The Hasimoto Transformation for a Finite Length Vortex Filament and its Application

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1 Introduction

A vortex filament is a space curve on which the vorticity of the fluid is concentrated. Vortex filaments are used to model very thin vortex structures such as vortices that trail off airplane wings or propellers. The model equation we consider in this paper is the Localized Induction Equation (LIE) given by

$$\boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss}$$

where $\boldsymbol{x}(s,t) = {}^t(x_1(s,t), x_2(s,t), x_3(s,t))$ is the position vector of the vortex filament parametrized by its arc length s at time t, $\times$ is the exterior product in the three-dimensional Euclidean space, and subscripts s and t are differentiations with the respective variables.

The LIE, which is derived by applying the localized induction approximation to the Biot–Savart integral, was first derived by Da Rios [13] in 1906 and was re-derived twice independently by Murakami et al. [30] in 1937 and by Arms and Hama [4] in 1965. Since then, many researches have been done on the LIE and many results have been obtained. Nishiyama and Tani [32, 33] proved the unique solvability of initial and initial-boundary value problems in Sobolev spaces. The author [1] and the author and Iguchi [3] proved the unique solvability of initial-boundary value problems in Sobolev spaces with different boundary conditions. In the above papers, the equation for the tangent vector, $\boldsymbol{v} := \boldsymbol{x}_s$, given by

$$\boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} \tag{1.1}$$

is introduced in the analysis and plays an important role. Equation (1.1) is sometimes called the Vortex Filament Equation (VFE).

Koiso [29] considered a geometrically generalized setting in which he rigorously proved the equivalence of the solvability of the initial value problem for the VFE and the cubic nonlinear Schrödinger equation. This equivalence was first shown by Hasimoto [23] in which he studied the formation of solitons on a vortex filament. He defined a transformation of variable known as the Hasimoto transformation to transform the VFE into a nonlinear Schrödinger equation. The Hasimoto transformation is a change of variable given by

$$q(s,t) = \kappa(s,t) \exp\left(\mathrm{i} \int_0^s \tau(r,t)\,\mathrm{d}r \right),$$

where i is the imaginary unit, κ is the curvature, and τ is the torsion of the filament. Defined as such, it is well known that q satisfies the nonlinear Schrödinger equation given by

$$\mathrm{i}q_t = q_{ss} + \frac{1}{2}|q|^2 q. \tag{1.2}$$

The original transformation proposed by Hasimoto uses the torsion of the filament in its definition, which means that the transformation is undefined at points where the curvature of the filament is zero. Koiso [29] constructed a transformation, sometimes referred to as the generalized Hasimoto transformation, and gave a mathematically rigorous proof of the equivalence of the VFE and (1.2). More precisely, Koiso proved that the solvability of initial value problems for the VFE and (1.2) are equivalent. More recently, Banica and Vega [5, 6, 7] and Gutiérrez, Rivas, and Vega [22] utilized the generalized Hasimoto transformation to construct and analyze a family of self-similar solutions of the LIE which forms a corner in finite time. In Chang, Shatah, and Uhlenbeck [12] and Nahmod, Shatah, Vega, and Zeng [28], they considered the Schrödinger maps and employed a Hasimoto type transformation to prove the correspondence between the solution of the Schrödinger maps and the solution of a nonlinear Schrödinger type equation. The Schrödinger maps are a generalization of the Heisenberg model for a ferromagnetic spin system given by

$$\boldsymbol{m}_t = \boldsymbol{m} \times \Delta \boldsymbol{m},$$

where Δ is the Laplacian in $\mathbf{R}^n$ and the unknown variable $\boldsymbol{m}$ takes values in $\mathbf{S}^2$. The Heisenberg model can be seen as a multi-dimensional version of the VFE.

All of the results mentioned above which utilizes the Hasimoto transformation consider either an infinitely long filament defined on the whole line or a closed filament defined on the torus. As far as the author knows, the rigorous justification and application of the Hasimoto transformation for problems describing the motion of filaments with end-points have not been done. In this paper, we justify the Hasimoto transformation and prove the equivalence of the solvability of the following initial-boundary value problems.

$$(1.3)\qquad \begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss}, & s \in I_L,\ t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0, & s \in I_L,\ t > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_1,\ \boldsymbol{v}(L,t) = \boldsymbol{b}, & t > 0, \end{cases}$$

$$(1.4)\qquad \begin{cases} \mathrm{i}q_t = q_{ss} + \dfrac{1}{2}|q|^2 q, & s \in I_L, t > 0, \\ q(s,0) = q_0(s), & s \in I_L, \\ q_s(0,t) = q_s(L,t) = 0, & t > 0. \end{cases}$$

Problem (1.3) is an initial-boundary value problem for the VFE which describes the motion of a vortex filament on a slanted plane, considered in a previous paper by the author [1]. Here, $L > 0$ is the length of the initial filament, $I_L \subset \mathbf{R}$ is the interval $(0, L)$, $\boldsymbol{e}_1 = {}^t(1, 0, 0)$, and $\boldsymbol{b} \in \mathbf{R}^3$ is an arbitrary constant vector with unit length. The boundary datum at $s = 0$ was chosen as $\boldsymbol{e}_1$ without loss of generality, because the VFE is invariant under rotation. Problem (1.4) is an initial-boundary value problem for the focusing cubic nonlinear Schrödinger equation, where $q = q(s,t)$ is a complex-valued function.

As an application of this equivalence, we will prove the orbital stability of the plane wave solution q_R of (1.4) given in the form

$$(1.5)\qquad q_R(t) = -\frac{1}{R}\exp\left\{-\frac{\mathrm{i}t}{2R^2}\right\}$$

for $R > L/\pi$ in the Sobolev space $H^2(I_L)$. This will be done by considering problem (1.3) with appropriate initial and boundary data, utilizing energy estimates for the solution $\boldsymbol{v}$ of (1.3) obtained by the author in [1, 2], and transferring the estimates for $\boldsymbol{v}$ into estimates for solutions of (1.4).

As far as the author knows, the results of this paper is the first time the generalized Hasimoto transformation is utilized to give new insight on the nonlinear Schrödinger equation from known facts about the VFE. All of the preceding works utilizing the generalized Hasimoto transformation did so to analyze the solution of the VFE utilizing known facts about the nonlinear Schrödinger equation.

The initial value problem for equation (1.2) on the torus, explicitly given by

$$(1.6)\qquad \begin{cases} \mathrm{i}u_t = u_{ss} + \dfrac{1}{2}|u|^2 u, & s \in \mathbf{T}, t > 0, \\ u(s,0) = u_0(s), & s \in \mathbf{T}, \end{cases}$$

where $\mathbf{T} = \mathbf{R}/[-L, L]$, is closely related to problem (1.4) since the solvability of problem (1.4) can be reduced to the solvability of problem (1.6) by reflection and periodic extension.

It is known by Zakharov and Shabat [38] that equation (1.2) is completely integrable, and the solution to problem (1.6) possess infinitely many conserved quantities. This in turn implies that solutions in the Sobolev space H^m for $m \in \mathbf{N}$ is bounded in H^m for all time. The solvability of problem (1.6) in Lebesgue or Sobolev spaces are known, for example, by Bourgain [9]. Hence, it is natural to ask if particular types of solutions are stable in Sobolev spaces. Namely, the stability of plane wave solutions of equation (1.2) has been studied by many researchers.

Zhidkov [39] gives a detailed analysis of the plane wave solutions for the initial value problem on the whole space $\mathbf{R}$. The stability of plane wave solutions and periodic wave solutions for problem (1.6) was investigated by Rowlands [35], Gallay and Hărăguş [18, 19], Faou, Gauckler, and Lubich [14], and Wilson [37].

In particular, Gallay and Hărăguş [18, 19] considered problem (1.6) and proved the orbital stability of periodic wave solutions in H^1_{per}, i.e., the orbital stability in H^1 with the perturbation restricted to periodic perturbations with the same period as the periodic wave solution. By definition, periodic wave solutions include plane wave solutions and hence, the stability results in [18, 19] are valid for plane wave solutions as well.

Faou, Gauckler, and Lubich [14] considered the initial value problem on the torus with general dimension $d \geq 1$, and proved the long-time orbital stability of plane wave solutions in H^r. By long-time they mean stability up to time of order $O(\frac{1}{\varepsilon^N})$ where $\varepsilon > 0$ is the size of the perturbation in H^r and $N \in \mathbf{N}$. The index $r > 0$ must be chosen sufficiently large, depending on N and the L^2 norm of the perturbation.

Wilson [37] considered the stability problem of plane wave solutions for the nonlinear Schrödinger equation with general power nonlinearities, and obtained similar results as [14].

Other results on the initial–boundary value problems for the nonlinear Schrödinger equation have been obtained by Holmer [24], Fokas and Its [15], Fokas, Its, and Sung [17], Lenells and Fokas [25, 26], Bona, Sun, and Zhang [8], and Fokas, Himonas, and Mantzavinos [16]. These results prove the well-posedness of initial-boundary value problems under various boundary conditions as well as obtain representation formulas for boundary values which represent unknown boundary values of a solution by known boundary values, but do not address the problem of stability of specific solutions.

In summary, we see that up until this paper, the stability of plane wave solutions in higher order Sobolev spaces is only partially known. Specifically, the regularity for which the stability is proved in [14, 37] is not given explicitly, and the time-span for which the stability holds is not global. On the other hand, the results in this paper give the time-global orbital stability of plane wave solutions in Sobolev space H^2. This is possible greatly due to the fact that the method of the proof given in this paper is vastly different from the preceding works.

The methods utilized to prove the stability of solutions in [39, 35, 18, 19, 14, 37] can be broadly categorized into two types. Direct variational methods and methods utilizing the Hamiltonian structure of the equation. The variational approach was utilized in Cazenave and Lions [11] to prove orbital stability of standing waves for the nonlinear Schrödinger equation. They formulate the orbital stability of standing waves as a minimizing problem for an energy functional associated to the nonlinear Schrödinger equation.

The approach utilizing the Hamiltonian structure of the equation was introduced by Grillakis, Shatah, and Strauss [20, 21] for a broad range of equations having a Hamiltonian structure. In this case, the orbital stability of particular solutions is often times proved utilizing the Birkhoff normal forms method to obtain estimates for the perturbation, which is the case for [14, 37].

These methods are widely adopted to approach stability problems for a wide variety of dispersive equations. In contrast, the method employed in this paper is tailor-made specifically for problems (1.3) and (1.4). The rough outline of our method is as follows. We utilize the generalized Hasimoto transformation to show that the plane wave solution (1.5) corresponds to a particular solution of the VFE, which we call an arc-shaped solution, and prove Lyapunov type stability estimates for the arc-shaped solution. The stability estimates are obtained through a standard energy method, along with a conserved quantity, for the perturbation. Then, the stability estimates for the arc-shaped solution is transferred to the nonlinear Schrödinger equation to obtain orbital stability estimates for the plane wave solution.

Comparing our method and traditional methods, one can say that our method is a lot more direct and simple, but is not as widely applicable to other problems.

2 Function Spaces, Notations, and Main Theorem

We introduce some function spaces that will be used throughout this paper, and notations associated with the spaces. For a non-negative integer m and $1 \leq p \leq \infty$, $W^{m,p}(I_L)$ is the Sobolev space containing all real-valued functions that have derivatives in the sense of distribution up to order m belonging to $L^p(I_L)$. We set $H^m(I_L) := W^{m,2}(I_L)$ as the Sobolev space equipped with the usual inner product, and set $H_0^1(I_L)$ as the closure, with respect to the H^1-norm, of the set of smooth functions with compact support. The norm in $H^m(I_L)$ is denoted by $\|\cdot\|_m$ and we simply write $\|\cdot\|$ for $\|\cdot\|_0$. Otherwise, for a Banach space X, the norm in X is written as $\|\cdot\|_X$. The inner product in $L^2(I_L)$ is denoted by

$(\cdot,\cdot)$.

For $0 < T \leq \infty$ and a Banach space X, $C^m([0,T];X)$ ($C^m\big([0,\infty);X\big)$ when $T=\infty$), denotes the space of functions that are m times continuously differentiable in t with respect to the norm of X. The space $L^\infty\big(0,\infty;X\big)$ denotes the space of functions that are essentially bounded in t with respect to the norm of X

For any function space described above, we say that a vector valued function belongs to the function space if each of its components does, and the same for complex-valued functions if both the real and imaginary parts do.

Finally, vectors $\boldsymbol{e}_j \in \mathbf{R}^3$ for $j=1,2,3$ denote the standard basis of $\mathbf{R}^3$. In other words, $\boldsymbol{e}_1 = {}^t(1,0,0)$, $\boldsymbol{e}_2 = {}^t(0,1,0)$, and $\boldsymbol{e}_3 = {}^t(0,0,1)$. Additionally, we denote the unit sphere in $\mathbf{R}^3$ by $\mathbf{S}^2$.

We next introduce some definitions in order to state the main theorems of this paper. First we define the compatibility conditions for both (1.3) and (1.4).

2.1 Compatibility Conditions for (1.3) and (1.4)

First we define the compatibility conditions needed in this paper for problem (1.3).

Definition 2.1. *For $m=0$ or 1, we say that $\boldsymbol{v}_0 \in H^{2m+1}(I_L)$ and $\boldsymbol{b} \in \mathbf{S}^2$ satisfy the m-th order compatibility condition for* (1.3) *if*

$$\boldsymbol{v}_0(0) = \boldsymbol{e}_1, \quad \boldsymbol{v}_0(L) = \boldsymbol{b},$$

when $m=0$, and

$$\boldsymbol{v}_0(0) \times \boldsymbol{v}_{0ss}(0) = \boldsymbol{v}_0(L) \times \boldsymbol{v}_{0ss}(L) = \mathbf{0}$$

when $m=1$. We also say that $\boldsymbol{v}_0$ and $\boldsymbol{b}$ satisfy the compatibility conditions for (1.3) *up to order* 1 *if $\boldsymbol{v}_0$ and $\boldsymbol{b}$ satisfy both the* 0*-th order and the* 1*-st order compatibility condition for* (1.3).

Next we define the corresponding compatibility condition for (1.4).

Definition 2.2. *For $q_0 \in H^2(I_L)$, we say that q_0 satisfies the* 0*-th order compatibility condition for* (1.4) *if*

$$q_{0s}(0) = q_{0s}(L) = 0$$

are satisfied.

2.2 The Hasimoto Transformation and the Main Theorems

To state our main theorem, we give a brief explanation of the Hasimoto transformation. The Hasimoto transformation is a map that relates the solution of equation (1.1) to the solution of equation (1.2) proposed by Hasimoto [23]. The original transformation proposed by Hasimoto isn't always well-defined because the transformation is defined using the torsion of the filament, which is not defined at points where the curvature of the filament is zero. Later, Koiso [29] proved that the solvability of the initial value problem on the torus for (1.1) and (1.2) is equivalent. Koiso did so by constructing a modified transformation, to which we refer to as the generalized Hasimoto transformation, which maps solutions of equation (1.1) to solutions of equation (1.2). This transformation is invertible, and hence, the solvability is equivalent.

One of the aims of this paper is to prove that the generalized Hasimoto transformation given by Koiso [29] can be further modified to prove the equivalence of the solvability of problem (1.3) and problem (1.4). More precisely, we prove the following.

Theorem 2.3. *For* $q_0 \in H^2(I_L)$ *satisfying the* 0*-th order compatibility condition for* (1.4)*, there exists* $\boldsymbol{v}_0 \in H^3(I_L)$ *and* $\boldsymbol{b} \in \mathbf{S}^2$ *satisfying* $|\boldsymbol{v}_0| \equiv 1$ *and the compatibility conditions for* (1.3) *up to order* 1 *such that the following holds. The solution* $\boldsymbol{v} \in C\big([0,\infty); H^3(I_L)\big) \cap C^1\big([0,\infty); H^1(I_L)\big)$ *of problem* (1.3) *with initial datum* $\boldsymbol{v}_0$ *and boundary datum* $\boldsymbol{b}$ *corresponds to the solution* $q \in C\big([0,\infty); H^2(I_L)\big) \cap C^1\big([0,\infty); L^2(I_L)\big)$ *of problem* (1.4) *with initial datum* q_0 *through the generalized Hasimoto transformation.*

Theorem 2.4. *For* $\boldsymbol{v}_0 \in H^3(I_L)$ *and* $\boldsymbol{b} \in \mathbf{S}^2$ *satisfying* $|\boldsymbol{v}_0| \equiv 1$ *and the compatibility conditions for* (1.3) *up to order* 1*, there exists* $q_0 \in H^2(I_L)$ *satisfying the* 0*-th order compatibility condition for* (1.4) *such that the following holds. The solution* $q \in C\big([0,\infty); H^2(I_L)\big) \cap C^1\big([0,\infty); L^2(I_L)\big)$ *of problem* (1.4) *with initial datum* q_0 *corresponds to the solution* $\boldsymbol{v} \in C\big([0,\infty); H^3(I_L)\big) \cap C^1\big([0,\infty); H^1(I_L)\big)$ *of problem* (1.3) *with initial datum* $\boldsymbol{v}_0$ *and boundary datum* $\boldsymbol{b}$ *through the inverse generalized Hasimoto transformation.*

Theorem 2.3 and 2.4 together states that the solvability of problem (1.3) and problem (1.4) are equivalent in suitable Sobolev spaces. The two theorems also imply that the compatibility conditions for (1.3) and (1.4) correspond to each other through the generalized Hasimoto transformation. Recall from the introduction that the solvability of problem (1.3) and problem (1.4) are already known. Hence, the solvability of either problem in itself is not new, but the fact that the solvability of the two problems are equivalent is new.

As an application of the above two theorems, we prove the following theorem.

Theorem 2.5. *The plane wave solution q_R of problem* (1.4) *given by*

$$q_R(t) = -\frac{1}{R}\exp\left\{-\frac{\mathrm{i}t}{2R^2}\right\}$$

with $R > \dfrac{L}{\pi}$ is orbitally stable in $H^2(I_L)$. More specifically, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $\phi_0 \in H^2(I_L)$ satisfying the 0-th order compatibility condition for (1.4) *and $\|\phi_0\|_2 \leq \delta$, the solution $q \in C\big([0,\infty); H^2(I_L)\big) \cap C^1\big([0,\infty); L^2(I_L)\big)$ of problem* (1.4) *with initial datum $q_0 = q_R(0) + \phi_0$ satisfies*

$$\sup_{t \geq 0} \inf_{\theta \in \mathbf{R}} \|\exp(\mathrm{i}\theta)q(t) - q_R(t)\|_2 < \varepsilon.$$

From here on, we will refer to the maps $q_0 \mapsto \boldsymbol{v}_0$ and $\boldsymbol{v} \mapsto q$ implied in Theorem 2.3 as the generalized Hasimoto transformation, and the maps $\boldsymbol{v}_0 \mapsto q_0$ and $q \mapsto \boldsymbol{v}$ implied in Theorem 2.4 as the inverse generalized Hasimoto transformation.

3 Outline of the Proof

In this section, we give the outline of the proof of the main theorems.

3.1 Proof of Theorem 2.3

Let $q_0 \in H^2(I_L)$ satisfy the 0-th order compatibility condition for problem (1.4) and define q_1^0 and q_2^0 by

$$q_0(s) = q_1^0(s) + \mathrm{i}q_2^0(s).$$

Furthermore, we define $\boldsymbol{v}_0, \boldsymbol{e}^0$, and $\boldsymbol{w}^0$ as the solution of the following system of ordinary differential equations.

$$\begin{cases} \boldsymbol{v}_{0s} = q_1^0\boldsymbol{e}^0 + q_2^0\boldsymbol{w}^0, & s \in I_L, \\ \boldsymbol{e}_s^0 = -q_1^0\boldsymbol{v}_0, & s \in I_L, \\ \boldsymbol{w}_s^0 = -q_0^2\boldsymbol{v}_0, & s \in I_L, \\ (\boldsymbol{v}_0, \boldsymbol{e}^0, \boldsymbol{w}^0)(0) = (\boldsymbol{e}_1, -\boldsymbol{e}_2, \boldsymbol{e}_3). \end{cases} \tag{3.1}$$

By direct calculations, we see that the triplet $\{\boldsymbol{v}_0, \boldsymbol{e}^0, \boldsymbol{w}^0\}$ is an orthonormal basis of $\mathbf{R}^3$ for all $s \in I_L$. Set $\boldsymbol{b} := \boldsymbol{v}_0(L)$, and we see that by definition, $\boldsymbol{v}_0$ and $\boldsymbol{b}$ satisfy the 0-th order compatibility condition of problem (1.3). We further calculate and see that

$$\boldsymbol{v}_{0ss} = (-(q_1^0)^2 - (q_2^0)^2)\boldsymbol{v}_0 + q_{1s}^0\boldsymbol{e}^0 + q_{2s}^0\boldsymbol{w}^0,$$

and hence,

$$\boldsymbol{v}_0 \times \boldsymbol{v}_{0ss} = q_{1s}^0 \boldsymbol{v}_0 \times \boldsymbol{e}^0 + q_{2s}^0 \boldsymbol{v}_0 \times \boldsymbol{w}^0$$

$$= -q_{1s}^0 \boldsymbol{w}^0 + q_{2s}^0 \boldsymbol{e}^0.$$

Here, we have used the fact that $\{\boldsymbol{v}_0, \boldsymbol{e}^0, \boldsymbol{w}^0\}$ is an orthonormal basis of $\mathbf{R}^3$ and the orientation given at $s = 0$ yields $\boldsymbol{v}_0 \times \boldsymbol{e}^0 = -\boldsymbol{w}^0$ and $\boldsymbol{v}_0 \times \boldsymbol{w}^0 = \boldsymbol{e}^0$. Since q_0 satisfies the 0-th order compatibility condition for (1.4), we see that

$$\boldsymbol{v}_0(0) \times \boldsymbol{v}_{0ss}(0) = \boldsymbol{v}_0(L) \times \boldsymbol{v}_{0ss}(L) = \mathbf{0}$$

and $\boldsymbol{v}_0$ and $\boldsymbol{b}$ satisfies the 1-st order compatibility condition for (1.3).

Let, $\boldsymbol{v} \in C\big([0,\infty); H^3(I_L)\big) \cap C^1\big([0,\infty); H^1(I_L)\big)$ be the solution of problem (1.3) with initial datum $\boldsymbol{v}_0$ and boundary datum $\boldsymbol{b}$ just obtained. From [1], we know that $|\boldsymbol{v}(s,t)| = 1$ for all $s \in I_L$ and $t > 0$. We now define $\boldsymbol{e}$ as the solution of

$$\begin{cases} \boldsymbol{e}_s = -(\boldsymbol{v}_s \cdot \boldsymbol{e})\boldsymbol{v}, & s \in I_L, \\ \boldsymbol{e}(0) = \boldsymbol{e}_2, \end{cases}$$

and set $\boldsymbol{w} = \boldsymbol{v} \times \boldsymbol{e}$. Here, $\cdot$ is the standard inner product of $\mathbf{R}^3$. Note that $\boldsymbol{e}$ and $\boldsymbol{w}$ depend on $t > 0$ as a parameter through $\boldsymbol{v}$. Again, we see that $\{\boldsymbol{v}, \boldsymbol{e}, \boldsymbol{w}\}$ is an orthonormal basis of $\mathbf{R}^3$ for all $s \in I_L$ and $t > 0$. Since $\boldsymbol{v} \cdot \boldsymbol{v}_s \equiv 0$, we have the decomposition

$$\boldsymbol{v}_s = \psi_1 \boldsymbol{e} + \psi_2 \boldsymbol{w} \tag{3.2}$$

for some $\psi_1(s,t)$ and $\psi_2(s,t)$. Then we also see that

$$\boldsymbol{e}_s = -\psi_1 \boldsymbol{v}, \tag{3.3}$$

$$\boldsymbol{w}_s = -\psi_2 \boldsymbol{v}. \tag{3.4}$$

Taking the inner product of $\boldsymbol{e}$ and $\boldsymbol{w}$ with equation (3.2) we have

$$\psi_1 = \boldsymbol{e} \cdot \boldsymbol{v}_s,$$

$$\psi_2 = \boldsymbol{w} \cdot \boldsymbol{v}_s,$$

from which we deduce that $\psi_1, \psi_2 \in C\big([0,\infty); H^2(I_L)\big)$ since $\boldsymbol{v} \in C\big([0,\infty); H^3(I_L)\big)$. From equation (3.2), we see that

$$|\psi_1|^2 + |\psi_2|^2 = |\boldsymbol{v}_s|^2,$$

which implies

$$\|\psi_1(t)\|^2 + \|\psi_2(t)\|^2 = \|\boldsymbol{v}_s(t)\|^2.$$

Furthermore, we have

$$\psi_{1s}\boldsymbol{e} + \psi_{2s}\boldsymbol{w} = \boldsymbol{v}_{ss} + (\psi_1)^2\boldsymbol{v} + (\psi_2)^2\boldsymbol{v},$$

$$\psi_{1ss}\boldsymbol{e} + \psi_{2ss}\boldsymbol{w} = \boldsymbol{v}_{sss} + 3\psi_1\psi_{1s}\boldsymbol{v} + 3\psi_2\psi_{2s}\boldsymbol{v} + \psi_1\big((\psi_1)^2 + (\psi_2)^2\big)\boldsymbol{e}$$
$$+ \psi_2\big((\psi_1)^2 + (\psi_2)^2\big)\boldsymbol{w},$$

and after taking the inner product of the above equations with $\boldsymbol{e}$ and $\boldsymbol{w}$, we derive

$$\|\psi_{1s}(t)\| \le \|\boldsymbol{v}_{ss}(t)\|,$$
$$\|\psi_{2s}(t)\| \le \|\boldsymbol{v}_{ss}(t)\|,$$
$$\|\psi_{1ss}(t)\| \le \|\boldsymbol{v}_{sss}(t)\| + C\|\boldsymbol{v}_s(t)\|_1^2\|\boldsymbol{v}_s(t)\|,$$
$$\|\psi_{2ss}(t)\| \le \|\boldsymbol{v}_{sss}(t)\| + C\|\boldsymbol{v}_s(t)\|_1^2\|\boldsymbol{v}_s(t)\|,$$

where $C > 0$ is determined from the embedding $H^1(I_L) \hookrightarrow L^\infty(I_L)$. Hence, we have

$$\|\psi_1(t)\|_2 + \|\psi_2(t)\|_2 \le C\big(\|\boldsymbol{v}(t)\|_3 + \|\boldsymbol{v}(t)\|_3^3\big), \tag{3.5}$$

for all $t > 0$, where $C > 0$ is independent of t and $\boldsymbol{v}$.

Since $\boldsymbol{v}\cdot\boldsymbol{v}_t \equiv 0$, a decomposition of the form $\boldsymbol{v}_t = p_1\boldsymbol{e} + p_2\boldsymbol{w}$ holds. From the equation $(\boldsymbol{v}_t)_s = (\boldsymbol{v}_s)_t$ and $(\boldsymbol{e}_t)_s = (\boldsymbol{e}_s)_t$, we deduce that $p_1 = -\psi_{2s}$ and $p_2 = \psi_{1s}$. Furthermore,

$$\boldsymbol{v}_t = -\psi_{2s}\boldsymbol{e} + \psi_{1s}\boldsymbol{w}, \tag{3.6}$$
$$\boldsymbol{e}_t = \psi_{2s}\boldsymbol{v} + \left\{ -\frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big) + \frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big)\Big|_{s=0}\right\}\boldsymbol{w},$$
$$\boldsymbol{w}_t = -\psi_{1s}\boldsymbol{v} - \left\{ -\frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big) + \frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big)\Big|_{s=0}\right\}\boldsymbol{e},$$

holds. Here, $\big|_{s=0}$ denotes the trace at $s = 0$. The above three equations along with the fact that $\boldsymbol{v} \in C^1\big([0,\infty); H^1(I_L)\big)$, $\psi_1 = \boldsymbol{e}\cdot\boldsymbol{v}_s$, and $\psi_2 = \boldsymbol{w}\cdot\boldsymbol{v}_s$ implies that $\psi_1, \psi_2 \in C^1\big([0,\infty); L^2(I_L)\big)$. This in turn allows us to calculate as follows.

$$\psi_{1t} = -\psi_{2ss} + \left\{ -\frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big) + \frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big)\Big|_{s=0}\right\}\psi_2,$$
$$\psi_{2t} = \psi_{1ss} - \left\{ -\frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big) + \frac{1}{2}\big((\psi_1)^2 + (\psi_2)^2\big)\Big|_{s=0}\right\}\psi_1.$$

Setting $\psi = \psi_1 + \mathrm{i}\psi_2$, we see that

$$\mathrm{i}\psi_t - \psi_{ss} = \frac{1}{2}|\psi|^2\psi - \frac{1}{2}|\psi(0,t)|^2\psi,$$

and after a gauge transform given by

$$q(s,t) = \psi(s,t)\exp\left\{\frac{\mathrm{i}}{2}\int_0^t |\psi(0,\tau)|^2\, d\tau\right\}, \tag{3.7}$$

we see that q satisfies

$$\mathrm{i}q_t = q_{ss} + \frac{1}{2}|q|^2 q.$$

As t tends to zero in equation (3.2), we see from the uniqueness of the solution of ordinary differential equations that

$$q(s,0) = q_0(s).$$

Finally, we see that by taking the derivative of the boundary condition in problem (1.3) with respect to t, we see that $\boldsymbol{v}_t(0,t) = \boldsymbol{v}_t(L,t) = \mathbf{0}$, and from equation (3.6), we see that

$$q_s(0,t) = q_s(L,t) = 0.$$

Also note that $q \in C\big([0,\infty); H^2(I_L)\big) \cap C^1\big([0,\infty); L^2(I_L)\big)$ from the definition of q along with the fact that $\psi_1, \psi_2 \in C\big([0,\infty); H^2(I_L)\big) \cap C^1\big([0,\infty); L^2(I_L)\big)$. Additionally, from equations (3.5) and (3.7) we have

$$\|q(t)\|_2 \le C\big(\|\boldsymbol{v}(t)\|_3 + \|\boldsymbol{v}(t)\|_3^3\big), \tag{3.8}$$

for all $t \ge 0$, where $C > 0$ is independent of t and $\boldsymbol{v}$. Estimate (3.8) will be utilized later. This finishes the proof of Theorem 2.3.

□

3.2 Proof of Theorem 2.4

The proof of Theorem 2.4 is similar to the proof of Theorem 2.3. Let $\boldsymbol{v}_0 \in H^3(I_L)$ and $\boldsymbol{b} \in \mathbf{S}^2$ satisfy $|\boldsymbol{v}_0| \equiv 1$ and the compatibility conditions for (1.3) up to order 1. Then, equations (3.2), (3.3), and (3.4) along with appropriate boundary conditions act as the definition of ψ_1 and ψ_2. From here, direct calculations based on these explicit equations, similar to Section 3.1, proves Theorem 2.4. □

3.3 Proof of Theorem 2.5

As an application of Theorems 2.3 and 2.4, we prove Theorem 2.5. The explicit nature of the generalized Hasimoto transformation for problems (1.3) and (1.4) allows us to prove the following.

Proposition 3.1. *For $q_0 \in H^2(I_L)$ and $\varphi_0 \in H^2(I_L)$ which satisfy the* 0*-th order compatibility condition for* (1.4)*, let $\boldsymbol{v}_0$ and $\tilde{\boldsymbol{v}}_0$ be initial data constructed by the generalized Hasimoto transformation from q_0 and $q_0 + \varphi_0$, respectively. Then, $\boldsymbol{v}_0, \tilde{\boldsymbol{v}}_0 \in H^3(I_L)$ and satsify*

$$\|\tilde{\boldsymbol{v}}_0 - \boldsymbol{v}_0\|_3 \leq C\big(\|\tilde{q}_0 - q_0\|_2 + \|\tilde{q}_0 - q_0\|_2^3\big),$$

where $C > 0$ depends on $\|q_0\|_2 + \|\varphi_0\|_2$, and is non-decreasing with respect to $\|q_0\|_2 + \|\varphi_0\|_2$.

Proposition 3.1 is proved by a direct energy method applied to equations (3.2), (3.3), and (3.4), along with equation (3.7). This shows that perturbations made to q_0 corresponds to perturbations to $\boldsymbol{v}_0$ through the generalized Hasimoto transformation. Furthermore, we can show stability estimates for the arc-shaped filament $\boldsymbol{v}^R$ given by

$$\boldsymbol{v}^R(s) := {}^t\big(\cos(s/R), -\sin(s/R), 0\big).$$

$\boldsymbol{v}^R$ is a stationary solution of problem (1.3) and corresponds to the plane wave solution q_R of problem (1.4) through the inverse generalized Hasimoto transformation. This can be confirmed by directly solving system (3.1) for $q_0 = q_R(0)$.

For the initial perturbation $\boldsymbol{\varphi}_0 \in H^3(I_L)$ of the arc-shaped filament, we assume the following.

Assumption 3.2. *For the initial perturbation $\boldsymbol{\varphi}_0$, we assume the following.*

(A1) $|\boldsymbol{v}^R(s) + \boldsymbol{\varphi}_0(s)| = 1$ *for all $s \in I_L$.*

(A2) $\boldsymbol{v}^R + \boldsymbol{\varphi}_0$ *and $\boldsymbol{b}$ satisfy the compatibility conditions for problem* (1.3) *up to order* 1*.*

These assumptions are necessary conditions for problem (1.3) to be well-posed for a perturbed arc-shaped filament. Then, we can show the following.

Proposition 3.3. *For $R > L/\pi$, the arc-shaped solution $\boldsymbol{v}^R$ of problem* (1.3) *is stable in the following sense. For $\boldsymbol{\varphi}_0 \in H^3(I_L)$ satisfying Assumption* 3.2*, the perturbed arc-shaped solution $\boldsymbol{v} \in C\big([0,\infty); H^3(I_L)\big) \cap C^1\big([0,\infty); H^1(I_L)\big)$ satisfies*

$$\|\boldsymbol{v}(t) - \boldsymbol{v}^R\|_3 \leq C\big(\|\boldsymbol{\varphi}_0\|_3 + \|\boldsymbol{\varphi}_0\|_3^3\big)$$

for all $t > 0$. Here, $C > 0$ depends on L and R.

Here, the perturbed arc-shaped solution is the solution of problem (1.3) with initial datum $\boldsymbol{v}^R + \boldsymbol{\varphi}_0$. Proposition 3.3 is proved by a direct energy estimate for the perturbation $\boldsymbol{\varphi} := \boldsymbol{v} - \boldsymbol{v}^R$, along with the utilization of conserved quantities known for problem (1.3).

Finally, combining all the estimates proves Theorem 2.5. □

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