}(s)$ is a unit spacelike vector and pseudo orthogonal to $\mathsf{T}(s)$ and $\mathsf{B}(s)$.

At this point, we call γ a null type Cartan curve. The Cartan-type frame can be regarded as a special case of the lightcone frame. The Frenet-type formula is

$$\begin{pmatrix} \mathsf{T}'(s) \\ \mathsf{B}'(s) \\ \mathsf{N}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \tau_l(s) \\ -\tau_l(s) & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{T}(s) \\ \mathsf{B}(s) \\ \mathsf{N}(s) \end{pmatrix},$$

where $\tau_l(s) = \frac{1}{2}\langle \mathsf{T}''(s), \mathsf{T}''(s)\rangle$. (τ_l, α) is called the curvature of γ.

In [4], Li, Pei and Takahashi defined lightcone framed surfaces, which are smooth surfaces with a lightcone frame on it.

Definition 2.2 ([4]) *U is an open domain in $\mathbb{R}^2$. We call $(\boldsymbol{x}, \ell^+, \ell^-) : U \to \mathbb{R}^3_1 \times \Delta_4$ a lightcone framed surface if there exist two functions $\beta_1, \beta_2 : U \to \mathbb{R}$ such that*

$$\boldsymbol{x}_u(u,v) \wedge \boldsymbol{x}_v(u,v) = \beta_1(u,v)\ell^+(u,v) + \beta_2(u,v)\ell^-(u,v)$$

for any $(u,v) \in U$. We call $\boldsymbol{x} : U \to \mathbb{R}^3_1$ a lightcone framed base surface if there exists $(\ell^+, \ell^-) : U \to \Delta_4$ such that $(\boldsymbol{x}, \ell^+, \ell^-)$ is a lightcone framed surface.

Define $\boldsymbol{t}(u,v) = -\frac{1}{2}(\ell^+(u,v) \wedge \ell^-(u,v))$. It is a unit spacelike vector. Then $\{\ell^+(u,v), \ell^-(u,v), \boldsymbol{t}(u,v)\}$ is a lightcone frame on $\boldsymbol{x}$ and we have

$$\begin{pmatrix} \boldsymbol{x}_u(u,v) \\ \boldsymbol{x}_v(u,v) \end{pmatrix} = \begin{pmatrix} a_1(u,v) & b_1(u,v) & c_1(u,v) \\ a_2(u,v) & b_2(u,v) & c_2(u,v) \end{pmatrix} \begin{pmatrix} \ell^+(u,v) \\ \ell^-(u,v) \\ \boldsymbol{t}(u,v) \end{pmatrix},$$

$$\begin{pmatrix} \ell^+_u(u,v) \\ \ell^-_u(u,v) \\ \boldsymbol{t}_u(u,v) \end{pmatrix} = \begin{pmatrix} e_1(u,v) & 0 & 2g_1(u,v) \\ 0 & -e_1(u,v) & 2f_1(u,v) \\ f_1(u,v) & g_1(u,v) & 0 \end{pmatrix} \begin{pmatrix} \ell^+(u,v) \\ \ell^-(u,v) \\ \boldsymbol{t}(u,v) \end{pmatrix},$$

$$\begin{pmatrix} \ell^+_v(u,v) \\ \ell^-_v(u,v) \\ \boldsymbol{t}_v(u,v) \end{pmatrix} = \begin{pmatrix} e_2(u,v) & 0 & 2g_2(u,v) \\ 0 & -e_2(u,v) & 2f_2(u,v) \\ f_2(u,v) & g_2(u,v) & 0 \end{pmatrix} \begin{pmatrix} \ell^+(u,v) \\ \ell^-(u,v) \\ \boldsymbol{t}(u,v) \end{pmatrix},$$

where

$$\begin{array}{lll} a_1(u,v) = -\frac{1}{2}\langle \boldsymbol{x}_u, \ell^-\rangle(u,v), & b_1(u,v) = -\frac{1}{2}\langle \boldsymbol{x}_u, \ell^+\rangle(u,v), & c_1(u,v) = \langle \boldsymbol{x}_u, \boldsymbol{t}\rangle(u,v), \\ a_2(u,v) = -\frac{1}{2}\langle \boldsymbol{x}_v, \ell^-\rangle(u,v), & b_2(u,v) = -\frac{1}{2}\langle \boldsymbol{x}_v, \ell^+\rangle(u,v), & c_2(u,v) = \langle \boldsymbol{x}_v, \boldsymbol{t}\rangle(u,v), \\ e_1(u,v) = -\frac{1}{2}\langle \ell^+_u, \ell^-\rangle(u,v), & f_1(u,v) = -\frac{1}{2}\langle \boldsymbol{t}_u, \ell^-\rangle(u,v), & g_1(u,v) = -\frac{1}{2}\langle \boldsymbol{t}_u, \ell^+\rangle(u,v), \\ e_2(u,v) = -\frac{1}{2}\langle \ell^+_v, \ell^-\rangle(u,v), & f_2(u,v) = -\frac{1}{2}\langle \boldsymbol{t}_v, \ell^-\rangle(u,v), & g_2(u,v) = -\frac{1}{2}\langle \boldsymbol{t}_v, \ell^+\rangle(u,v). \end{array}$$

When one of $\beta_1(u,v)$ and $\beta_2(u,v)$ is zero and another is non-zero for any $(u,v) \in U$, the surface $\boldsymbol{x}$ is null.

3 Null type framed surfaces

In this section, we study null type framed base surfaces, which are special mixed type surfaces. However, since the base surfaces are null except singularities, we give a more useful tool to describe them.

Let $\Delta = \{(\boldsymbol{\nu_1}, \boldsymbol{\nu_2}) \in NC^* \times NC^* \mid \langle \boldsymbol{\nu_1}, \boldsymbol{\nu_2}\rangle = 1\}$ and U be an open domain in $\mathbb{R}^2$.

Definition 3.1 *A surface $(\boldsymbol{x}, \boldsymbol{n}_1, \boldsymbol{n}_2) : U \to \mathbb{R}^3_1 \times \Delta$ is called a null type framed surface (generalized null framed surface) if $\langle \boldsymbol{x}_u(u,v), \boldsymbol{n}_1(u,v)\rangle = \langle \boldsymbol{x}_v(u,v), \boldsymbol{n}_1(u,v)\rangle = 0$ for any $(u,v) \in U$. We call $\boldsymbol{x} : U \to \mathbb{R}^3_1$ a null type base surface if there exists $(\boldsymbol{n}_1, \boldsymbol{n}_2) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is a null type framed surface.*

Define $\boldsymbol{s}(u,v) = \boldsymbol{n}_1(u,v) \wedge \boldsymbol{n}_2(u,v)$. Then $\langle \boldsymbol{s}(u,v), \boldsymbol{s}(u,v)\rangle = 1$ and $\{\boldsymbol{n}_1(u,v), \boldsymbol{n}_2(u,v), \boldsymbol{s}(u,v)\}$ is a moving frame on the surface $\boldsymbol{x}$. We have the following formulas

$$\begin{pmatrix} \boldsymbol{x}_u(u,v) \\ \boldsymbol{x}_v(u,v) \end{pmatrix} = \begin{pmatrix} a_1(u,v) & b_1(u,v) \\ a_2(u,v) & b_2(u,v) \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(u,v) \\ \boldsymbol{s}(u,v) \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{n}_{1u}(u,v) \\ \boldsymbol{n}_{2u}(u,v) \\ \boldsymbol{s}_u(u,v) \end{pmatrix} = \begin{pmatrix} e_1(u,v) & 0 & f_1(u,v) \\ 0 & -e_1(u,v) & g_1(u,v) \\ -g_1(u,v) & -f_1(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(u,v) \\ \boldsymbol{n}_2(u,v) \\ \boldsymbol{s}(u,v) \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{n}_{1v}(u,v) \\ \boldsymbol{n}_{2v}(u,v) \\ \boldsymbol{s}_v(u,v) \end{pmatrix} = \begin{pmatrix} e_2(u,v) & 0 & f_2(u,v) \\ 0 & -e_2(u,v) & g_2(u,v) \\ -g_2(u,v) & -f_2(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(u,v) \\ \boldsymbol{n}_2(u,v) \\ \boldsymbol{s}(u,v) \end{pmatrix},$$

where

$$\begin{array}{ll} a_1(u,v) = \langle \boldsymbol{x}_u(u,v), \boldsymbol{n}_2(u,v)\rangle, & a_2(u,v) = \langle \boldsymbol{x}_v(u,v), \boldsymbol{n}_2(u,v)\rangle, \\ b_1(u,v) = \langle \boldsymbol{x}_u(u,v), \boldsymbol{s}(u,v)\rangle, & b_2(u,v) = \langle \boldsymbol{x}_v(u,v), \boldsymbol{s}(u,v)\rangle, \\ e_1(u,v) = \langle \boldsymbol{n}_{1u}(u,v), \boldsymbol{n}_2(u,v)\rangle, & e_2(u,v) = \langle \boldsymbol{n}_{1v}(u,v), \boldsymbol{n}_2(u,v)\rangle, \\ f_1(u,v) = \langle \boldsymbol{n}_{1u}(u,v), \boldsymbol{s}(u,v)\rangle, & f_2(u,v) = \langle \boldsymbol{n}_{1v}(u,v), \boldsymbol{s}(u,v)\rangle, \\ g_1(u,v) = \langle \boldsymbol{n}_{2u}(u,v), \boldsymbol{s}(u,v)\rangle, & g_2(u,v) = \langle \boldsymbol{n}_{2v}(u,v), \boldsymbol{s}(u,v)\rangle. \end{array}$$

The ten functions are called basic invariants of the null type framed surface $(\boldsymbol{x}, \boldsymbol{n}_1, \boldsymbol{n}_2)$.

A point $(u_0, v_0) \in U$ is the singularity of $\boldsymbol{x}$ if $\boldsymbol{x}_u(u_0, v_0) \wedge \boldsymbol{x}_v(u_0, v_0) = \boldsymbol{0}$. We define a set S as all singularities of $\boldsymbol{x}$, then

$$S = \{(u, v) \in U \mid a_1(u, v)b_2(u, v) - a_2(u, v)b_1(u, v) = 0\}.$$

Next, we discuss what the null type framed base surfaces look like.

Theorem 3.2 *Locally, the null type framed base surface is a null plane, a nullcone or a tangent developable of a null type curve.*

The null plane and the nullcone are trivial. Next, we only consider the null tangent developable.

4 Bertrand offsets of null tangent developables

In this section, we always suppose that γ, $\bar{\gamma} : I \to \mathbb{R}^3_1$ are two different null type Cartan curves and $\boldsymbol{x}(u, v) = \gamma(u) + v\mathsf{T}(u)$ is the null tangent developable of γ, where $u \in I$ is the pseudo arc-length parameter of γ. $\{\mathsf{T}(u), \mathsf{N}(u), \mathsf{B}(u)\}$ is a moving frame on $\boldsymbol{x}$. We call it the Blaschke frame and $\mathsf{N}(u)$ the central normal vector to the surface $\boldsymbol{x}$. For the Blaschke frame in the Euclidean space case, please see [9].

For the curve $\bar{\gamma}$ and a regular pseudo spherical curve $\boldsymbol{e}_1 : I \to \mathbb{R}^3_1$, we construct a ruled surface $\bar{\boldsymbol{x}}(u, v) = \bar{\gamma}(u) + v\boldsymbol{e}_1(u)$, whose striction curve is $\bar{\gamma}$. Define $\boldsymbol{e}_2(u) = \dfrac{\boldsymbol{e}_1'(u)}{||\boldsymbol{e}_1'(u)||}$. Let $\boldsymbol{e}_3(u)$ be a pseudo spherical vector field which is linearly independent to $\boldsymbol{e}_1(u)$ and $\boldsymbol{e}_2(u)$. Then $\{\boldsymbol{e}_1(u), \boldsymbol{e}_2(u), \boldsymbol{e}_3(u)\}$ is a moving frame of $\bar{\boldsymbol{x}}$, called the Blaschke frame (Figure 2). $\boldsymbol{e}_2(u)$ is called the central normal vector to the surface $\bar{\boldsymbol{x}}$.

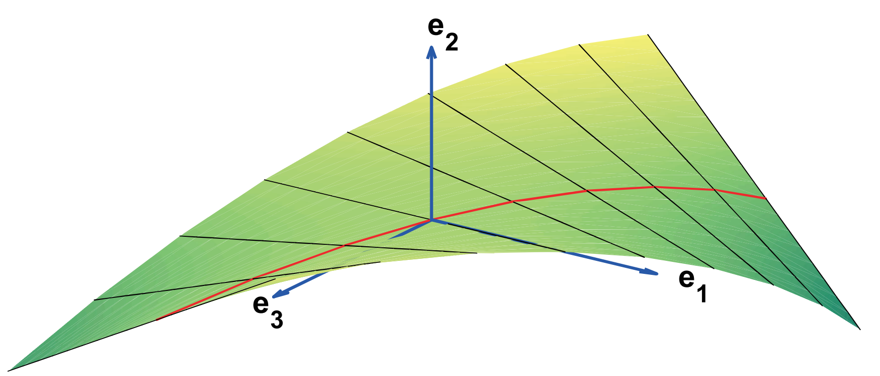

Figure 2: Blaschke frame.

Definition 4.1 *Let $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ are two ruled surfaces we define above. We say $\bar{\boldsymbol{x}}$ is a Bertrand offset of $\boldsymbol{x}$ if there exists a correspondence between their rulings such that $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ have the common central normal line at the striction point of their correspondent rulings.*

We introduce a signature ε such that $\mathsf{N}(u) = \varepsilon \boldsymbol{e}_2(u)$, where $\varepsilon = 1$ or -1. For the Bertrand offsets with non-null rulings and null rulings, they have different properties. We discuss them respectively.

4.1 Bertrand offsets with non-null rulings

Denote that $M_\delta = \begin{cases} S_1^2, \ \delta = 1, \\ H_+^2, \ \delta = -1 \end{cases}$ is a 2-dimensional manifold. $\boldsymbol{e}_1 : I \to M_\delta$ is a spacelike curve on M_δ. Let $\boldsymbol{e}_2(u) = \dfrac{\boldsymbol{e}_1'(u)}{||\boldsymbol{e}_1'(u)||}$ and $\boldsymbol{e}_3(u) = \boldsymbol{e}_1(u) \wedge \boldsymbol{e}_2(u)$. Then

$$\langle \boldsymbol{e}_1(u), \boldsymbol{e}_1(u)\rangle = \delta, \ \langle \boldsymbol{e}_2(u), \boldsymbol{e}_2(u)\rangle = 1, \langle \boldsymbol{e}_3(u), \boldsymbol{e}_3(u)\rangle = -\delta.$$

The Blaschke formula is

$$\begin{pmatrix} \boldsymbol{e}_1'(u) \\ \boldsymbol{e}_2'(u) \\ \boldsymbol{e}_3'(u) \end{pmatrix} = \begin{pmatrix} 0 & p(u) & 0 \\ -\delta p(u) & 0 & -\delta q(u) \\ 0 & -q(u) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1(u) \\ \boldsymbol{e}_2(u) \\ \boldsymbol{e}_3(u) \end{pmatrix},$$

where

$$p(u) = ||\boldsymbol{e}_1'(u)||, \ q(u) = \frac{\det(\boldsymbol{e}_1(u), \boldsymbol{e}_1'(u), \boldsymbol{e}_1''(u))}{||\boldsymbol{e}_1'(u)||^2}$$

are called Blaschke invariants.

Now, we give some properties of Bertrand offsets with non-null rulings.

Proposition 4.2 *Let $\bar{\boldsymbol{x}}(u,v) = \bar{\gamma}(u) + v\boldsymbol{e}_1(u)$ be a Bertrand offset of $\boldsymbol{x}(u,v) = \gamma(u) + v\mathsf{T}(u)$. (τ_l, α) is the curvature of γ. Then there exists a non-zero constant λ such that $\bar{\gamma}(u) = \gamma(u) + \lambda \mathsf{N}(u)$ and $\alpha(u) = \lambda \tau_l(u)$. $\bar{\gamma}$ is a regular null curve.*

For a surface $\boldsymbol{x}$ and its Bertrand offset $\bar{\boldsymbol{x}}$, since their central normal vectors N and $\boldsymbol{e}_2$ are parallel, the other vectors in their Blaschke frames have the following relation.

Proposition 4.3 *Let $\bar{\boldsymbol{x}}$ be a Bertrand offset of $\boldsymbol{x}$. Their Blaschke frames are $\{\boldsymbol{e}_1(u), \boldsymbol{e}_2(u), \boldsymbol{e}_3(u)\}$ and $\{\mathsf{T}(u), \mathsf{N}(u), \mathsf{B}(u)\}$, respectively. Then there exist four non-zero constants c_1, c_2, c_3, c_4 such that*

$$\begin{pmatrix} \mathsf{T}(u) \\ \mathsf{B}(u) \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1(u) \\ \boldsymbol{e}_3(u) \end{pmatrix}$$

and

$$c_1^2 = c_2^2, \ c_3^2 = c_4^2, \ c_1c_4 + c_2c_3 = 0, \ c_1c_3 = \frac{1}{2}\delta, \ c_2c_4 = -\frac{1}{2}\delta,$$

where

$$\delta = \langle \boldsymbol{e}_1(u), \boldsymbol{e}_1(u)\rangle.$$

We give the relation between the curvature of γ and the Blaschke invariants.

Corollary 4.4 *Let $\bar{\boldsymbol{x}}$ be a Bertrand offset of $\boldsymbol{x}$. $p(u)$ and $q(u)$ are Bertrand invariants of $\bar{\boldsymbol{x}}$. Then there exist two non-zero constants A, B, such that $Ap(u) + Bq(u) = 1$.*

Proposition 4.5 *Let $\bar{\boldsymbol{x}}$ be a Bertrand offset of $\boldsymbol{x}(u,v) = \gamma(u) + v\,\mathsf{T}(u)$. (τ_l, α) is the curvature of γ. $p(u)$ and $q(u)$ are Bertrand invariants of $\bar{\boldsymbol{x}}$. $\delta = \langle \boldsymbol{e}_1(u), \boldsymbol{e}_1(u)\rangle$. c_3 and c_4 are two constants in Proposition 4.3. Then*

$$\tau_l(u) = \varepsilon(c_3 p(u) - c_4 q(u)) = \frac{1}{2}\delta(p^2(u) - q^2(u)).$$

4.2 Bertrand offsets with null rulings

In this subsection, we research the Bertrand offsets with null rulings.

Denote $S^1_+ = \{\boldsymbol{y} = (1, y_2, y_3) \mid y_2^2 + y_3^2 = 1\}$. It is a spacelike circle on the nullcone. Let $\theta : I \to \mathbb{R}$ satisfy $\theta'(u) > 0$ for any $u \in I$ and

$$\begin{aligned}
\boldsymbol{e}_1(u) &= (1, \cos\theta(u), \sin\theta(u)),\\
\boldsymbol{e}_2(u) &= \frac{\boldsymbol{e}_1'(u)}{||\boldsymbol{e}_1'(u)||} = (0, -\sin\theta(u), \cos\theta(u)),\\
\boldsymbol{e}_3(u) &= \frac{1}{2}(-1, \cos\theta(u), \sin\theta(u)).
\end{aligned}$$

The Blaschke formula is

$$\begin{pmatrix} \boldsymbol{e}_1'(u) \\ \boldsymbol{e}_2'(u) \\ \boldsymbol{e}_3'(u) \end{pmatrix} = \begin{pmatrix} 0 & \theta'(u) & 0 \\ -\frac{1}{2}\theta'(u) & 0 & -\theta'(u) \\ 0 & \frac{1}{2}\theta'(u) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1(u) \\ \boldsymbol{e}_2(u) \\ \boldsymbol{e}_3(u) \end{pmatrix}.$$

$\theta'(u)$ is called the Blaschke invariant.

Proposition 4.6 *Let $\bar{\boldsymbol{x}}$ be a Bertrand offset of $\boldsymbol{x}$. Then the Blaschke invariant $\theta'(u)$ is a non-zero constant.*

Theorem 4.7 *Let $\bar{\boldsymbol{x}}(u,v) = \bar{\gamma}(u) + v\boldsymbol{e}_1(u)$ be a Bertrand offset of $\boldsymbol{x}(u,v) = \gamma(u) + v\,\mathsf{T}(u)$. Then γ is a regular null helix.*

5 Examples

Here, we give two examples to show the Bertrand offsets with null rulings and non-null rulings, respectively.

Example 5.1 Let $(\gamma, \mathsf{T}) : (0,1) \to \mathbb{R}^3_1 \times NC^*$ be

$$\gamma(u) = \left(\frac{1}{u}, \sin\frac{1}{u}, -\cos\frac{1}{u}\right), \ \mathsf{T}(u) = -u^2\left(1, \cos\frac{1}{u}, \sin\frac{1}{u}\right).$$

Then we know

$$\mathsf{N}(u) = -2u\left(1, \cos\frac{1}{u}, \sin\frac{1}{u}\right) + \left(0, -\sin\frac{1}{u}, \cos\frac{1}{u}\right),$$

$$\mathsf{B}(u) = \left(2 + \frac{1}{2u^2}\right)\left(1, \cos\frac{1}{u}, \sin\frac{1}{u}\right) + \frac{2}{u}\left(0, \sin\frac{1}{u}, \cos\frac{1}{u}\right) - \frac{1}{u^2}\left(0, \cos\frac{1}{u}, \sin\frac{1}{u}\right),$$

$$\alpha(u)=\frac{1}{u^4},\ \tau_l(u)=\frac{1}{2u^4}.$$

Let $\bar{\gamma}(u)=\gamma(u)+2\mathsf{N}(u)$ and $\boldsymbol{e}_1(u)=-\frac{1}{2}\mathsf{T}(u)+\mathsf{B}(u)$. Then we can get two ruled surfaces:

$$\boldsymbol{x}(u,v)=\gamma(u)+v\mathsf{T}(u)=\left(\frac{1}{u},\sin\frac{1}{u},-\cos\frac{1}{u}\right)-u^2v\left(1,\cos\frac{1}{u},\sin\frac{1}{u}\right)$$

and

$$\begin{aligned}\bar{\boldsymbol{x}}(u,v)=&\bar{\gamma}(u)+v\boldsymbol{e}_1(u)\\ =&\left(\frac{1}{u},-\sin\frac{1}{u},\cos\frac{1}{u}\right)-4u\left(1,\cos\frac{1}{u},\sin\frac{1}{u}\right)\\ &+v\left(\left(\frac{u^2}{2}+2+\frac{1}{2u^2}\right)\left(1,\cos\frac{1}{u},\sin\frac{1}{u}\right)+\frac{2}{u}\left(0,\sin\frac{1}{u},-\cos\frac{1}{u}\right)\right.\\ &\left.-\frac{1}{u^2}\left(0,\cos\frac{1}{u},\sin\frac{1}{u}\right)\right).\end{aligned}$$

$\bar{\boldsymbol{x}}$ is the Bertrand offset of $\boldsymbol{x}$.

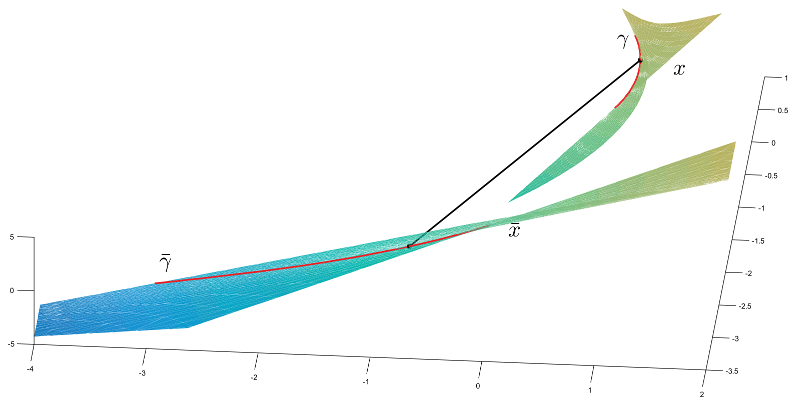

Figure 3: $\boldsymbol{x}$ and its Bertrand offset $\bar{\boldsymbol{x}}$ in Example 5.1.

Example 5.2 Let $\gamma(u)=(u,\cos u,\sin u)$, $u\in\mathbb{R}$ be a regular null helix and $\bar{\gamma}(u)=(u,-\cos u,-\sin u)$. $\boldsymbol{e}_1(u)=\mathsf{T}(u)=(1,-\sin u,\cos u)$. We have two ruled surfaces:

$$\boldsymbol{x}(u,v)=\gamma(u)+v\mathsf{T}(u)=(u,\cos u,\sin u)+v(1,-\sin u,\cos u)$$

and

$$\bar{\boldsymbol{x}}(u,v)=\bar{\gamma}(u)+v\boldsymbol{e}_1(u)=(u,-\cos u,-\sin u)+v(1,-\sin u,\cos u).$$

$\bar{\boldsymbol{x}}$ is the Bertrand offset of $\boldsymbol{x}$.

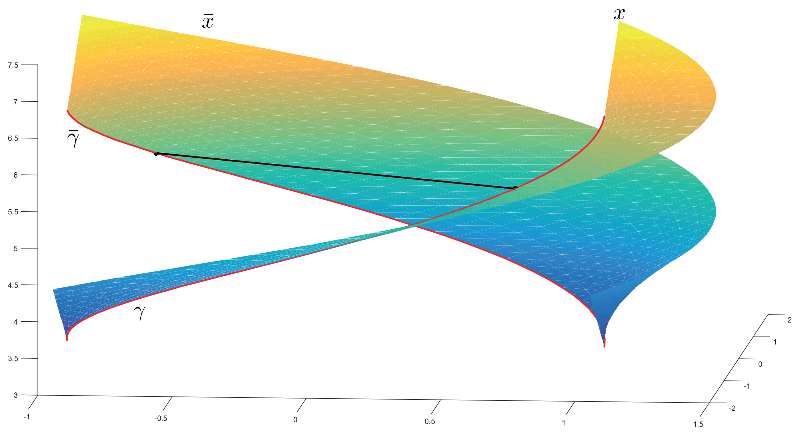

Figure 4: $\boldsymbol{x}$ and its Bertrand offset $\bar{\boldsymbol{x}}$ in Example 5.2.

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