On a two-phase free boundary problem for inhomogeneous incompressible viscous fluids

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1 Introduction

This is a brief survey of the joint work [2] with Kenta Oishi, Waseda Univeristy.

Let us consider a two-phase free boundary problem for inhomogeneous incompressible viscous fluids in the N-dimensional Euclidean space $\mathbf{R}^N$ for $N \geq 3$. Define

$$\mathbf{R}^N_\pm = \{\xi = (\xi', \xi_N) : \xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbf{R}^{N-1}, \pm\xi_N > 0\}.$$

An inhomogeneous incompressible viscous fluid occupies $\mathbf{R}^N_+$ at time $t = 0$, and the fluid is denoted by fluid$_+$. On the other hand, another inhomogeneous incompressible viscous fluid occupies $\mathbf{R}^N_-$ at $t = 0$, and the fluid is denoted by fluid$_-$. The two fluids are thus initially separated from one another by the flat interface

$$\mathbf{R}^N_0 = \{\xi = (\xi', \xi_N) : \xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbf{R}^{N-1}, \xi_N = 0\}.$$

Let $\xi \in \mathbf{R}^N_\pm$. Suppose that $\rho^0_\pm(\xi)$ are given functions and $\rho^*_\pm$ are positive constants. The initial densities of fluid$_\pm$ are given by $\rho^0_\pm(\xi) + \rho^*_\pm$. We suppose that $\mu_\pm = \mu_\pm(s) > 0$ are smooth functions on $s > 0$ and that the viscosity coefficients of fluid$_\pm$ are given by $\mu_\pm(\rho^0_\pm(\xi) + \rho^*_\pm)$ at $t = 0$. Furthermore, the initial velocities of fluid$_\pm$ are given by $\mathbf{u}^0_\pm(\xi) = (u^0_{1\pm}(\xi), \dots, u^0_{N\pm}(\xi))^\mathsf{T}$, where $\mathbf{M}^\mathsf{T}$ stands for the transpose of $\mathbf{M}$.

Our unknowns will be a transformation $\Theta(\cdot, t) : \mathbf{R}^N \ni \xi \mapsto x = \Theta(\xi, t) \in \mathbf{R}^N$, densities $\rho_\pm = \rho_\pm(x, t)$, pressures $\mathfrak{q}_\pm = \mathfrak{q}_\pm(x, t)$, and velocities

$$\mathbf{v}_\pm = \mathbf{v}_\pm(x, t) = (v_{1\pm}(x, t), \dots, v_{N\pm}(x, t))^\mathsf{T}$$

satisfying the following two-phase free boundary problem:

$$\partial_t \Theta = \mathbf{v}_\pm \circ \Theta, \quad \Theta(\xi, 0) = \xi, \quad \xi \in \mathbf{R}^N_\pm, \tag{1.1}$$

$$\Omega_\pm(t) = \Theta(\mathbf{R}^N_\pm, t), \quad \Gamma(t) = \Theta(\mathbf{R}^N_0, t), \tag{1.2}$$

$$\partial_t\rho_\pm + \mathbf{v}_\pm\cdot\nabla\rho_\pm = 0, \quad x\in\Omega_\pm(t), \tag{1.3}$$

$$\operatorname{div}\mathbf{v}_\pm = 0, \quad x\in\Omega_\pm(t), \tag{1.4}$$

$$\rho_\pm(\partial_t\mathbf{v}_\pm + \mathbf{v}_\pm\cdot\nabla\mathbf{v}_\pm) = \operatorname{Div}(\mu_\pm(\rho_\pm)\mathbf{D}(\mathbf{v}_\pm) - \mathfrak{q}_\pm\mathbf{I}), \quad x\in\Omega_\pm(t), \tag{1.5}$$

$$(\mu_+(\rho_+)\mathbf{D}(\mathbf{v}_+) - \mathfrak{q}_+\mathbf{I})\mathbf{n}_{\Gamma(t)} = (\mu_-(\rho_-)\mathbf{D}(\mathbf{v}_-) - \mathfrak{q}_-\mathbf{I})\mathbf{n}_{\Gamma(t)}, \quad x\in\Gamma(t), \tag{1.6}$$

$$\mathbf{v}_+ = \mathbf{v}_-, \quad x\in\Gamma(t), \tag{1.7}$$

$$\mathbf{v}_\pm(\xi,0) = \mathbf{u}^0_\pm(\xi), \quad \xi\in\mathbf{R}^N_\pm, \tag{1.8}$$

$$\rho_\pm(\xi,0) = \rho^0_\pm(\xi) + \rho^*_\pm, \quad \xi\in\mathbf{R}^N_\pm, \tag{1.9}$$

where $\partial_t = \partial/\partial t$ and $\mathbf{v}_\pm\circ\Theta = (\mathbf{v}_\pm\circ\Theta)(\xi,t) = \mathbf{v}_\pm(\Theta(\xi,t),t)$.

Here $\mathbf{n}_{\Gamma(t)}$ is the unit normal vector on $\Gamma(t)$ pointing from $\Omega_-(t)$ into $\Omega_+(t)$ and $\mathbf{I}$ is the $N\times N$ identity matrix. For $\mathbf{u} = (u_1(x),\dots,u_N(x))^\top$, $\mathbf{D}(\mathbf{u})$ is the doubled deformation rate tensor, i.e.,

$$\mathbf{D}(\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^\top, \quad \nabla\mathbf{u} = \begin{pmatrix}\partial_1 u_1 & \dots & \partial_N u_1\\ \vdots & \ddots & \vdots\\ \partial_1 u_N & \dots & \partial_N u_N\end{pmatrix},$$

where $\partial_j = \partial/\partial x_j$ for $j=1,\dots,N$. Let $\mathbf{M} = (M_{ij}(x))_{1\le i,j\le N}$ be a matrix-valued function, and let

$$f = f(x), \quad \mathbf{g} = (g_1(x),\dots,g_N(x))^\top, \quad \mathbf{h} = (h_1(x),\dots,h_N(x))^\top.$$

One then defines

$$\nabla f = (\partial_1 f,\dots,\partial_N f)^\top, \quad \Delta f = \sum_{j=1}^N \partial_j^2 f, \quad \operatorname{div}\mathbf{g} = \sum_{j=1}^N \partial_j g_j,$$

$$\Delta\mathbf{g} = (\Delta g_1,\dots,\Delta g_N), \quad \nabla^2\mathbf{g} = \{\partial_i\partial_k g_k : i,j,k = 1,\dots,N\},$$

$$\mathbf{g}\cdot\nabla f = \sum_{j=1}^N g_j\partial_j f, \quad \mathbf{g}\cdot\nabla\mathbf{h} = (\mathbf{g}\cdot\nabla h_1,\dots,\mathbf{g}\cdot\nabla h_N)^\top,$$

$$\operatorname{Div}\mathbf{M} = \left(\sum_{j=1}^N \partial_j M_{1j},\cdots\sum_{j=1}^N \partial_j M_{Nj}\right)^\top.$$

In particular,

$$\begin{aligned}&\operatorname{Div}(\mu_\pm(\rho_\pm)\mathbf{D}(\mathbf{v}_\pm) - \mathfrak{q}_\pm\mathbf{I})\\ &= \mu_\pm(\rho_\pm)(\Delta\mathbf{v}_\pm + \nabla\operatorname{div}\mathbf{v}_\pm) + \mu'(\rho_\pm)\mathbf{D}(\mathbf{v}_\pm)\nabla\rho_\pm - \nabla\mathfrak{q}_\pm \quad \text{in } \Omega_\pm(t).\end{aligned}$$

Two-phase free boundary problems for inhomogeneous incompressible viscous fluids were studied by Tanaka [4], Xu and Zhang [5] in an L_2 setting for both space and time. Those papers proved global existence theorems for small initial data. On the other hand, Saito, Shibata, and Zhang [3] proved a local existence theorem on general unbounded domains for large initial data in an L_p-in-time and L_q-in-space setting. Our work in this article is a continuation of [3] and gives a global existence theorem of (1.1)–(1.9) for small initial data with suitable p, q.

2 Formulation in Lagrangian coordinates

This section transforms (1.1)–(1.9) into a system in Lagrangian coordinates.

Let $\mathbf{u}_\pm(\xi,t) = \mathbf{v}_\pm(\Theta(\xi,t),t)$ for $(\xi,t)\in\mathbf{R}^N_\pm\times\mathbf{R}_+$, where $\mathbf{R}_+=(0,\infty)$. The solution Θ to (1.1) is then given by

$$\Theta(\xi,t) = \xi + \int_0^t \mathbf{u}_\pm(\xi,s)\,ds, \quad (\xi,t)\in\mathbf{R}^N_\pm\times\mathbf{R}_+.$$

It follows from (1.1) and (1.3) that $\partial_t[\rho_\pm(\Theta(\xi,t),t)]=0$, and thus integrating this equation over $[0,t]$ with respect to time variable shows

$$\rho_\pm(\Theta(\xi,t),t) = \rho_\pm(\Theta(\xi,0),0), \quad (\xi,t)\in\mathbf{R}^N_\pm\times\mathbf{R}_+.$$

Combining this with $\Theta(\xi,0)=\xi$ in (1.1) and $\rho_\pm(\xi,0)=\rho^0_\pm(\xi)+\rho^*_\pm$ in (1.9), we obtain

$$\rho_\pm(\Theta(\xi,t),t) = \rho^0_\pm(\xi)+\rho^*_\pm, \quad (\xi,t)\in\mathbf{R}^N_\pm\times\mathbf{R}_+.$$

From the above observation, our new unknowns in Lagrangian coordinates will be the Lagrangian velocities $\mathbf{u}_\pm(\xi,t)=\mathbf{v}_\pm(\Theta(\xi,t),t)$ and pressures $\mathfrak{p}_\pm(\xi,t)=\mathfrak{q}_\pm(\Theta(\xi,t),t)$ for $(\xi,t)\in\mathbf{R}^N_\pm\times\mathbf{R}_+$. Let us define

$$\dot{\mathbf{R}}^N = \mathbf{R}^N_+\cup\mathbf{R}^N_-$$

and let $(\xi,t)\in\dot{\mathbf{R}}^N\times\mathbf{R}_+$. One sets

$$\mathbf{u}=\mathbf{u}(\xi,t)=\begin{cases}\mathbf{u}_+(\xi,t), & (\xi,t)\in\mathbf{R}^N_+\times\mathbf{R}_+,\\ \mathbf{u}_-(\xi,t), & (\xi,t)\in\mathbf{R}^N_-\times\mathbf{R}_+,\end{cases}$$
$$\mathfrak{p}=\mathfrak{p}(\xi,t)=\begin{cases}\mathfrak{p}_+(\xi,t), & (\xi,t)\in\mathbf{R}^N_+\times\mathbf{R}_+,\\ \mathfrak{p}_-(\xi,t), & (\xi,t)\in\mathbf{R}^N_-\times\mathbf{R}_+,\end{cases}$$

and also

$$\sigma_0=\sigma_0(\xi)=\begin{cases}\rho^0_+(\xi), & \xi\in\mathbf{R}^N_+,\\ \rho^0_-(\xi), & \xi\in\mathbf{R}^N_-,\end{cases}\qquad \sigma=\sigma(\xi)=\begin{cases}\rho^*_+, & \xi\in\mathbf{R}^N_+,\\ \rho^*_-, & \xi\in\mathbf{R}^N_-.\end{cases}$$

Furthermore,

$$\mu(s,\xi)=\mu_+(s)\mathbb{1}_{\mathbf{R}^N_+}(\xi)+\mu_-(s)\mathbb{1}_{\mathbf{R}^N_-}(\xi),$$

where $\mathbb{1}_A$ is the indicator function of $A \subset \mathbf{R}^N$, i.e., $\mathbb{1}_A(\xi) = 1$ for $\xi \in A$ and $\mathbb{1}_A(\xi) = 0$ for $\xi \notin A$. It then holds that

$$\mu(\sigma_0(\xi) + \sigma, \xi) = \begin{cases} \mu_+(\rho_+^0(\xi) + \rho_+^*), & \xi \in \mathbf{R}_+^N, \\ \mu_-(\rho_-^0(\xi) + \rho_-^*), & \xi \in \mathbf{R}_+^N, \end{cases}$$

$$\mu(\sigma(\xi), \xi) = \begin{cases} \mu_+(\rho_+^*), & \xi \in \mathbf{R}_+^N, \\ \mu_-(\rho_-^*), & \xi \in \mathbf{R}_-^N. \end{cases}$$

Let us denote $\mu(\sigma_0(\xi)+\sigma, \xi)$ by $\mu(\sigma_0+\sigma)$ and $\mu(\sigma(\xi), \xi)$ by $\mu(\sigma)$ for short in what follows. Notice that σ and $\mu(\sigma)$ are piecewise constants.

Let $f = f(\xi)$ be a function defined on $\dot{\mathbf{R}}^N$. Then $[\![f]\!]$ stands for the jump of the quantity f across the flat interface $\xi_N = 0$, i.e.,

$$[\![f]\!] = [\![f]\!](\xi') = \lim_{\xi_N \downarrow 0} \Big(f(\xi', \xi_N) - f(\xi', -\xi_N) \Big),$$

where $\xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbf{R}^{N-1}$.

Let us now substitute the new unknowns $(\mathbf{u}, \mathfrak{p})$ into (1.4)–(1.8). We then achieve the following set of equations:

$$\begin{cases} \partial_t \mathbf{u} - \sigma^{-1} \operatorname{Div}(\mu(\sigma)\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \sigma^{-1}\mathsf{F}(\mathbf{u}) & \text{in } \dot{\mathbf{R}}^N \times \mathbf{R}_+, \\ \operatorname{div} \mathbf{u} = \mathsf{G}(\mathbf{u}) = \operatorname{div} \widetilde{\mathsf{G}}(\mathbf{u}) & \text{in } \dot{\mathbf{R}}^N \times \mathbf{R}_+, \\ [\![(\mu(\sigma)\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{e}_N]\!] = [\![\mathsf{H}(\mathbf{u})]\!] & \text{on } \mathbf{R}^{N-1} \times \mathbf{R}_+, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \mathbf{R}^{N-1} \times \mathbf{R}_+, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot{\mathbf{R}}^N, \end{cases} \tag{2.1}$$

where $\mathbf{e}_N = (0, \ldots, 0, 1)^\mathsf{T}$ and

$$\mathbf{u}_0 = \mathbf{u}_0(\xi) = \begin{cases} \mathbf{u}_+^0(\xi), & \xi \in \mathbf{R}_+^N, \\ \mathbf{u}_-^0(\xi), & \xi \in \mathbf{R}_-^N. \end{cases}$$

Here the right members $\mathsf{F}(\mathbf{u})$, $\mathsf{G}(\mathbf{u})$, $\widetilde{\mathsf{G}}(\mathbf{u})$, and $\mathsf{H}(\mathbf{u})$ stand for nonlinear terms, see [2] for their exact formulas.

3 Global solvability in Lagrangian coordinates

System (2.1) leads us to the following linearized problem:

$$\begin{cases} \partial_t \mathbf{u} - \sigma^{-1}\operatorname{Div}(\mu(\sigma)\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I}) = \sigma^{-1}\mathbf{f} & \text{in } \dot{\mathbf{R}}^N \times \mathbf{R}_+, \\ \operatorname{div}\mathbf{u} = g = \operatorname{div}\mathbf{g} & \text{in } \dot{\mathbf{R}}^N \times \mathbf{R}_+, \\ [\![(\mu(\sigma)\mathbf{D}(\mathbf{u}) - \mathfrak{p}\mathbf{I})\mathbf{e}_N]\!] = [\![\mathbf{h}]\!] & \text{on } \mathbf{R}^{N-1} \times \mathbf{R}_+, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \mathbf{R}^{N-1} \times \mathbf{R}_+, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot{\mathbf{R}}^N. \end{cases} \tag{3.1}$$

To state a main result for (3.1), we introduce the notation. Let $\mathbf{N}$ be the set of all positive integers. For $\mathbf{a} = (a_1, \dots, a_N)^\mathsf{T}$ and $\mathbf{b} = (b_1, \dots, b_N)^\mathsf{T}$, we set

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^N a_j b_j, \quad \mathbf{a}_{\tan} = \mathbf{a} - \mathbf{e}_N \langle \mathbf{e}_N, \mathbf{a} \rangle.$$

Let $p \in [1, \infty]$, $q \in (1, \infty)$, $n \in \mathbf{N}$, and $s \in \mathbf{R}_+$. Let G be an open set in $\mathbf{R}^N$. Then $L_p(G)$, $H_p^n(G)$, and $B_{q,p}^s(G)$ are the standard Lebesgue space, Sobolev space, and Besov space on G, respectively. Their respective norms are denoted by $\|\cdot\|_{L_p(G)}$, $\|\cdot\|_{H_p^n(G)}$, and $\|\cdot\|_{B_{q,p}^s(G)}$. The homogemeous Sobolev space $\widehat{H}_q^1(G)$ is given by

$$\widehat{H}_q^1(G) = \{u \in L_{q,\mathrm{loc}}(G) : \nabla u \in L_q(G)^N\}.$$

Define for $\mathbf{f} = \mathbf{f}(x) = (f_1(x), \dots, f_N(x))^\mathsf{T}$ and $\mathbf{g} = \mathbf{g}(x) = (g_1(x), \dots, g_N(x))^\mathsf{T}$

$$(\mathbf{f}, \mathbf{g})_G = \int_G \langle \mathbf{f}(x), \mathbf{g}(x) \rangle \, dx = \sum_{j=1}^N \int_G f_j(x) g_j(x) \, dx.$$

The space $J_q(\dot{\mathbf{R}}^N)$ of solenoidal vector fields is then defined as

$$J_q(\dot{\mathbf{R}}^N) = \{\mathbf{f} \in L_q(\dot{\mathbf{R}}^N)^N : (\mathbf{f}, \nabla\varphi)_{\dot{\mathbf{R}}^N} = 0 \text{ for any } \varphi \in \widehat{H}_{q'}^1(\mathbf{R}^N)\},$$

where $q' = q/(q-1)$.

Let X be a Banach space. Then X^M, $M \geq 2$, is the M-product space of X, while the norm of X^M is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^M}$ for the sake of simplicity. Let $L_p(\mathbf{R}_+, X)$ and $H_p^1(\mathbf{R}_+, X)$ be the X-valued Lebesgue space on $\mathbf{R}_+$ and the X-valued Sobolev space on $\mathbf{R}_+$, respectively. Their respective norms are denoted by $\|\cdot\|_{L_p(\mathbf{R}_+,X)}$ and $\|\cdot\|_{H_p^1(\mathbf{R}_+,X)}$. Furthermore, we set

$${}_0H_p^1(\mathbf{R}_+, X) = \{f \in H_p^1(\mathbf{R}_+, X) : f|_{t=0} = 0 \text{ in } X\}$$

endowed with the norm $\|\cdot\|_{{}_0H_p^1(\mathbf{R}_+,X)} := \|\cdot\|_{H_p^1(\mathbf{R}_+,X)}$. Let $[\cdot,\cdot]_\theta$ be the complex interpolation functor for $\theta \in (0,1)$. Define

$$H_p^{1/2}(\mathbf{R}_+, X) = [L_p(\mathbf{R}_+, X), H_p^1(\mathbf{R}_+, X)]_{1/2}.$$

Let $w(t)$ be a function of time t and

$$\|w(t)f\|_{L_p(\mathbf{R}_+,X)} = \left(\int_0^\infty \left(w(t)\|f(t)\|_X\right)^p dt\right)^{1/p},$$

where $p \in (1,\infty)$. For the right members $\mathbf{f}$, g, $\mathbf{g}$, and $\mathbf{h}$ of (3.1), we introduce $\|\cdot\|_{F_{p,q}(w(t))}$ as follows:

$$\begin{aligned}\|(\mathbf{f},g,\mathbf{g},\mathbf{h})\|_{F_{p,q}(w(t))} &= \|w(t)\mathbf{f}\|_{L_p(\mathbf{R}_+,L_q(\dot{\mathbf{R}}^N)^N)} + \|w(t)g\|_{L_p(\mathbf{R}_+,H^1_q(\dot{\mathbf{R}}^N))}\\ &\quad + \|w(t)\partial_t\mathbf{g}\|_{L_p(\mathbf{R}_+,L_q(\dot{\mathbf{R}}^N)^N)} + \|w(t)\mathbf{h}\|_{L_p(\mathbf{R}_+,H^1_q(\dot{\mathbf{R}}^N)^N)}\\ &\quad + \|w(t)\mathbf{h}\|_{H^{1/2}_p(\mathbf{R}_+,L_q(\dot{\mathbf{R}}^N)^N)}.\end{aligned}$$

Then we have the following theorem for (3.1).

Theorem 3.1. *Suppose $N \geq 3$. Let $p, q_1, q_2 \in (1,\infty)$ satisfy*

$$2 < q_1 < N, \quad q_1 \leq q_2 < \infty, \quad p\left(\frac{N}{2q_1} - \frac{1}{2}\right) > 1$$

and

$$2 - \frac{2}{p} - \frac{1}{q_i} \neq 0, \quad 1 - \frac{2}{p} - \frac{1}{q_i} \neq 0 \quad \text{for } i = 1,2. \tag{3.2}$$

Let $\mathbf{u}_0$, $\mathbf{f}$, g, $\mathbf{g}$, and $\mathbf{h}$ satisfy the following conditions:

(a) $\langle t\rangle\mathbf{f} \in \bigcap_{r\in\{q_1/2,q_2\}} L_p(\mathbf{R}_+, L_r(\dot{\mathbf{R}}^N)^N)$;

(b) $\langle t\rangle g \in \bigcap_{r\in\{q_1/2,q_2\}} L_p(\mathbf{R}_+, H^1_r(\dot{\mathbf{R}}^N))$;

(c) $\langle t\rangle^{1/2}\mathbf{g} \in \bigcap_{r\in\{q_1/2,q_2\}} {}_0H^1_p(\mathbf{R}_+, L_r(\dot{\mathbf{R}}^N)^N)$ *with* $[\![\langle\mathbf{g},\mathbf{e}_N\rangle]\!] = 0$ *on* $\mathbf{R}^{N-1}\times\mathbf{R}_+$, *while* $\langle t\rangle\partial_t\mathbf{g} \bigcap_{r\in\{q_1/2,q_2\}} L_p(\mathbf{R}_+, L_r(\dot{\mathbf{R}}^N)^N)$;

(d) $\langle t\rangle\mathbf{h} \in \bigcap_{r\in\{q_1/2,q_2\}} (H^{1/2}_p(\mathbf{R}_+ L_r(\dot{\mathbf{R}}^N)^N) \cap L_p(\mathbf{R}_+, H^1_r(\dot{\mathbf{R}}^N)^N))$;

(e) $\mathbf{u}_0 \in B^{2-2/p}_{q_1,p}(\dot{\mathbf{R}}^N)^N \cap B^{2-2/p}_{q_2,p}(\dot{\mathbf{R}}^N)^N \cap J_{q_1/2}(\dot{\mathbf{R}}^N)$;

(f) $[\![(\mu(\sigma)\mathbf{D}(\mathbf{u}_0)\mathbf{e}_N)_{\tan}]\!] = [\![(\mathbf{h}|_{t=0})_{\tan}]\!]$ *in* $B^{1-2/p-1/q_2}_{q_2,p}(\mathbf{R}^{N-1})$ *if* $1 - 2/p - 1/q_2 > 0$;

(g) $[\![\mathbf{u}_0]\!] = 0$ *in* $B^{2-2/p-1/q_2}_{q_2,p}(\mathbf{R}^{N-1})$ *if* $2 - 2/p - 1/q_2 > 0$.

Then (3.1) *admits a unique solution* $\mathbf{u}$ *with some pressure* $\mathfrak{p}$, *and there holds for* $q = q_1$ *or* $q = q_2$

$$\begin{aligned}&\|\langle t\rangle^{1/2}\mathbf{u}\|_{L_p(\mathbf{R}_+,L_q(\dot{\mathbf{R}}^N)^N)} + \|\langle t\rangle\partial_t\mathbf{u}\|_{L_p(\mathbf{R}_+,L_q(\dot{\mathbf{R}}^N)^N)}\\ &+ \|\langle t\rangle\nabla\mathbf{u}\|_{L_p(\mathbf{R}_+,L_q(\dot{\mathbf{R}}^N)^{N\times N})} + \|\langle t\rangle\nabla\mathbf{u}\|_{H^{1/2}_p(\mathbf{R}_+,L_q(\dot{\mathbf{R}}^N)^{N\times N})}\\ &\leq C\Big[\|\mathbf{u}_0\|_{B^{2-2/p}_{q_1,p}(\dot{\mathbf{R}}^N)} + \|\mathbf{u}_0\|_{B^{2-2/p}_{q_2,p}(\dot{\mathbf{R}}^N)} + \|\mathbf{u}_0\|_{L_{q_1/2}(\dot{\mathbf{R}}^N)}\end{aligned}$$

$$+ \sum_{r \in \{q_1/2, q_2\}} \Big(\|(\mathbf{f}, g, \mathbf{g}, \mathbf{h})\|_{F_{p,r}(\langle t \rangle)} + \|\langle t \rangle^{1/2} \mathbf{g}\|_{L_p(\mathbf{R}_+, L_r(\dot{\mathbf{R}}^N)^N)} \Big) \Big],$$

where $C = C(N, p, q_1, q_2)$ is a positive constant.

To state a global existence theorem for (2.1), we introduce function spaces. Define

$$Z_{N,p,q} = H^1_p(\mathbf{R}_+, L_q(\dot{\mathbf{R}}^N)^N) \cap L_p(\mathbf{R}_+, H^2_q(\dot{\mathbf{R}}^N)^N).$$

Let δ be a positive number and $\langle t \rangle = \sqrt{t^2+1}$. We set

$$Z^\delta_{N,p,q} = \{\mathbf{u} \in Z_{N,p,q} : \langle t \rangle^\delta \partial_t \mathbf{u} \in L_p(\mathbf{R}_+, L_q(\dot{\mathbf{R}}^N)^N), \langle t \rangle^\delta \mathbf{u} \in L_p(\mathbf{R}_+, H^2_q(\dot{\mathbf{R}}^N)^N)\}$$

with the norm

$$\|\mathbf{u}\|_{Z^\delta_{N,p,q}} = \|\langle t \rangle^\delta \partial_t \mathbf{u}\|_{L_p(\mathbf{R}_+, L_q(\dot{\mathbf{R}}^N)^N)} + \|\langle t \rangle^\delta \mathbf{u}\|_{L_p(\mathbf{R}_+, H^2_q(\dot{\mathbf{R}}^N)^N)}.$$

Furthermore, the auxiliary function space $A^\delta_{N,p,q}$ is defined by

$$\begin{aligned} A^\delta_{N,p,q} = \{\mathbf{u} : \langle t \rangle^\delta \partial_t \mathbf{u} \in L_p(\mathbf{R}_+, L_q(\dot{\mathbf{R}}^N)^N), \\ \langle t \rangle^\delta \nabla \mathbf{u} \in L_p(\mathbf{R}_+, H^1_q(\dot{\mathbf{R}}^N)^{N\times N}), \\ \langle t \rangle^\delta \nabla \mathbf{u} \in H^{1/2}_p(\mathbf{R}_+, L_q(\dot{\mathbf{R}}^N)^{N\times N})\} \end{aligned}$$

with the semi-norm

$$\begin{aligned} \|\mathbf{u}\|_{A^\delta_{N,p,q}} &= \|\langle t \rangle^\delta \partial_t \mathbf{u}\|_{L_p(\mathbf{R}_+, L_q(\dot{\mathbf{R}}^N)^N)} + \|\langle t \rangle^\delta \nabla \mathbf{u}\|_{L_p(\mathbf{R}_+, H^1_q(\dot{\mathbf{R}}^N)^{N\times N})} \\ &\quad + \|\langle t \rangle^\delta \nabla \mathbf{u}\|_{H^{1/2}_p(\mathbf{R}_+, L_q(\dot{\mathbf{R}}^N)^{N\times N})}. \end{aligned}$$

Let us now introduce an assumption of p, q.

Assumption 3.2. *Suppose $N \geq 3$. Let $p, q_1, q_2 \in (1, \infty)$ satisfy*

$$2 < q_1 < N < q_2 < \infty, \quad p\left(\frac{N}{2q_1} - \frac{1}{2}\right) > 1.$$

For p, q_1, and q_2 satisfying Assumption 3.2, we define

$$\begin{aligned} K_{N,p,q_1,q_2} &= Z^{1/2}_{N,p,q_1} \cap Z^{1/2}_{N,p,q_2} \cap A^1_{N,p,q_1} \cap A^1_{N,p,q_2}, \\ \|\mathbf{u}\|_{K_{N,p,q_1,q_2}} &= \sum_{q \in \{q_1, q_2\}} \Big(\|\mathbf{u}\|_{Z^{1/2}_{N,p,q}} + \|\mathbf{u}\|_{A^1_{N,p,q}} \Big). \end{aligned}$$

In addition, we set for $\delta_0 > 0$ and the initial velocity $\mathbf{u}_0$

$$\begin{aligned} K_{N,p,q_1,q_2;\mathbf{u}_0}(\delta_0) = \{\mathbf{u} \in K_{N,p,q_1,q_2} : \|\mathbf{u}\|_{K_{N,p,q_1,q_2}} \leq \delta_0, \\ [\![\mathbf{u}]\!] = 0 \text{ on } \mathbf{R}^{N-1}, \mathbf{u}|_{t=0} = \mathbf{u}_0 \text{ in } \dot{\mathbf{R}}^N\}. \end{aligned}$$

Combining our linear theory, Theorem 3.1, with the contraction mapping principle shows the following global existence theorem for (2.1).

Theorem 3.3. *Suppose that Assumption* 3.2 *and* (3.2) *hold. Then there exist constants* $\delta_0, \varepsilon_1, \varepsilon_2 \in (0,1)$ *such that* (2.1) *admits a unique solution* $\mathbf{u} \in K_{N,p,q_1,q_2;\mathbf{u}_0}(\delta_0)$ *with some pressure* $\mathfrak{p}$ *for any*

$$\sigma_0 \in H^1_{q_1}(\dot{\mathbf{R}}^N) \cap H^1_{q_2}(\dot{\mathbf{R}}^N) \tag{3.3}$$

and for any

$$\mathbf{u}_0 \in B^{2-2/p}_{q_1,p}(\dot{\mathbf{R}}^N)^N \cap B^{2-2/p}_{q_2,p}(\dot{\mathbf{R}}^N)^N \cap J_{q_1/2}(\dot{\mathbf{R}}^N) \tag{3.4}$$

satisfying the smallness conditions:

$$\sum_{i=1}^{2} \|\sigma_0\|_{H^1_{q_i}(\dot{\mathbf{R}}^N)} \le \varepsilon_1, \quad \|\mathbf{u}_0\|_{L_{q_1/2}(\dot{\mathbf{R}}^N)} + \sum_{i=1}^{2} \|\mathbf{u}_0\|_{B^{2-2/p}_{q_i,p}(\dot{\mathbf{R}}^N)} \le \varepsilon_2$$

and the compatibility conditions (a) *and* (b) :

(a) $[\![(\mu(\sigma_0 + \sigma)\mathbf{D}(\mathbf{u}_0)\mathbf{e}_N)_{\tan}]\!] = 0$ *on* $\mathbf{R}^{N-1}$ *if* $1 - 2/p - 1/q_2 > 0$,

(b) $[\![\mathbf{u}_0]\!] = 0$ *on* $\mathbf{R}^{N-1}$ *if* $2 - 2/p - 1/q_2 > 0$.

4 Global solvability for the original system

This section shows a global existence theorem for (1.1)–(1.9). Following [1], we introduce definition of solutions to (1.1)–(1.9).

Definition 4.1. *We call* $(\Theta, \rho_\pm, \mathbf{v}_\pm, \mathfrak{q}_\pm)$ *a global-in-time solution to* (1.1)–(1.9) *if the following assertions hold for some* $p, q \in (1, \infty)$, $\rho^0_\pm \in H^1_q(\mathbf{R}^N_\pm)$, *and* $\mathbf{u}^0_\pm \in B^{2-2/p}_{q,p}(\mathbf{R}^N_\pm)^N$.

(1) *Let* $\Omega_\pm(t) = \Theta(\mathbf{R}^N_\pm, t)$ *for* $t > 0$. *Then* $\Theta(\cdot, t)$ *is a* C^1*-diffeomorphism from* $\mathbf{R}^N_+$ *onto* $\Omega_+(t)$ *and from* $\mathbf{R}^N_-$ *onto* $\Omega_-(t)$ *for each* $t > 0$.

(2) $\Theta = \Theta(\xi, t)$ *is a solution to* (1.1) *in the classical sense.*

(3) $\rho_\pm = \rho_\pm(x,t)$ *is given by* $\rho_\pm(x,t) = \rho^\pm_0(\Theta^{-1}_t(x)) + \rho^*_\pm$ *for* $x \in \Omega_\pm(t)$ *and* $t > 0$, *where* Θ^{-1}_t *is the inverse mapping of* $\Theta(\cdot, t) : \dot{\mathbf{R}}^N \to \dot{\Omega}(t)$ *with* $\dot{\Omega}(t) = \Omega_+(t) \cup \Omega_-(t)$.

(4) (2.1) *admits a solution* $(\mathbf{u}, \mathfrak{p})$, *and* $(\mathbf{v}_\pm, \mathfrak{q}_\pm)$ *are given by*

$$\mathbf{v}_\pm(x,t) = \mathbf{u}(\Theta^{-1}_t(x), t), \quad \mathfrak{q}_\pm(x,t) = \mathfrak{p}(\Theta^{-1}_t(x), t), \quad x \in \Omega_\pm(t),\ t > 0.$$

The following theorem then holds.

Theorem 4.2. *Suppose that Assumption* 3.2 *and* (3.2) *holds. Furthermore, we assume*

$$\frac{2}{p} + \frac{N}{q_2} < 1.$$

Let ε_1, ε_2 *be the positive numbers given by Theorem* 3.3*, and let* σ_0, $\mathbf{u}_0$ *satisfy* (3.3), (3.4)*, respectively, together with the smallness conditions and the compatibility conditions stated in Theorem* 3.3*. Then there exists a global-in-time solution* $(\Theta, \rho_\pm, \mathbf{v}_\pm, \mathfrak{q}_\pm)$ *to* (1.1)–(1.9)*, and also*

$$\|\mathbf{v}_\pm(t)\|_{B^{2-2/p}_{q,p}(\Omega_\pm(t))} = O(t^{-1/2}) \quad \text{as } t \to \infty$$

for $q = q_1$ *or* $q = q_2$*.*

References

[1] J. Escher, J. Prüss, and G. Simonett, Analytic solutions for a Stefan problem with Gibbs-Thomson correction, J. Reine Angew. Math., **563** (2003), 1–52.

[2] K. Oishi and H. Saito, Global solvability for two-phase flows of inhomogeneous incompressible viscous fluids, preprint.

[3] H. Saito, Y. Shibata, and X. Zhang, Some free boundary problem for two-phase inhomogeneous incompressible flows, SIAM J. Math. Anal., **52** (2020), 3397–3443.

[4] N. Tanaka. Global existence of two phase nonhomogeneous viscous incompressible fluid flow, Comm. Partial Differential Equations, **18** (1993), 41–81.

[5] L. Xu and Z. Zhang, On the free boundary problem to the two viscous immiscible fluids. J. Differential Equations, **248** (2010), 1044–1111.

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