Energy conservation in a scaling limit of the 2D filtered-Euler equations

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1 Introduction

According to the Kolmogorov theory [13], energy dissipation in inviscid flows is closely related to 3D turbulence and we expect that energy dissipating solutions of the 3D Euler equations could describe 3D turbulent flows. Onsager conjectured that weak solutions of the 3D Euler equations acquiring a Hölder continuity with the order greater than $1/3$ conserve the energy, and the energy dissipation could occur when the order is less than $1/3$ [7, 19, 20]; Onsager's conjecture has been proven mathematically [2, 3, 5]. For 2D inviscid flows, the Kraichnan-Leith-Batchelor theory [1, 14, 15] indicates that two inertial ranges corresponding to a backward energy cascade and a forward enstrophy cascade appear in 2D turbulent flows. That is to say, 2D turbulent flows are characterized by conservation of the energy and dissipation of the enstrophy. In this work, we study energy conservation in inviscid flows by introducing the filtered-Euler equations that is a regularized model of the Euler equations.

Motions of incompressible and inviscid flows are often described by the Euler equations:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{1.1}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t) = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t))$ is the fluid velocity field and $p = p(\boldsymbol{x},t)$ is the scalar pressure. A definition of a classical weak solution for (1.1) with $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x})$ is as follows, see [6].

Definition 1.1. *A velocity field $\boldsymbol{u} \in L^\infty(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^2))$ vanishing at infinity is a weak solution of (1.1) with initial data $\boldsymbol{u}_0$ provided that*

(i) *for any vector $\Psi \in C_c^\infty(\mathbb{R}^2\times(0,T))$ with $\nabla\cdot\Psi = 0$,*

$$\iint_{\mathbb{R}^2\times(0,T)} (\partial_t\Psi\cdot\boldsymbol{u} + \nabla\Psi : \boldsymbol{u}\otimes\boldsymbol{u})\, d\boldsymbol{x}dt = 0,$$

where $\boldsymbol{v}\otimes\boldsymbol{v} = (v_i v_j)$, $\nabla\Psi = (\partial_j\psi_i)$ and $A : B = \sum_{i,j} a_{ij}b_{ij}$,

(ii) *for any scalar $\psi \in C_c^\infty(\mathbb{R}^2\times(0,T))$, $\iint_{\mathbb{R}^2\times(0,T)} \nabla\psi\cdot\boldsymbol{u}d\boldsymbol{x}dt = 0$,*

(iii) *$\boldsymbol{u} \in \mathrm{Lip}([0,T]; H^{-L}_{\mathrm{loc}}(\mathbb{R}^2))$ for some $L > 0$ and $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot)$ in $H^{-L}_{\mathrm{loc}}(\mathbb{R}^2)$.*

We introduce vorticity defined by $\omega := \operatorname{curl} \boldsymbol{u} = \partial_{x_1} u_2 - \partial_{x_2} u_1$. Taking the curl of (1.1), we obtain the vorticity form of (1.1):

$$\partial_t \omega + (\boldsymbol{u} \cdot \nabla)\omega = 0 \tag{1.2}$$

with initial vorticity $\omega_0 := \operatorname{curl} \boldsymbol{u}_0$. The velocity field $\boldsymbol{u}$ is recovered from ω via the Biot-Savart law:

$$\boldsymbol{u}(\boldsymbol{x}, t) = (\boldsymbol{K} * \omega)(\boldsymbol{x}, t) := \int_{\mathbb{R}^2} \boldsymbol{K}(\boldsymbol{x} - \boldsymbol{y})\omega(\boldsymbol{y}, t) d\boldsymbol{y}, \tag{1.3}$$

where $\boldsymbol{K}$ is the singular kernel defined by

$$\boldsymbol{K}(\boldsymbol{x}) := \nabla^{\perp} G(\boldsymbol{x}) = \frac{1}{2\pi} \frac{\boldsymbol{x}^{\perp}}{|\boldsymbol{x}|^2}, \qquad G(\boldsymbol{x}) := \frac{1}{2\pi} \log |\boldsymbol{x}|.$$

Here, we have used the notations $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ and $\boldsymbol{x}^{\perp} = (-x_2, x_1)$. In this paper, we treat weak solutions of (1.2) with $\omega_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $p > 1$; the existence of a global weak solution has been established for $1 < p \leq \infty$ and the uniqueness holds only for $p = \infty$ [6, 18, 21]. As it is mentioned in [17], a weak solution for $\omega_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $p \geq 4/3$ satisfies (1.2) in the following sense.

$$\iint_{\mathbb{R}^2 \times (0,T)} \left(\partial_t \psi(\boldsymbol{x}, t) + \nabla \psi(\boldsymbol{x}, t) \cdot \boldsymbol{u}(\boldsymbol{x}, t)\right) \omega(\boldsymbol{x}, t) d\boldsymbol{x} dt = 0$$

for any $\psi \in C_c^{\infty}(\mathbb{R}^2 \times (0, T))$. For a weak solution of (1.2), we consider the kinetic energy,

$$\frac{1}{2} \int |\boldsymbol{u}(\boldsymbol{x}, t)|^2 d\boldsymbol{x}, \tag{1.4}$$

where $\boldsymbol{u}$ is given by (1.3), though (1.4) is not finite on $\mathbb{R}^2$ except for specific vorticity, see [6] for an example. Cheskidov et al. [4] have shown that a weak solution of the 2D Euler equations on the flat torus $\mathbb{T}^2$, for which (1.4) is finite, conserves the energy for $\omega_0 \in L^{3/2}(\mathbb{T}^2)$ by using a spatial mollification. They have also shown energy conservation for the weak solution obtained by an inviscid limit of the 2D Navier-Stokes equations for $\omega_0 \in L^p(\mathbb{T}^2)$, $p > 1$. In this paper, we consider another regularization of the Euler equations, which we call the *filtered-Euler equations*, and show energy conservation on $\mathbb{R}^2$ in the limit of the regularization parameter. Although the energy for the filtered inviscid model is still infinite, we formally extract a finite time-dependent term and see the convergence of its time-derivative. We also see that the weak solution of the 2D filtered-Euler equations converges weakly to a weak solution of the 2D Euler equations and the limit satisfies a local energy balance.

2 The filtered-Euler equations

The filtered-Euler equations are given by

$$\partial_t \boldsymbol{v}^{\varepsilon} + (\boldsymbol{u}^{\varepsilon} \cdot \nabla)\boldsymbol{v}^{\varepsilon} - (\nabla \boldsymbol{v}^{\varepsilon})^T \cdot \boldsymbol{u}^{\varepsilon} + \nabla p^{\varepsilon} = 0, \qquad \nabla \cdot \boldsymbol{v}^{\varepsilon} = 0, \tag{2.1}$$

where $\boldsymbol{v}^\varepsilon$ and p^ε denote the velocity field and the generalized pressure, respectively. Another field $\boldsymbol{u}^\varepsilon$ is a spatially filtered velocity of $\boldsymbol{v}^\varepsilon$, that is,

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x},t) = (h^\varepsilon * \boldsymbol{v}^\varepsilon)(\boldsymbol{x},t), \qquad h^\varepsilon(\boldsymbol{x}) := \frac{1}{\varepsilon^2} h\left(\frac{\boldsymbol{x}}{\varepsilon}\right), \qquad \varepsilon > 0, \tag{2.2}$$

in 2D case. As for the derivation of the filtered-Euler equations, we consider the Hamiltonian structure with the Hamiltonian,

$$\frac{1}{2}\int_{\mathbb{R}^2} \boldsymbol{v}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{u}^\varepsilon(\boldsymbol{x}) d\boldsymbol{x},$$

and the Lagrangian flow map $\boldsymbol{\eta}^\varepsilon$ associated with $\boldsymbol{u}^\varepsilon$:

$$\partial_t \boldsymbol{\eta}^\varepsilon(\boldsymbol{x},t) = \boldsymbol{u}^\varepsilon\left(\boldsymbol{\eta}^\varepsilon(\boldsymbol{x},t),t\right), \qquad \boldsymbol{\eta}^\varepsilon(\boldsymbol{x},0) = \boldsymbol{x}. \tag{2.3}$$

Then, the filtered Euler equations arise from an application of Hamilton's principle, see [8, 12] for the details. Here, $h \in L^1(\mathbb{R}^2)$, which we call the *filter function*, is a radial function satisfying $\int_{\mathbb{R}^2} h(\boldsymbol{x}) d\boldsymbol{x} = 1$. For simplicity, we assume $h \in C_0^1(\mathbb{R}^2 \setminus \{0\})$: a continuously differentiable function that vanishes at infinity and may have a singularity at the origin.

Note that, considering specific filter functions, we obtain two well-known regularizations: the Euler-α model and the vortex blob model, see [9, 12]. Indeed, the Euler-α equations are derived by considering the filter function,

$$h^\varepsilon(\boldsymbol{x}) = \frac{1}{2\pi\varepsilon^2} K_0\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right),$$

where K_0 denotes the modified Bessel function of the second kind. It is important to remark that K_0 has a singularity at the origin like $K_0(r) \sim -\log r$ as $r \to 0$ and decays exponentially. A remarkable property of the Euler-α model is that the filtered velocity $\boldsymbol{u}^\varepsilon$ is written by $\boldsymbol{u}^\varepsilon = (1-\varepsilon^2\Delta)^{-1}\boldsymbol{v}^\varepsilon$. Thus, the filter function h^ε is a fundamental solution for the operator $1-\varepsilon^2\Delta$. Since $\boldsymbol{v}^\varepsilon$ is explicitly expressed by $\boldsymbol{u}^\varepsilon$, we find the following equations for $\boldsymbol{u}^\varepsilon$.

$$(1-\varepsilon^2\Delta)\partial_t \boldsymbol{u}^\varepsilon + \boldsymbol{u}^\varepsilon \cdot \nabla(1-\varepsilon^2\Delta)\boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^T \cdot (1-\varepsilon^2\Delta)\boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = 0,$$

which are known as the Euler-α equations (replace ε with α). In the vortex blob model, the filter functions is given by

$$h^\varepsilon(\boldsymbol{x}) = \frac{1}{2\pi\varepsilon^2}\psi\left(\frac{|\boldsymbol{x}|}{\varepsilon}\right), \qquad \psi(r) := \frac{2}{(r^2+1)^2}.$$

Since the filter function in the vortex blob model is a bounded function and decays algebraically, it is less singular than the Euler-α model.

Similarly to the Euler equations, taking the curl of (2.1) with the incompressible condition, we obtain the equation for $q^\varepsilon := \operatorname{curl} \boldsymbol{v}^\varepsilon$,

$$\partial_t q^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla) q^\varepsilon = 0, \qquad \boldsymbol{u}^\varepsilon = \boldsymbol{K}^\varepsilon * q^\varepsilon, \qquad \boldsymbol{K}^\varepsilon := \boldsymbol{K} * h^\varepsilon. \tag{2.4}$$

Note that the Biot-Savart law for the filtered vorticity $\omega^\varepsilon := \operatorname{curl} \boldsymbol{u}^\varepsilon$ gives $\boldsymbol{u}^\varepsilon = \boldsymbol{K} * \omega^\varepsilon$ and we have $\nabla \cdot \boldsymbol{u}^\varepsilon = 0$ and $\omega^\varepsilon = h^\varepsilon * q^\varepsilon$. The preceding study [9] has shown that the 2D filtered-Euler equations have a unique global weak solution for $q_0 \in \mathcal{M}(\mathbb{R}^2)$, the space of finite Radon measures on $\mathbb{R}^2$, under certain conditions for the filter function:

Theorem 2.1. [9] *Suppose that $h \in C_0^1(\mathbb{R}^2 \setminus \{0\})$ satisfies*

$$(1 - \log|\cdot|)^{-1} h,\ |\cdot|\,\nabla h \in L^\infty(\mathbb{R}^2), \qquad |\cdot|\,h,\ \nabla h \in L^1(\mathbb{R}^2).$$

Then, for any $q_0 \in \mathcal{M}(\mathbb{R}^2)$, there exists a unique global weak solution of (2.4) *and* (2.3) *such that*

$$\boldsymbol{\eta}^\varepsilon \in C^1([0,T]; \mathscr{G}), \qquad q^\varepsilon \in C_w([0,T]; \mathcal{M}(\mathbb{R}^2)), \qquad \boldsymbol{u}^\varepsilon \in C([0,T]; C_0(\mathbb{R}^2)), \tag{2.5}$$

where $\mathscr{G}$ denotes the group of all homeomorphisms of $\mathbb{R}^2$ preserving the Lebesgue measure and C_w does the weak continuity.

The assumption in Theorem 2.1 indicates that the filter function h may have a singularity such that

$$h(\boldsymbol{x}) \sim \mathcal{O}\left(-\log|\boldsymbol{x}|\right), \qquad \nabla h(\boldsymbol{x}) \sim \mathcal{O}\left(|\boldsymbol{x}|^{-1}\right),$$

as $|\boldsymbol{x}| \to 0$. The condition in Theorem 2.1 is described in terms of the filter function, which allows us to apply the theorem to various filtered models including the Euler-α model and the vortex blob method. Note that $\boldsymbol{\eta}^\varepsilon$, q^ε and $\boldsymbol{u}^\varepsilon$ are related to each other, since we have $q^\varepsilon(\boldsymbol{x},t) = q_0\left(\boldsymbol{\eta}^\varepsilon(\boldsymbol{x},-t)\right)$ and $\boldsymbol{u}^\varepsilon = \boldsymbol{K}^\varepsilon * q^\varepsilon$. The weak solution (2.5) satisfies (2.4) in the sense that

$$\iint_{\mathbb{R}^2\times(0,T)} \left(\partial_t \psi(\boldsymbol{x},t) + \nabla\psi(\boldsymbol{x},t)\cdot \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\right) q^\varepsilon(\boldsymbol{x},t)\, d\boldsymbol{x}dt = 0$$

for any $\psi \in C_0^\infty(\mathbb{R}^2 \times (0,T))$, the space of smooth functions vanishing at infinity in $\mathbb{R}^2$ and the boundary of $(0,T)$. We mention the convergence of weak solutions of the 2D filtered-Euler equations to those of the 2D Euler equations in the $\varepsilon \to 0$ limit. For $q_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, the weak solution of (2.4) strongly converges to a unique global weak solution of (1.1) with the same initial vorticity, since the filtered flow map $\boldsymbol{\eta}^\varepsilon$ converges to a flow map induced by the 2D Euler equations:

Theorem 2.2. [9] *Let $q_0 = \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Under the assumption for h in Theorem 2.1, for any $T > 0$, there exists $C(T) > 0$ such that*

$$\sup_{t\in[0,T]} \sup_{\boldsymbol{x}\in\mathbb{R}^2} |\boldsymbol{\eta}^\varepsilon(\boldsymbol{x},t) - \boldsymbol{\eta}(\boldsymbol{x},t)| \le C(T)\varepsilon^{e^{-T}},$$

where $\boldsymbol{\eta}$ is the Lagrangian flow map induced by the weak solution of the 2D Euler equations with initial vorticity ω_0.

For $q_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $1 < p < \infty$, as we show in Theorem 3.1, the filtered weak solution converges weakly to a weak solution of (1.1), which is constructed in [6]. The convergence result has been extended to initial vorticity in $\mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$ with a distinguished sign:

Theorem 2.3. [10] *In addition to the assumption of Theorem 2.1, suppose that h is a positive function satisfying*

$$|\cdot|h \in L^1(\mathbb{R}^2), \qquad |\cdot|^3 h \in L^\infty(\mathbb{R}^2).$$

Let $(\boldsymbol{u}^\varepsilon, q^\varepsilon)$ be a solution of the 2D filtered Euler equations for $q_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$ with a distinguished sign and compact support. Then, for any $T > 0$, there exist subsequences $\{\boldsymbol{u}^{\varepsilon_j}\}$, $\{q^{\varepsilon_j}\}$ and their limits $\boldsymbol{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \times [0,T])$, $q = \operatorname{curl} \boldsymbol{u} \in L^\infty([0,T]; \mathcal{M}(\mathbb{R}^2))$ such that

$$q^{\varepsilon_j} \overset{*}{\rightharpoonup} q \quad \text{in } L^\infty([0,T]; \mathcal{M}(\mathbb{R}^2)), \qquad \boldsymbol{u}^{\varepsilon_j} \rightharpoonup \boldsymbol{u} \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^2 \times [0,T]),$$

and $(\boldsymbol{u}, q)$ are a weak solution of the 2D Euler equations with initial vorticity q_0 in the sense of Definition 1.1.

In this paper, we use the following notations. A open ball is denoted by $B_r := \{\boldsymbol{x} \in \mathbb{R}^2 \mid |\boldsymbol{x}| < r\}$. For the exponent p in the Lebesgue or Sobolev space, p' is the conjugate exponent of p, that is, $1 = 1/p + 1/p'$ for $p \in [1,\infty]$, and $p^* \in (2,\infty)$ is defined by $p^* := 2p/(2-p)$, that is, $1/p^* = 1/p - 1/2$ for $p \in (1,2)$. We omit the domain in the norm when it is the entire space $\mathbb{R}^2$.

3 Main results

We first see basic properties of a weak solution of (2.4) with $q_0 \in \mathcal{M}(\mathbb{R}^2)$. Considering the Lagrangian flow map $\boldsymbol{\eta}^\varepsilon$, we have

$$\|q^\varepsilon(\cdot,t)\|_{\mathcal{M}} = \|q_0\|_{\mathcal{M}}$$

and $\{q^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $C([0,T]; \mathcal{M}(\mathbb{R}^2))$. We also have

$$\|\omega^\varepsilon(\cdot,t)\|_{L^p} \le \|h^\varepsilon\|_{L^p}\|q_0\|_{\mathcal{M}} = \varepsilon^{-2(1-1/p)}\|h\|_{L^p}\|q_0\|_{\mathcal{M}}$$

for any $1 \le p < \infty$, which implies that $\{\omega^\varepsilon\} \subset C([0,T]; L^1(\mathbb{R}^2))$ is uniformly bounded. As for the filtered velocity $\boldsymbol{u}^\varepsilon$, it follows that

$$\|\boldsymbol{u}^\varepsilon(\cdot,t)\|_{L^\infty} \le C\varepsilon^{-2(1-1/r)}\|q_0\|_{\mathcal{M}} \tag{3.1}$$

for any $2 < r \le \infty$. Next, we consider the energy for a filtered solution by replacing $\boldsymbol{u}$ with $\boldsymbol{u}^\varepsilon$ in (1.4). Note that the energy for $\boldsymbol{u}^\varepsilon$ is not finite in general. Indeed, $\boldsymbol{u}^\varepsilon = \boldsymbol{K}^\varepsilon * q^\varepsilon$ implies $\boldsymbol{u}^\varepsilon(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-1}$ as $|\boldsymbol{x}| \to \infty$ and thus $|\boldsymbol{u}^\varepsilon|^2$ is not integrable on $\mathbb{R}^2$. We now formally derive the energy dissipation rate by dividing the energy into two parts: a time-invariant term and a time-dependent term. In particular, we focus on the time-dependent term, which is well-defined for weak solutions of (2.4) with $q_0 \in \mathcal{M}(\mathbb{R}^2)$, and differentiate it with respect to t. We start by substituting $\boldsymbol{u}^\varepsilon = \boldsymbol{K}^\varepsilon * q^\varepsilon$ into the energy:

$$\frac{1}{2}\int_{\mathbb{R}^2} |\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)|^2\, d\boldsymbol{x} = \frac{1}{2}\iiint \boldsymbol{K}^\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{K}^\varepsilon(\boldsymbol{x}-\boldsymbol{z}) q^\varepsilon(\boldsymbol{y},t) q^\varepsilon(\boldsymbol{z},t) d\boldsymbol{y} d\boldsymbol{z} d\boldsymbol{x}.$$

Note that $\boldsymbol{K}^\varepsilon = \nabla^\perp G^\varepsilon$ and $\Delta G^\varepsilon = h^\varepsilon$, Then, integration by parts formally yields

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^2} |\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)|^2\, d\boldsymbol{x} &= -\frac{1}{2}\iiint h^\varepsilon(\boldsymbol{x}-\boldsymbol{y})\cdot G^\varepsilon(\boldsymbol{x}-\boldsymbol{z}) q^\varepsilon(\boldsymbol{y},t) q^\varepsilon(\boldsymbol{z},t) d\boldsymbol{x} d\boldsymbol{y} d\boldsymbol{z} \\ &= -\frac{1}{2}\iint \left(h^\varepsilon * G^\varepsilon\right)(\boldsymbol{y}-\boldsymbol{z}) q^\varepsilon(\boldsymbol{y},t) q^\varepsilon(\boldsymbol{z},t) d\boldsymbol{y} d\boldsymbol{z}.\end{aligned}$$

Here, we introduce the *pseudo-energy* :

$$\mathscr{H}^\varepsilon := -\frac{1}{2}\iint G^\varepsilon(\boldsymbol{x}-\boldsymbol{y}) q^\varepsilon(\boldsymbol{x},t) q^\varepsilon(\boldsymbol{y},t) d\boldsymbol{x} d\boldsymbol{y}.$$

Although $\mathscr{H}^\varepsilon$ is not finite in general, we find that $\mathscr{H}^\varepsilon$ is a invariant quantity. Indeed, for the point-vortex initial vorticity, $\mathscr{H}^\varepsilon$ gives the Hamiltonian of the *filtered point-vortex system*, see [11]. On the basis of the above calculation, we divide the energy into two parts as follows.

$$\frac{1}{2}\int_{\mathbb{R}^2} |\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)|^2\, d\boldsymbol{x} = \mathscr{H}^\varepsilon + \mathscr{E}(t),$$

and

$$\mathscr{E}(t) := -\frac{1}{2}\iint H_G^\varepsilon(\boldsymbol{x}-\boldsymbol{y}) q^\varepsilon(\boldsymbol{x},t) q^\varepsilon(\boldsymbol{y},t) d\boldsymbol{x} d\boldsymbol{y},$$

where

$$H_G^\varepsilon(\boldsymbol{x}) := \left(h^\varepsilon * G^\varepsilon\right)(\boldsymbol{x}) - G^\varepsilon(\boldsymbol{x}) = \left(h^\varepsilon * (G^\varepsilon - G)\right)(\boldsymbol{x}). \tag{3.2}$$

Since H_G^ε belongs to $C_0(\mathbb{R}^2)$ for any fixed ε, we find

$$|\mathscr{E}(t)| \le \|H_G^\varepsilon\|_{L^\infty} \|q^\varepsilon(\cdot,t)\|_{\mathcal{M}}^2 = C_\varepsilon \|q_0\|_{\mathcal{M}}^2,$$

where C_ε is the constant depending on ε. Thus, the time-dependent term $\mathscr{E}(t)$ is finite for any $q_0 \in \mathcal{M}(\mathbb{R}^2)$. Note that

$$\iint H_G^\varepsilon(\boldsymbol{x}-\boldsymbol{y}) q^\varepsilon(\boldsymbol{x},t) q^\varepsilon(\boldsymbol{y},t) d\boldsymbol{y} d\boldsymbol{x} = \iint H_G^\varepsilon(\boldsymbol{\eta}^\varepsilon(\boldsymbol{x},t) - \boldsymbol{\eta}^\varepsilon(\boldsymbol{y},t)) q_0(\boldsymbol{x}) q_0(\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y}.$$

Then, the time-derivative of $\mathscr{E}(t)$ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t) &= -\frac{1}{2}\iint (\nabla H_G^\varepsilon)(\boldsymbol{\eta}^\varepsilon(\boldsymbol{x},t) - \boldsymbol{\eta}^\varepsilon(\boldsymbol{y},t)) \\ &\qquad \cdot \left(\boldsymbol{u}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\boldsymbol{x},t),t) - \boldsymbol{u}^\varepsilon(\boldsymbol{\eta}^\varepsilon(\boldsymbol{y},t),t)\right) q_0(\boldsymbol{x}) q_0(\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} \\ &= -\frac{1}{2}\iint (\nabla H_G^\varepsilon)(\boldsymbol{x}-\boldsymbol{y}) \cdot \left(\boldsymbol{u}^\varepsilon(\boldsymbol{x},t) - \boldsymbol{u}^\varepsilon(\boldsymbol{y},t)\right) q^\varepsilon(\boldsymbol{x},t) q^\varepsilon(\boldsymbol{y},t) d\boldsymbol{x} d\boldsymbol{y}.\end{aligned}$$

Hence, we define the energy dissipation rate by

$$\mathscr{D}_E^\varepsilon(t) := -\frac{1}{2}\iint (\nabla H_G^\varepsilon)(\boldsymbol{x}-\boldsymbol{y}) \cdot \left(\boldsymbol{u}^\varepsilon(\boldsymbol{x},t) - \boldsymbol{u}^\varepsilon(\boldsymbol{y},t)\right) q^\varepsilon(\boldsymbol{x},t) q^\varepsilon(\boldsymbol{y},t) d\boldsymbol{x} d\boldsymbol{y}.$$

It follows from $\nabla H_G^\varepsilon \in C_0(\mathbb{R}^2)$ and (3.1) that

$$|\mathscr{D}_E^\varepsilon(t)| \le \|\nabla H_G^\varepsilon\|_{L^\infty} \|\boldsymbol{u}^\varepsilon(\cdot,t)\|_{L^\infty} \|q^\varepsilon(\cdot,t)\|_{\mathcal{M}}^2 \le C_\varepsilon \|q_0\|_{\mathcal{M}}^3.$$

Thus, $\mathscr{D}_E^\varepsilon$ is well-defined for weak solutions of (2.4) with $q_0 \in \mathcal{M}(\mathbb{R}^2)$. However, the boundedness of $\mathscr{D}_E^\varepsilon$ depends on the filter parameter ε and, in the $\varepsilon \to 0$ limit, $\mathscr{D}_E^\varepsilon$ is not finite in general. Our concern is the set of initial vorticity that provides the uniform boundedness of $\{\mathscr{D}_E^\varepsilon\}_{\varepsilon>0}$. In this study, we consider weak solutions of (2.4) with $q_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, and give a sufficient condition for p that yields energy conservation. In the following theorems, we assume the same condition as Theorem 2.1 for the filter function h.

Theorem 3.1. *Suppose that $h \in C_0^1(\mathbb{R}^2 \setminus \{0\})$ is a radial function satisfying*

$$|\cdot|h,\ \nabla h \in L^1(\mathbb{R}^2), \qquad |\cdot|^\alpha h,\ |\cdot|^3 h,\ |\cdot|\nabla h \in L^\infty(\mathbb{R}^2)$$

for some $\alpha \in [0,1)$. Let $(\boldsymbol{u}^\varepsilon, q^\varepsilon)$ be a weak solution of the 2D filtered-Euler equations with $q_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $3/2 < p \leq \infty$. Then, we have

$$\lim_{\varepsilon \to 0} \|\mathscr{D}_E^\varepsilon\|_{L^\infty(0,T)} = 0.$$

Moreover, there exists a weak solution of the 2D Euler equations,

$$\boldsymbol{u} \in L^\infty(0,T; L^{p^*}(\mathbb{R}^2) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)), \quad \omega = \operatorname{curl} \boldsymbol{u} \in L^\infty(0,T; L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)),$$

such that, taking subsequences as needed, we have

$$q^\varepsilon \rightharpoonup \omega \quad \text{in } L^p(\mathbb{R}^2 \times (0,T)), \qquad \boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{in } C([0,T]; L^r_{\mathrm{loc}}(\mathbb{R}^2))$$

for any $r \in [1, p^*)$ in the $\varepsilon \to 0$ limit, and there exists $P \in L^\infty(0,T; L^{p^*/2}(\mathbb{R}^2))$ such that the following local energy balance holds in the sense of distributions.

$$\partial_t \left(\frac{|\boldsymbol{u}|^2}{2} \right) + \nabla \cdot \left(\boldsymbol{u} \left(\frac{|\boldsymbol{u}|^2}{2} + P \right) \right) = 0.$$

The conditions for h in Theorem 3.1 imply $h \in L^p(\mathbb{R}^2)$ and $\nabla h \in L^q(\mathbb{R}^2)$ for any $p \in [1,\infty)$ and $q \in [1,2)$. The filter functions for the Euler-α model and the vortex blob model satisfy these conditions. The convergence to the Euler equations for $q_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ is proven in the same way as [10]. For the case $p = 3/2$, the same result as Theorem 3.1 holds under an additional condition for the regularity of $\boldsymbol{u}^\varepsilon$:

Theorem 3.2. *Let $(\boldsymbol{u}^\varepsilon, q^\varepsilon)$ be a weak solution of the 2D filtered-Euler equations with $q_0 \in L^1(\mathbb{R}^2) \cap L^{3/2}(\mathbb{R}^2)$ and $\boldsymbol{u}^\varepsilon$ satisfy*

$$\|\boldsymbol{u}^\varepsilon(\cdot - \boldsymbol{y}, t) - \boldsymbol{u}^\varepsilon(\cdot, t)\|_{L^3} \leq C(T)|\boldsymbol{y}|^\alpha, \quad (\boldsymbol{y}, t) \in \mathbb{R}^2 \times (0,T), \tag{3.3}$$

for some $\alpha \in (1/3, 1]$, where $C(T)$ is independent of ε. Then, we have the same result as Theorem 3.1 with $p = 3/2$.

We remark that (3.3) is related to Onsager's critical condition. Although Onsager conjectured for the 3D Euler equations, the energy conservation holds for the weak solution of the Euler equations satisfying (3.3) regardless of the dimension [3]. As it is mentioned in [4], weak solutions of the 2D Euler equations with $\omega_0 \in L^{3/2}$ satisfy (3.3). Our main theorems are consistent with these preceding results, though we require the family $\{\boldsymbol{u}^\varepsilon\}$ to satisfy (3.3) uniformly.

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