Fast rotation limit for the MHD equations in a 3D infinite layer

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1 Introduction

This note is a survey of our paper [6]. We consider the initial value problem for the incompressible magnetohydrodynamics (MHD) equations with the Coriolis force in a 3D infinite layer $\mathbb{D} = \mathbb{R}^2 \times \mathbb{T}$:

$$\begin{cases} \partial_t u - \nu\Delta u + \Omega e_3 \times u + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla p = 0 & t>0,\ (x,z)\in\mathbb{D}, \\ \partial_t B - \Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0 & t>0,\ (x,z)\in\mathbb{D}, \\ \nabla\cdot u = \nabla\cdot B = 0 & t\geq 0,\ (x,z)\in\mathbb{D}, \\ u(0,x,z) = u_0(x,z), \quad B(0,x,z) = B_0(x,z) & (x,z)\in\mathbb{D}. \end{cases} \tag{1}$$

Here $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1]$ represents the 1D torus. The point of $\mathbb{D}$ is denoted by (x,z) with $x = (x_1,x_2) \in \mathbb{R}^2$ and $z \in \mathbb{T}$. Let $u = u(t,x,z) = (u_1(t,x,z), u_2(t,x,z), u_3(t,x,z))$, $B = B(t,x,z) = (B_1(t,x,z), B_2(t,x,z), B_3(t,x,z))$ and $p = p(t,x,z)$ be the unknown velocity field, the unknown magnetic field and the unknown pressure of the fluids, respectively. The vector fields $u_0 = u_0(x,z) = (u_{0,1}(x,z), u_{0,2}(x,z), u_{0,3}(x,z))$ and $B_0 = B_0(x,z) = (B_{0,1}(x,z), B_{0,2}(x,z), B_{0,3}(x,z))$ denote the given initial velocity field and the given initial magnetic field satisfying the divergence-free conditions $\nabla\cdot u_0 = \nabla\cdot B_0 = 0$, respectively. The constants $\Omega \in \mathbb{R}$ and $\nu > 0$ are the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$ and the viscosity coefficient, respectively.

Let us first review the known results for the rotating MHD equations (1) in $\mathbb{R}^3$. Ahn–Kim–Lee [1] proved the unique existence of global solution to (1) for the initial data $u_0 \in H^s(\mathbb{R}^3)$ $(\frac{1}{2} < s < \frac{3}{4})$, $B_0 \in (L^2 \cap L^q(\mathbb{R}^3))$ $(q > 3)$, when the rotating speed $|\Omega|$ is sufficiently large. Takada–Yoneda [8] showed that (1) has a unique global solution for sufficiently large $|\Omega|$ when $u_0, B_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. It follows from the proof of [8] that the global solution (u,B) converges to $(0, e^{t\Delta}B_0)$ as the size of rotating speed $|\Omega|$ tends to infinity.

In the case $B \equiv 0$, (1) corresponds to the Navier–Stokes equation with the Coriolis force. We review the known results for the rotating Navier–Stokes equations in $\mathbb{D}$. We decompose a function f as $f = \bar{f} + \tilde{f}$, where the 2D part $\bar{f}(x) = \mathcal{Q}f(x) := \int_{\mathbb{T}} f(x,z)dz$ and the 3D part $\tilde{f} = (I - \mathcal{Q})f(x,z) := f - \bar{f}$. Gallay and Roussier-Michon [4] showed that for the initial velocity $u_0 \in H^1_{\mathrm{loc}}(\mathbb{D})$ satisfying $\tilde{u}_0 \in (I-\mathcal{Q})H^1(\mathbb{D})$, $\bar{u}_{0,3} \in H^1(\mathbb{R}^2)$, $\partial_1 \bar{u}_{0,2} - \partial_2 \bar{u}_{0,1} \in (L^1 \cap L^2)(\mathbb{R}^2)$, there exists a unique global solution for the sufficiently large $|\Omega|$, and the solution converges to 2D Lamb–Oseen vortex in $L^1(\mathbb{R}^2)$ as $t \to \infty$. The authors in [5] showed that there exists a unique global solution for the sufficiently large $|\Omega|$ if $u_0 = \bar{u}_0 + \tilde{u}_0 \in L^2(\mathbb{R}^2) + (I - \mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})$, and the solution converges to the solution to 2D incompressible Navier–Stokes equations associated with the initial data $\bar{u}_0$ as $|\Omega| \to \infty$.

We first introduce limit equations. The limit equations are represented as the coupled system of the 2D incompressible MHD equations in $\mathbb{R}^2$:

$$\begin{cases} \partial_t \bar{u}^\infty - \nu \Delta_h \bar{u}^\infty & \\ \quad + \mathbb{P}[(\bar{u}_h^\infty \cdot \nabla_h)\bar{u}^\infty - (\bar{B}_h^\infty \cdot \nabla_h)\bar{B}^\infty - \mathcal{Q}(\tilde{B}^\infty \cdot \nabla)\tilde{B}^\infty] = 0 & t > 0,\ x \in \mathbb{R}^2, \\ \partial_t \bar{B}^\infty - \Delta_h \bar{B}^\infty + (\bar{u}_h^\infty \cdot \nabla_h)\,\bar{B}^\infty - (\bar{B}_h^\infty \cdot \nabla_h)\bar{u}^\infty = 0 & t > 0,\ x \in \mathbb{R}^2, \\ \nabla_h \cdot \bar{u}_h^\infty = \nabla_h \cdot \bar{B}_h^\infty = 0 & t \geq 0,\ x \in \mathbb{R}^2, \\ \bar{u}^\infty(0,x) = \bar{u}_0(x), \quad \bar{B}^\infty(0,x) = \bar{B}_0(x) & x \in \mathbb{R}^2 \end{cases} \tag{2}$$

and the induction equation in the 3D layer $\mathbb{D}$:

$$\begin{cases} \partial_t \tilde{B}^\infty - \Delta \tilde{B}^\infty + (\bar{u}^\infty \cdot \nabla)\tilde{B}^\infty - (\tilde{B}_h^\infty \cdot \nabla_h)\bar{u}^\infty = 0 & t > 0,\ (x,z) \in \mathbb{D}, \\ \nabla \cdot \tilde{B}^\infty = 0 & t \geq 0,\ (x,z) \in \mathbb{D}, \\ \tilde{B}^\infty(0,x,z) = \tilde{B}_0(x,z) & (x,z) \in \mathbb{D}. \end{cases} \tag{3}$$

Here $\Delta_h = \partial_1^2 + \partial_2^2$, $\nabla_h = (\partial_1, \partial_2)$, $v_h = (v_1, v_2)$, and $\mathbb{P}$ is the Helmholtz projection onto the divergence free vector fields defined in (5).

In this manuscript, we prove the global well-posedness of (1) for $(u_0, B_0) \in L^2(\mathbb{R}^2) + (I - \mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})$ when the rotating speed $|\Omega|$ is sufficiently large. Furthermore, we shall consider the global solution (u, B) to (1) converges to that of the limit system (2) and (3) as $|\Omega| \to \infty$ in the space-time norm $L^p(0,\infty; L^q(\mathbb{D}))$ for $2 < p, q < \infty$ with $\frac{2}{p} + \frac{2}{q} = 1$.

The main results of [6] read as follows:

Theorem 1. *Let* $(u_0, B_0) = (\bar{u}_0, \bar{B}_0) + (\tilde{u}_0, \tilde{B}_0) \in L^2(\mathbb{R}^2) + (I - \mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})$.

(I) *There exists a constant* $\omega = \omega(\nu, \bar{u}_0, \tilde{u}_0, \bar{B}_0, \tilde{B}_0) > 0$ *such that for every* $\Omega \in \mathbb{R}$ *with*

$|\Omega| \geq \omega$, (1) *has a unique global solution* $(u, B) = (\bar{u}, \bar{B}) + (\tilde{u}, \tilde{B})$ *in the class*

$$\bar{u}, \bar{B} \in C([0,\infty); L^2(\mathbb{R}^2)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^2)),$$
$$\tilde{u}, \tilde{B} \in C([0,\infty); (I-\mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})) \cap L^2(0,\infty; \dot{H}^{\frac{3}{2}}(\mathbb{D})).$$

(II) *Let* $2 < p, q < \infty$ *satisfy* $\frac{2}{p} + \frac{2}{q} = 1$. *Then,*

$$\lim_{|\Omega|\to\infty} \|u - \bar{u}^\infty\|_{L^p(0,\infty;L^q(\mathbb{D}))} = \lim_{|\Omega|\to\infty} \|B - B^\infty\|_{L^p(0,\infty;L^q(\mathbb{D}))} = 0. \tag{4}$$

Here, $(\bar{u}^\infty, B^\infty)$ *is the global solution to* (2), (3) *associated with the initial data* $(\bar{u}_0, B_0)$ *in the class*

$$\bar{u}^\infty, \bar{B}^\infty \in C([0,\infty); L^2(\mathbb{R}^2)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^2)),$$
$$\tilde{B}^\infty \in C([0,\infty); (I-\mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})) \cap L^2(0,\infty; \dot{H}^{\frac{3}{2}}(\mathbb{D})).$$

Remark 2. Let $2 < p, q < \infty$ satisfy $2/p + 2/q = 1$. Then, by the Sobolev embedding $\dot{H}^{1-2/q}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ and the interpolation inequality, it holds

$$L^\infty(0,\infty; L^2(\mathbb{R}^2)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^2)) \hookrightarrow L^p(0,\infty; L^q(\mathbb{R}^2)).$$

Moreover, it follows from the Sobolev embedding $H^{\frac{3}{2}(1-2/q)}(\mathbb{D}) \hookrightarrow L^q(\mathbb{D})$, and the Poincaré inequality (6) that

$$\begin{aligned} &L^\infty(0,\infty; (I-Q)\dot{H}^{\frac{1}{2}}(\mathbb{D})) \cap L^2(0,\infty; \dot{H}^{\frac{3}{2}}(\mathbb{D})) \\ &\quad \hookrightarrow L^\infty(0,\infty; (I-Q)L^2(\mathbb{D})) \cap L^2(0,\infty; \dot{H}^{\frac{3}{2}}(\mathbb{D})) \hookrightarrow L^p(0,\infty; (I-Q)L^q(\mathbb{D})). \end{aligned}$$

Remark 3. In the proof of [8], it is shown that for $2 < q < 6$

$$\lim_{|\Omega|\to\infty} \|(u, B) - (0, e^{t\Delta}B_0)\|_{L^4(0,\infty;\dot{W}^{\frac{1}{2},3}(\mathbb{R}^3))} = 0.$$

On the other hand, our result yields

$$\lim_{|\Omega|\to\infty} \|(u, B) - (\bar{u}^\infty, B^\infty)\|_{L^p(0,\infty;L^q(\mathbb{D}))} = 0.$$

This manuscript is organized as follows. In Section 2, we introduce some notation and prepare the some inequalities. In Section 3, we prove the global regularity to the limit system (2) and (3). In Section 4, we consider the linear solution and state the space-time estimate. In Section 5, we introduce the modified linear solution $\tilde{u}^L$, and state the global space-time estimates. In Section 6, we establish the energy estimates for the perturbations. Finally in Section 7, we present the proof of Theorem 1.

2 Preliminaries

In this section, We follow the notation in [4], state several notation in $\mathbb{D}$. We define the Fourier transforms in $\mathbb{D}$ as

$$f_n(k) := \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_0^1 f(x,z)e^{-i(k\cdot x+2\pi nz)}\,dzdx, \quad k\in\mathbb{R}^2,\ n\in\mathbb{Z}$$

for $f\in L^2(\mathbb{D})^3$. Then, the inverse Fourier transform is represented as

$$f(x,z) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}} f_n(k)e^{i(k\cdot x+2\pi nz)}\,dk, \quad x\in\mathbb{R}^2,\ z\in\mathbb{T}.$$

Also, we introduce the inner-product of the homogeneous Sobolev space $\dot{H}^s(\mathbb{D})$ for $s\in\mathbb{R}$ as

$$\langle f,g\rangle_{\dot{H}^s(\mathbb{D})} := \int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}}(|k|^2+4\pi^2n^2)^s f_n(k)\overline{g_n(k)}\,dk.$$

Then, we define the norm of f in $\dot{H}^s(\mathbb{D})$ as

$$\|f\|^2_{\dot{H}^s(\mathbb{D})} := \langle f,f\rangle_{\dot{H}^s(\mathbb{D})}.$$

Note that $(I-\mathcal{Q})H^s(\mathbb{D})$ is equivalent to $(I-\mathcal{Q})\dot{H}^s(\mathbb{D})$ in the sense of equivalent norms:

$$\min\{1,2^{s/2}\}\|\tilde{f}\|_{\dot{H}^s(\mathbb{D})} \leqslant \|\tilde{f}\|_{H^s(\mathbb{D})} \leqslant \max\{1,2^{s/2}\}\|\tilde{f}\|_{\dot{H}^s(\mathbb{D})}$$

for all $\tilde{f}\in(I-\mathcal{Q})\dot{H}^s(\mathbb{D})$ and $s\in\mathbb{R}$. Indeed for $k\in\mathbb{R}^2$ and $n\in\mathbb{Z}\setminus\{0\}$, there hold $\tilde{f}_0(k)=0$ and

$$|k|^2+4\pi^2n^2 \leqslant 1+|k|^2+4\pi^2n^2 \leqslant 2(|k|^2+4\pi^2n^2).$$

Next, we define the Helmholtz projection $\mathbb{P}$ in the 3D layer $\mathbb{D}$ onto the divergence-free vector fields by

$$(\mathbb{P}f)_n(k) := \left(I-\frac{\xi\otimes\xi}{|k|^2+4\pi^2n^2}\right)f_n(k), \tag{5}$$

where $\xi=(k,2\pi n)\in\mathbb{R}^2\times 2\pi\mathbb{Z}$ and $k=(k_1,k_2)\in\mathbb{R}^2$ with $|k|^2=k_1^2+k_2^2$.

Finally, we state the following Poincaré inequality and the interpolation estimates in $\mathbb{D}$.

Lemma 4.

(i) *Let $-\infty<s_0\leqslant s_1<\infty$. Then, it holds for $\tilde{f}\in(I-\mathcal{Q})\dot{H}^{s_1}(\mathbb{D})$*

$$\|\tilde{f}\|_{\dot{H}^{s_0}(\mathbb{D})} \leqslant (2\pi)^{-(s_1-s_0)}\|\tilde{f}\|_{\dot{H}^{s_1}(\mathbb{D})}. \tag{6}$$

(ii) *Let $-\infty<s_0\leqslant s\leqslant s_1<\infty$ and $0\leqslant\theta\leqslant 1$ satisfy $s=\theta s_0+(1-\theta)s_1$. Then, it holds for $f\in H^{s_0}(\mathbb{D})\cap H^{s_1}(\mathbb{D})$*

$$\|f\|_{H^s(\mathbb{D})} \leqslant \|f\|^{\theta}_{H^{s_0}(\mathbb{D})}\|f\|^{1-\theta}_{H^{s_1}(\mathbb{D})}. \tag{7}$$

3 Limit System

In this section, we prove the result on the global existence and global energy estimate for the limit system (2) and (3). Let $\bar{U}^\infty := (\bar{u}^\infty, \bar{B}^\infty)$, and $\bar{U}_0 := (\bar{u}_0, \bar{B}_0)$.

Theorem 5. *For $\bar{U}_0 \in L^2(\mathbb{R}^2)$ with $\nabla_h \cdot (\bar{u}_0)_h = \nabla_h \cdot (\bar{B}_0)_h = 0$ and $\tilde{B}_0 \in (I-Q)\dot{H}^{\frac{1}{2}}(\mathbb{D})$ with $\nabla \cdot \tilde{B}_0 = 0$, (2), (3) has a unique global solutions $\bar{u}^\infty, \bar{B}^\infty, \tilde{B}^\infty$ with*

$$\bar{u}^\infty, \bar{B}^\infty \in C([0,\infty); L^2(\mathbb{R}^2))^3 \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^2))^3,$$
$$\tilde{B}^\infty \in C([0,\infty); (I-Q)\dot{H}^{\frac{1}{2}}(\mathbb{D}))^3 \cap L^2(0,\infty; \dot{H}^{\frac{3}{2}}(\mathbb{D}))^3.$$

Moreover, there exists a positive constant $C = C(\left\|\bar{U}_0\right\|_{L^2(\mathbb{R}^2)}, \left\|\tilde{B}_0\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})})$ such that

$$\sup_{t\geq 0}\left(\left\|\bar{U}^\infty(t)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde{B}^\infty(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\right) + 2\min\{1,\nu\}\int_0^\infty \|\bar{U}^\infty\|^2_{\dot{H}^1(\mathbb{R}^2)} + \left\|\tilde{B}^\infty\right\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{D})} dt$$
$$\leq K. \tag{8}$$

Proof. Taking the $L^2(\mathbb{R}^2)$-inner product of the first equation of (2) and $\bar{u}^\infty(t)$, the $L^2(\mathbb{R}^2)$-inner product of the second equation of (2) and $\bar{B}^\infty(t)$, and the $L^2(\mathbb{D})$-inner product of the equation of (3) and $\tilde{B}^\infty(t)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\bar{u}^\infty(t)\|^2_{L^2(\mathbb{R}^2)} + \left\|\bar{B}^\infty(t)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde{B}^\infty(t)\right\|^2_{L^2(\mathbb{D})}\right)\\
&\quad + \min\{1,\nu\}\left(\|\nabla_h\bar{u}^\infty(t)\|^2_{L^2(\mathbb{R}^2)} + \left\|\nabla_h\bar{B}^\infty(t)\right\|^2_{L^2(\mathbb{R}^2)} + \|\nabla\tilde{B}^\infty(t)\|^2_{L^2(\mathbb{D})}\right)\\
&\leq -\left\langle(\bar{u}_h^\infty\cdot\nabla_h)\bar{u}^\infty, \bar{u}^\infty\right\rangle_{L^2(\mathbb{R}^2)} + \left\langle(\bar{B}_h^\infty\cdot\nabla_h)\bar{B}^\infty, \bar{u}^\infty\right\rangle_{L^2(\mathbb{R}^2)}\\
&\qquad + \left\langle\mathcal{Q}(\tilde{B}^\infty\cdot\nabla)\tilde{B}^\infty, \bar{u}^\infty\right\rangle_{L^2(\mathbb{R}^2)}\\
&\quad - \left\langle(\bar{u}_h^\infty\cdot\nabla_h)\bar{B}^\infty, \bar{B}^\infty\right\rangle_{L^2(\mathbb{R}^2)} + \left\langle(\bar{B}_h^\infty\cdot\nabla_h)\bar{u}^\infty, \bar{B}^\infty\right\rangle_{L^2(\mathbb{R}^2)}\\
&\quad - \left\langle(\bar{u}^\infty\cdot\nabla)\tilde{B}^\infty, \tilde{B}^\infty\right\rangle_{L^2(\mathbb{D})} + \left\langle(\tilde{B}_h^\infty\cdot\nabla_h)\bar{u}^\infty, \tilde{B}^\infty\right\rangle_{L^2(\mathbb{D})}\\
&= 0,
\end{aligned}$$

which follows from the integration by parts and the divergence-free condition.

Therefore we have (8) and this complete the proof of this lemma. □

4 Linear Solution

In this section, we state the linear solution associated with (1) with $B \equiv 0$.

Let $\{T_\Omega(t)\}_{t\geq 0}$ denote the semigroup generated by the linear operator $-\nu\Delta + \Omega\mathbb{P}e_3 \times \mathbb{P}$ in $\mathbb{D}$, which is given explicitly by

$$
\begin{aligned}
&T_\Omega(t)\tilde{u}(x,z)\\
&\quad= \frac{1}{2\pi}\sum_{\sigma\in\{\pm\}}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-t(\nu|\xi|^2+\sigma i\Omega\frac{2\pi n}{|\xi|})}\left\langle \tilde{u}_n(k),(v^\sigma)_n(k)\right\rangle_{\mathbb{C}^3}(v^\sigma)_n(k)e^{-i(k\cdot x+2\pi nz)}\,dk\\
&\quad= \sum_{\sigma\in\{\pm\}} e^{t(\nu\Delta-\sigma i\Omega\frac{D_3}{|D|})}\tilde{u}^\sigma(x,z)
\end{aligned}
\tag{9}
$$

for $x\in\mathbb{R}^2$, $z\in\mathbb{T}$ and the vector field $\tilde{u}$ with $\nabla\cdot\tilde{u}=0$, where

$$
(\tilde{u}^\sigma)_n(k) := \left\langle \tilde{u}_n(k),(v^\sigma)_n(k)\right\rangle_{\mathbb{C}^3}(v^\sigma)_n(k), \quad (v^0)_n(k) := \frac{\xi}{|\xi|},
$$

and

$$
(v^\pm)_n(k) := \frac{1}{\sqrt{2}|\xi||k|}\begin{pmatrix} 2\pi k_1 n \pm ik_2|\xi| \\ 2\pi k_2 n \mp ik_1|\xi| \\ -|k|^2 \end{pmatrix}.
$$

Here, we remark that $\left\langle \tilde{u}_n(k),(v^0)_n(k)\right\rangle_{\mathbb{C}^3}=0$ from the divergence-free condition of $\tilde{u}$. For the derivation of the explicit form of (9), we refer to [4, 5].

Next, let $R>0$, and we define $\mathcal{P}_R$ as

$$
(\mathcal{P}_R f)_n(k) = \chi\left(\frac{|\xi|}{R}\right) f_n(k),
\tag{10}
$$

where $\chi\in C_0^\infty(\mathbb{R})$ satisfies $0\leq\chi\leq 1$, $\chi\equiv 1$ in $[-1,1]$ and $\chi\equiv 0$ in $\mathbb{R}\setminus[-2,2]$. Then, we recall the following space-time estimates for the semigroup $T_\Omega(t)$.

Lemma 6 ([3], Theorem 5.3, [4], Corollary 2.4)**.** *Let $R>0$, let $\nu>0$, and let $1\leq p\leq\infty$ and $2\leq q\leq\infty$ satisfy $\frac{1}{p}+\frac{2}{q}\leq 1$. Then, there exists a positive constant $C_R=C(\nu,R,p,q)$ such that if $\tilde{u}_0\in(I-\mathcal{Q})L^2(\mathbb{D})$ and $\tilde{f}\in L^1(0,\infty;(I-\mathcal{Q})L^2(\mathbb{D}))$ satisfy the divergence-free condition and if v is the solution to the linear equation:*

$$
\begin{cases}
\partial_t v - \nu\Delta v + \Omega e_3\times v + \nabla p = \mathcal{P}_R\tilde{f} & t>0,\ (x,z)\in\mathbb{D},\\
\nabla\cdot v = 0 & t\geq 0,\ (x,z)\in\mathbb{D},\\
v(0,x,z) = \mathcal{P}_R\tilde{u}_0(x,z) & (x,z)\in\mathbb{D},
\end{cases}
\tag{11}
$$

then it holds for $\Omega\in\mathbb{R}\setminus\{0\}$

$$
\|v\|_{L^p(0,\infty;L^q(\mathbb{D}))} \leq C_R|\Omega|^{-\frac{1}{4p}}\left(\|\tilde{u}_0\|_{L^2(\mathbb{D})} + \left\|\mathcal{P}_R\tilde{f}\right\|_{L^1(0,\infty;L^2(\mathbb{D}))}\right).
$$

Here, $\mathcal{P}_R$ is defined in (10).

5 Modified Linear Equation

In this section, we follow the ideas in [2, 7], and introduce the modified linear dispersive solution $\tilde{u}^L$ for the velocity fields. We shall state that the space-time norm $L^p(0,\infty;L^q(\mathbb{D}))$ of $\tilde{u}^L$ can be taken arbitrarily small when the rotating speed $|\Omega|$ is sufficiently high.

We consider the linear equation generated by (1) with the external force $\mathcal{N}_5$:

$$\begin{cases} \partial_t \tilde{u}^L - \nu\Delta\tilde{u}^L + \Omega\mathbb{P}\left(e_3 \times \tilde{u}^L\right) = \mathcal{P}_R\mathbb{P}\mathcal{N}_5 & t>0,\ (x,z)\in\mathbb{D}, \\ \nabla\cdot\tilde{u}^L = 0 & t\geq 0,\ (x,z)\in\mathbb{D}, \\ \tilde{u}^L(0,x,z) = \mathcal{P}_R\tilde{u}_0(x,z) & (x,z)\in\mathbb{D}, \end{cases} \tag{12}$$

where $\mathcal{P}_R$ denotes (10), and $\mathcal{N}_5$ is defined by

$$\mathcal{N}_5 = (I-\mathcal{Q})\left(\tilde{B}^\infty\cdot\nabla\right)\tilde{B}^\infty + \left(\bar{B}^\infty\cdot\nabla\right)\tilde{B}^\infty + \left(\tilde{B}_h^\infty\cdot\nabla_h\right)\bar{B}^\infty.$$

Also, in order to establish the space-time estimates for $\tilde{u}^L$, we introduce the following integral equations for (12), which follows from the Duhamel principle:

$$\tilde{u}^L(t) = T_\Omega(t)\mathcal{P}_R\tilde{u}_0 + \int_0^t T_\Omega(t-\tau)\mathcal{P}_R\mathbb{P}\mathcal{N}_5(\tau)d\tau. \tag{13}$$

Here, $T_\Omega(t)$ is defined in (9). Then, we shall state the global-in-time existence of the solution $\tilde{u}^L$ to (12), the global a priori $\dot{H}^{\frac{1}{2}}(\mathbb{D})$-estimates for $\tilde{u}^L$, and space-time estimates for $\tilde{u}^L$:

Lemma 7. *Let $\tilde{u}_0 \in (I-\mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})$ satisfy $\nabla\cdot\tilde{u}_0 = 0$, and $(\bar{B}^\infty, \tilde{B}^\infty)$ is a global-in-time solution to the system (2) and (3). Then, the equation (12) with the initial data $\mathcal{P}_R\tilde{u}_0$ has a unique global solution $\tilde{u}^L$ in the class*

$$\tilde{u}^L \in C([0,\infty);(I-\mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D}))^3 \cap L^2(0,\infty;\dot{H}^{\frac{3}{2}}(\mathbb{D}))^3.$$

Moreover, the solution $\tilde{u}^L$ to (12) satisfies the following estimates:

(i) *There exists a positive constant $C = C(\nu)$ such that*

$$\sup_{t\geq 0}\left\|\tilde{u}^L(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})} + \nu\int_0^\infty \left\|\nabla\tilde{u}^L(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})} dt \leq \left\|\tilde{u}_0\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})} + CK^2, \tag{14}$$

where the positive constant K is defined in (8).

(ii) *Let $1\leq p\leq\infty$, $2\leq q\leq\infty$ satisfy $\frac{1}{p}+\frac{2}{q}\leq 1$ and $R>0$. Then, there exists a positive constant $C_4 = C(\nu,p,q,R)$ such that for any $\Omega\in\mathbb{R}\setminus\{0\}$, there holds*

$$\left\|\tilde{u}^L\right\|_{L^p(0,\infty;L^q(\mathbb{D}))} \leq C_4|\Omega|^{-\frac{1}{4p}}\left(\left\|\tilde{u}_0\right\|_{L^2(\mathbb{D})} + K^{\frac{3}{4}}\left(K^{\frac{1}{4}}+1\right)\right). \tag{15}$$

6 Energy Estimate for the Perturbation terms

In this section, we establish the global a priori estimates for the perturbations $(\bar{u}^r, \bar{B}^r)$, and $(\tilde{u}^r, \tilde{B}^r)$, where $\bar{u}^r := \bar{u} - \bar{u}^\infty$, $\bar{B}^r := \bar{B} - \bar{B}^\infty$, $\tilde{u}^r := \tilde{u} - \tilde{u}^L$ and $\tilde{B}^r := \tilde{B} - \tilde{B}^\infty$. First, we have the time evolution equation for $(\bar{u}^r, \bar{B}^r)$:

$$\begin{cases} \partial_t \bar{u}^r - \nu \Delta_h \bar{u}^r + \mathbb{P}\mathcal{N}_6 = 0 & t > 0,\ x \in \mathbb{R}^2, \\ \partial_t \bar{B}^r - \Delta_h \bar{B}^r + \mathcal{N}_7 = 0 & t > 0,\ x \in \mathbb{R}^2, \\ \nabla_h \cdot \bar{u}^r_h = \nabla_h \cdot \bar{B}^r_h = 0 & t \geq 0,\ x \in \mathbb{R}^2, \\ \bar{u}^r(0,x) = \bar{B}^r(0,x) = 0 & x \in \mathbb{R}^2, \end{cases} \tag{16}$$

where $\mathcal{N}_6$ and $\mathcal{N}_7$ are denoted as

$$\begin{aligned} \mathcal{N}_6 := & \sum_{j,k \in \{r,L\}} \mathcal{Q}\left(\tilde{u}^j \cdot \nabla\right) \tilde{u}^k \\ & + \sum_{\substack{j,k \in \{r,\infty\} \\ (j,k) \neq (\infty,\infty)}} \left\{ \left(\bar{u}^j_h \cdot \nabla_h\right) \bar{u}^k - \left(\bar{B}^j_h \cdot \nabla_h\right) \bar{B}^k - \mathcal{Q}\left(\tilde{B}^j \cdot \nabla\right) \tilde{B}^k \right\}, \\ \mathcal{N}_7 := & \sum_{\substack{j,k \in \{r,\infty\} \\ (j,k) \neq (\infty,\infty)}} \left\{ \left(\bar{u}^j_h \cdot \nabla_h\right) \bar{B}^k - \left(\bar{B}^j_h \cdot \nabla_h\right) \bar{u}^k \right\} \\ & + \sum_{\substack{j=r,L \\ k=r,\infty}} \mathcal{Q}\left\{ \left(\tilde{u}^j \cdot \nabla\right) \tilde{B}^k - \left(\tilde{B}^k \cdot \nabla\right) \tilde{u}^j \right\}, \end{aligned}$$

respectively. Secondly, we state the time evolution equation for $(\tilde{u}^r, \tilde{B}^r)$:

$$\begin{cases} \partial_t \tilde{u}^r - \nu \Delta \tilde{u}^r + \Omega \mathbb{P}\left(e_3 \times \tilde{u}^r\right) + \mathbb{P}[\mathcal{N}_8 - (I - \mathcal{P}_R)\mathcal{N}_5] = 0 & t > 0,\ (x,z) \in \mathbb{D}, \\ \partial_t \tilde{B}^r - \Delta \tilde{B}^r + \mathcal{N}_9 = 0 & t > 0,\ (x,z) \in \mathbb{D}, \\ \nabla \cdot \tilde{u}^r = \nabla \cdot \tilde{B}^r = 0 & t \geq 0,\ (x,z) \in \mathbb{D}, \\ \tilde{u}^r(0,x,z) = (I - \mathcal{P}_R)\tilde{u}_0(x,z), \quad \tilde{B}^r(0,x,z) = 0 & (x,z) \in \mathbb{D}, \end{cases} \tag{17}$$

where $\mathcal{N}_5$, $\mathcal{N}_8$ and $\mathcal{N}_9$ are given by

$$\mathcal{N}_5 = (I-\mathcal{Q})\left(\tilde{B}^\infty\cdot\nabla\right)\tilde{B}^\infty + \left(\bar{B}^\infty\cdot\nabla\right)\tilde{B}^\infty + \left(\tilde{B}^\infty_h\cdot\nabla_h\right)\bar{B}^\infty,$$

$$\mathcal{N}_8 := \sum_{j,k\in\{r,L\}}(I-\mathcal{Q})\left(\tilde{u}^j\cdot\nabla\right)\tilde{u}^k + \sum_{\substack{j=r,L\\k=r,\infty}}\left\{\left(\tilde{u}^j_h\cdot\nabla_h\right)\bar{u}^k + \left(\bar{u}^k\cdot\nabla\right)\tilde{u}^j\right\}$$
$$- \sum_{\substack{j,k\in\{r,\infty\}\\(j,k)\neq(\infty,\infty)}}\left\{(I-\mathcal{Q})\left(\tilde{B}^j\cdot\nabla\right)\tilde{B}^k + \left(\tilde{B}^j_h\cdot\nabla_h\right)\bar{B}^k + \left(\bar{B}^j\cdot\nabla\right)\tilde{B}^k\right\},$$

$$\mathcal{N}_9 := \sum_{\substack{j,k\in\{r,\infty\}\\(j,k)\neq(\infty,\infty)}}\left\{\left(\bar{u}^j\cdot\nabla\right)\bar{B}^k - \left(\tilde{B}^k_h\cdot\nabla_h\right)\bar{u}^j\right\}$$
$$+ \sum_{\substack{j=r,L\\k=r,\infty}}\left\{(I-\mathcal{Q})\left(\tilde{u}^j\cdot\nabla\right)\tilde{B}^k - \left(\tilde{B}^k\cdot\nabla\right)\tilde{u}^j\right.$$
$$\left. + \left(\tilde{u}^j_h\cdot\nabla_h\right)\bar{B}^k - \left(\bar{B}^k\cdot\nabla\right)\tilde{u}^j\right\},$$

respectively.

Let $\bar{U}^r := (\bar{u}^r, \bar{B}^r)$ and $\tilde{U}^r := (\tilde{u}^r, \tilde{B}^r)$. Then, we establish the following energy estimates for the perturbations $\bar{U}^r$ and $\tilde{U}^r$:

Lemma 8. *Let $\nu > 0$, and let $\bar{U}^r$, $\tilde{U}^r$, $\tilde{u}^L$, $(\bar{u}^\infty, \bar{B}^\infty)$ and $\tilde{B}^\infty$ be solutions of* (16), (17), (12), (2) *and* (3), *respectively.*

(i) *There exists a positive constant $C = C(\nu)$ such that*

$$\frac{d}{dt}\left(\left\|\bar{U}^r\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde{U}^r\right\|^2_{L^2(\mathbb{D})}\right) + \min\{1,\nu\}\left(\left\|\bar{U}^r\right\|^2_{\dot{H}^1(\mathbb{R}^2)} + \left\|\tilde{U}^r\right\|^2_{\dot{H}^1(\mathbb{D})}\right)$$
$$\leq \left(\left\|\bar{U}^r\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde{U}^r\right\|^2_{L^2(\mathbb{D})}\right)F_1(t) + G_1(t) + C\left\|(I-\mathcal{P}_R)\mathcal{N}_5\right\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{D})}, \quad (18)$$

where

$$F_1(t) = C\left\{\left\|\bar{U}^\infty\right\|^2_{\dot{H}^1(\mathbb{R}^2)} + \left(1 + \left\|\tilde{B}^\infty\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\right)\left\|\tilde{B}^\infty\right\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{D})}\right\},$$

$$G_1(t) = C\left\|\tilde{u}^L\right\|^2_{L^\infty(\mathbb{D})}\left\{\sum_{j=r,\infty}\left\|\bar{U}^j\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde{U}^r\right\|^2_{L^2(\mathbb{D})} + \left\|\tilde{u}^L\right\|^2_{L^2(\mathbb{D})} + + \left\|\tilde{B}^\infty\right\|^2_{L^2(\mathbb{D})}\right\}.$$

(ii) *There exists a positive constant $C = C(\nu)$ such that*

$$\frac{d}{dt}\left\|\tilde{U}^r\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})} + \min\{1,\nu\}\left\|\tilde{U}^r\right\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{D})}$$
$$\leq C\left\|\tilde{U}^r(t)\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\left\|\tilde{U}^r(t)\right\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{D})} + \left\|\tilde{U}^r(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}F_{2.1}(t) + F_{2.2}(t) + G_2(t), \quad (19)$$

where

$$F_{2.1}(t) = C\left[\sum_{j=r,\infty} \left(1 + \left\|\bar U^j(t)\right\|^2_{L^2(\mathbb{R}^2)}\right) \left\|\bar U^j(t)\right\|^2_{\dot H^1(\mathbb{R}^2)} + \left\|\tilde B^\infty\right\|^2_{\dot H^{\frac12}(\mathbb{D})} \left\|\tilde B^\infty\right\|^2_{\dot H^{\frac32}(\mathbb{D})} + \left\|\tilde u^L\right\|^2_{\dot H^{\frac12}(\mathbb{D})} \left\|\tilde u^L\right\|^2_{\dot H^{\frac32}(\mathbb{D})} \right],$$

$$F_{2.2}(t) = C\Big\{ \left\|(I-\mathcal{P}_R)\mathcal{N}_5\right\|^2_{\dot H^{-\frac12}(\mathbb{D})} + \left\|\bar U^r(t)\right\|^2_{L^2(\mathbb{R}^2)} \left\|\tilde B^\infty(t)\right\|^2_{\dot H^{\frac32}(\mathbb{D})} + \left\|\tilde B^\infty(t)\right\|^2_{\dot H^{\frac12}(\mathbb{D})} \left\|\bar U^r(t)\right\|^2_{\dot H^1(\mathbb{R}^2)} \Big\},$$

$$G_2(t) = C\left[\left\|\tilde u^L\right\|^2_{L^\infty(\mathbb{D})} \left\{1 + \sum_{j=r,\infty} \left\|\bar U^j\right\|_{\dot H^1(\mathbb{R}^2)} + \left\|\tilde B^r(t)\right\|^2_{\dot H^{\frac12}(\mathbb{D})}\right\} + \left\|\nabla \tilde u^L\right\|_{L^\infty(\mathbb{D})} \sum_{j=r,\infty} \left\|\bar U^j\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\nabla \tilde u^L\right\|^2_{L^\infty(\mathbb{D})} \left(1 + \left\|\tilde U^r(t)\right\|^2_{\dot H^{\frac12}(\mathbb{D})}\right) \right].$$

7 Proof of Theorem 1

We present the proof of Theorem 1.

Proof of Theorem 1. Let $T^2_{\max}$ be the maximal existence time of the perturbation part of the solution $(\bar u^r, \bar B^r, \tilde u^r, \tilde B^r)$ to (16) and (17) with

$$\bar u^r,\ \bar B^r \in C([0,T^2_{\max}); L^2(\mathbb{R}^2))^3 \cap L^2(0,T^2_{\max}; \dot H^1(\mathbb{R}^2))^3,$$
$$\tilde u^r,\ \tilde B^r \in C([0,T^2_{\max}); (I-\mathcal{Q})\dot H^{\frac12}(\mathbb{D}))^3 \cap L^2(0,T^2_{\max}; \dot H^{\frac32}(\mathbb{D}))^3.$$

Let $0 < \varepsilon < 1$ be determined later, and we define the positive time T^2_* as

$$T^2_* := \left\{ 0 \le T \le T^2_{\max} \,\middle|\, \alpha_2(T) \le \varepsilon^2,\ \beta_2(T) \le \varepsilon \right\},$$

$$\alpha_2(T) := \sup_{0\le t\le T} \left(\left\|\bar U^r(t)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde U^r(t)\right\|^2_{L^2(\mathbb{D})} \right) + \min\{1,\nu\} \int_0^T \left(\left\|\bar U^r\right\|^2_{\dot H^1(\mathbb{R}^2)} + \left\|\tilde U^r\right\|^2_{\dot H^1(\mathbb{D})} \right) dt,$$

$$\beta_2(T) := \sup_{0\le t\le T} \left\|\tilde U^r(t)\right\|^2_{\dot H^{\frac12}(\mathbb{D})} + \frac12 \min\{1,\nu\} \int_0^T \left\|\tilde U^r(t)\right\|^2_{\dot H^{\frac32}(\mathbb{D})} dt. \tag{20}$$

Choosing a positive constant R_1 such that

$$\left\|(I-\mathcal{P}_{R_1})\tilde u_0\right\|_{\dot H^{\frac12}(\mathbb{D})} \le \frac{\varepsilon^2}{2},$$

we see that T_*^2 is positive.

Assume that $0 < T_*^2 < T_{\max}^2$. Integrating (18) on $[0,T]$ and using (6), we have for $0 \le T \le T_*^2$ and $R \ge R_1$

$$\|\bar{U}^r(t)\|_{L^2(\mathbb{R}^2)}^2 + \left\|\tilde{U}^r(t)\right\|_{L^2(\mathbb{D})}^2 + \min\{1,\nu\}\int_0^T \left(\|\bar{U}^r\|_{\dot{H}^1(\mathbb{R}^2)}^2 + \left\|\tilde{U}^r\right\|_{\dot{H}^1(\mathbb{D})}^2\right) dt$$
$$\le \frac{\varepsilon^4}{4}$$
$$+ \int_0^T \left(\left(\|\bar{U}^r(t)\|_{L^2(\mathbb{R}^2)}^2 + \left\|\tilde{U}^r(t)\right\|_{L^2(\mathbb{D})}^2\right) F_1(t) + G_1(t) + C\,\|(I-\mathcal{P}_R)\mathcal{N}_5(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{D})}^2\right) dt.$$

We see from (8) that

$$\int_0^T F_1(t)dt = C\int_0^T \left\{\|\nabla_h \bar{U}^\infty\|_{L^2(\mathbb{R}^2)}^2 + \left(1+\left\|\tilde{B}^\infty\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2\right)\left\|\nabla\tilde{B}^\infty\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2\right\} dt$$
$$\le CK(K+1) =: K_1. \tag{21}$$

It follows from (8), (14) and (15) that there exists a positive constant $C_{1.R}$ such that

$$\int_0^T G_1(t)dt = C\int_0^T \left\{\sum_{j=r,\infty} \|\bar{U}^j\|_{L^2(\mathbb{R}^2)}^2 + \left\|\tilde{U}^r\right\|_{L^2(\mathbb{D})}^2 + \|\tilde{u}^L\|_{L^2(\mathbb{D})}^2 + + \left\|\tilde{B}^\infty\right\|_{L^2(\mathbb{D})}^2\right\} dt$$
$$\le C\int_0^\infty \|\tilde{u}^L\|_{L^\infty(\mathbb{D})}^2\, dt$$
$$\le C_{1.R}|\Omega|^{-\frac{1}{4}}. \tag{22}$$

By the Gronwall lemma, combining (21) and (22) yields

$$\alpha_2(T) \le e^{K_1}\left(\frac{\varepsilon^4}{4} + C_{1.R}|\Omega|^{-\frac{1}{4}} + C\int_0^T \|(I-\mathcal{P}_R)\mathcal{N}_5(t)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{D})}^2\, dt\right). \tag{23}$$

Integrating (19) on $[0,T]$ and choosing an ε such that $C\varepsilon^{\frac{1}{2}} \le \frac{1}{2}\min\{1,\nu\}$ yield

$$\left\|\tilde{U}^r(T)\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2 + \min\{1,\nu\}\int_0^T \left\|\nabla\tilde{U}^r\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2 dt$$
$$\le \|(I-\mathcal{P}_R)\tilde{u}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2$$
$$+ \int_0^T \left\{C\left\|\tilde{U}^r(t)\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\left\|\tilde{U}^r(t)\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2 + \left\|\tilde{U}^r(t)\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2 F_{2.1}(t) + F_{2.2}(t) + G_2(t)\right\} dt$$
$$\le \frac{\varepsilon^4}{4} + \frac{1}{2}\min\{1,\nu\}\int_0^T \left\|\nabla\tilde{U}^r\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2 dt + \int_0^T \left\{\left\|\tilde{U}^r\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^2 F_{2.1}(t) + F_{2.2}(t) + G_2(t)\right\} dt.$$

It follows from (8) and (14) that

$$\int_0^T F_{2.1}(t)dt = C\int_0^T \Bigg[\sum_{j=r,\infty}\left(1+\|\bar{U}^j(t)\|^2_{L^2(\mathbb{R}^2)}\right)\|\nabla_h \bar{U}^j(t)\|^2_{L^2(\mathbb{R}^2)}$$

$$+\left\|\tilde{B}^\infty\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\left\|\nabla\tilde{B}^\infty\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}+\|\tilde{u}^L\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\|\nabla\tilde{u}^L\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\Bigg]dt$$

$$\leq C\left\{(1+K)^2+\left(\|\tilde{u}_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}+K^2\right)^2\right\} =: K_2. \tag{24}$$

We have by (8)

$$\int_0^T F_{2.2}(t)dt = \int_0^T C\Big\{\|(I-\mathcal{P}_R)\mathcal{N}_5\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{D})}+\|\bar{U}^r(t)\|^2_{L^2(\mathbb{R}^2)}\left\|\nabla\tilde{B}^\infty(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}$$

$$+\left\|\tilde{B}^\infty(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\|\nabla_h\bar{U}^r(t)\|^2_{L^2(\mathbb{R}^2)}\Big\}dt,$$

$$\leq C\int_0^T \|(I-\mathcal{P}_R)\mathcal{N}_5\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{D})}\,dt + CK\varepsilon^2. \tag{25}$$

We see from (8) and (15) that there exists a positive constant $C_{2.R}$ such that

$$\int_0^T G_2(t)dt = C\int_0^T \Bigg[\|\tilde{u}^L\|^2_{L^\infty(\mathbb{D})}\Bigg\{1+\sum_{j=r,\infty}+\|\nabla_h\bar{U}^j\|_{L^2(\mathbb{R}^2)}+\left\|\tilde{B}^r(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\Bigg\}$$

$$+\|\nabla\tilde{u}^L\|_{L^\infty(\mathbb{D})}\sum_{j=r,\infty}\|\bar{U}^j\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\tilde{u}^L\|^2_{L^\infty(\mathbb{D})}\left(1+\left\|\tilde{U}^r(t)\right\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}\right)\Bigg]dt$$

$$\leq C\int_0^\infty \left(\|\tilde{u}^L\|^2_{L^\infty(\mathbb{D})}+\|\nabla\tilde{u}^L\|_{L^\infty(\mathbb{D})}+\|\nabla\tilde{u}^L\|^2_{L^\infty(\mathbb{D})}\right)dt$$

$$+C\left(\int_0^T\|\tilde{u}^L\|^2_{L^\infty(\mathbb{D})}\,dt\right)^{\frac{1}{2}}\left\{\sum_{j=r,\infty}\int_0^T\|\nabla_h\bar{U}^j\|^2_{L^2(\mathbb{R}^2)}\,dt\right\}^{\frac{1}{2}}$$

$$\leq C_{2.R}\left(|\Omega|^{-\frac{1}{4}}+|\Omega|^{-\frac{1}{8}}\right). \tag{26}$$

It follows from (24), (25), (26) and the Gronwall lemma that

$$\beta_2(T)$$

$$\leq e^{K_2}\left\{\frac{\varepsilon^4}{2}+C\int_0^T\|(I-\mathcal{P}_R)\mathcal{N}_5\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{D})}\,dt+CK\varepsilon^2+C_{2.R}\left(|\Omega|^{-\frac{1}{4}}+|\Omega|^{-\frac{1}{8}}\right)\right\}. \tag{27}$$

Choosing an $\varepsilon > 0$ such that

$$\varepsilon \leq \min\left\{\frac{1}{2},\ \frac{1}{4C^2}\min\{1,\nu^2\},\ \frac{1}{e^{K_1}},\ \frac{1}{e^{K_2}},\ \frac{1}{12CKe^{K_2}}\right\},$$

we can take $R_0 = \max\{R_1, R_2\}$ satisfying

$$\int_0^\infty \|(I-\mathcal{P}_{R_0})\mathcal{N}_5\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{D})}^2 \, dt \le \min\left\{\frac{\varepsilon^2}{8Ce^{K_1}}, \frac{\varepsilon}{12Ce^{K_2}}\right\}.$$

Thus, we can choose positive ω such that

$$\omega \ge \max\left\{1, \left(\frac{8C_{1.R_0}e^{K_1}}{\varepsilon^2}\right)^4, \left(\frac{12C_{2.R_0}e^{K_2}}{\varepsilon}\right)^8\right\},$$

and we see from (23) and (27) that

$$\alpha_2(T) \le \frac{\varepsilon^2}{2}, \quad \beta_2(T) \le \frac{\varepsilon}{2} \tag{28}$$

for $\Omega \in \mathbb{R}$ with $|\Omega| \ge \omega$. This contradicts to the definition of T_*^2, and we complete the proof of the global well-posedness of (1).

It remains to prove (4). Let $2 < p, q < \infty$ satisfy $\dfrac{2}{p} + \dfrac{2}{q} = 1$, and we can decompose

$$u - \bar{u}^\infty = \bar{u}^r + \tilde{u}^r + \tilde{u}^L, \quad B - B^\infty = \bar{B}^r + \tilde{B}^r.$$

Then, we consider the estimates for the each term. We first establish the estimates for $(\bar{u}^r, \bar{B}^r)$. It follows from the Sobolev embeddings $\dot{H}^{1-\frac{2}{q}}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ and the interpolation inequality that

$$\begin{aligned} \|\bar{u}^r\|_{L^p(0,\infty;L^q(\mathbb{R}^2))} &\le C\|\bar{u}^r\|_{L^p(0,\infty;\dot{H}^{1-\frac{2}{q}}(\mathbb{R}^2))} \le C\left\|\|\bar{u}^r\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}} \|\nabla_h \bar{u}^r\|_{L^2(\mathbb{R}^2)}^{1-\frac{2}{q}}\right\|_{L^p(0,\infty)} \\ &\le C\|\bar{u}^r\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}^{\frac{2}{q}} \|\nabla_h \bar{u}^r\|_{L^2(0,\infty;L^2(\mathbb{R}^2))}^{1-\frac{2}{q}}, \end{aligned} \tag{29}$$

$$\left\|\bar{B}^r\right\|_{L^p(0,\infty;L^q(\mathbb{R}^2))} \le C\left\|\bar{B}^r\right\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}^{\frac{2}{q}} \left\|\nabla_h \bar{B}^r\right\|_{L^2(0,\infty;L^2(\mathbb{R}^2))}^{1-\frac{2}{q}}. \tag{30}$$

We next derive the estimates for $(\tilde{u}^r, \tilde{B}^r)$. We see by (6), (7) and the Sobolev embedding $H^{\frac{3}{2}(1-\frac{2}{q})}(\mathbb{D}) \hookrightarrow L^q(\mathbb{D})$, that

$$\begin{aligned} \|\tilde{u}^r\|_{L^p(0,\infty;L^q(\mathbb{D}))} &\le C\|\tilde{u}^r\|_{L^p(0,\infty;\dot{H}^{\frac{3}{2}(1-\frac{2}{q})}(\mathbb{D}))} \le C\left\|\|\tilde{u}^r\|_{L^2(\mathbb{D})}^{\frac{2}{q}} \|\nabla \tilde{u}^r\|_{\dot{H}^{\frac{1}{2}}(\mathbb{D})}^{1-\frac{2}{q}}\right\|_{L^p(0,\infty)} \\ &\le C\|\tilde{u}^r\|_{L^\infty(0,\infty;\dot{H}^{\frac{1}{2}}(\mathbb{D}))}^{\frac{2}{q}} \|\nabla \tilde{u}^r\|_{L^2(0,\infty;\dot{H}^{\frac{1}{2}}(\mathbb{D}))}^{1-\frac{2}{q}}, \end{aligned} \tag{31}$$

$$\left\|\tilde{B}^r\right\|_{L^p(0,\infty;L^q(\mathbb{D}))} \le C\left\|\tilde{B}^r\right\|_{L^\infty(0,\infty;\dot{H}^{\frac{1}{2}}(\mathbb{D}))}^{\frac{2}{q}} \left\|\nabla \tilde{B}^r\right\|_{L^2(0,\infty;\dot{H}^{\frac{1}{2}}(\mathbb{D}))}^{1-\frac{2}{q}}. \tag{32}$$

Finally, since $\frac{1}{p} + \frac{2}{q} \le \frac{2}{p} + \frac{2}{q} = 1$, we have by (15) in Lemma 7 that

$$\left\|\tilde{u}^L\right\|_{L^p(0,\infty;L^q(\mathbb{D}))} \le C|\Omega|^{-\frac{1}{4p}}. \tag{33}$$

Combining (29)–(33) with (28) yields that there exist $R > 0$ and $\omega > 0$ such that for $|\Omega| \geq \omega$

$$\begin{aligned}\|u - \bar{u}^\infty\|_{L^p(0,\infty;L^q(\mathbb{D}))} &\leq \|\bar{u}^r\|_{L^p(0,\infty;L^q(\mathbb{R}^2))} + \|\tilde{u}^r\|_{L^p(0,\infty;L^q(\mathbb{D}))} + \left\|\tilde{u}^L\right\|_{L^p(0,\infty;L^q(\mathbb{D}))} \\ &\leq C\left(\varepsilon + |\Omega|^{-\frac{1}{4p}}\right), \qquad (34)\end{aligned}$$

$$\|B - B^\infty\|_{L^p(0,\infty;L^q(\mathbb{D}))} \leq \left\|\bar{B}^r\right\|_{L^p(0,\infty;L^q(\mathbb{R}^2))} + \left\|\tilde{B}^r\right\|_{L^p(0,\infty;L^q(\mathbb{D}))} \leq C\varepsilon. \qquad (35)$$

Thus, we obtain (4) from (34) and (35), and this completes the proof of Theorem 1. □

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