On the Stability of Out-flowing Compressible Viscous Gas

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1 Introduction

For dimensions $n \geq 2$, the isentropic compressible flow is governed by the following system of partial differential equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho U) = 0, \\ \partial_t(\rho U) + \operatorname{div}(\rho U \otimes U) + \nabla P = \operatorname{div}\mathbb{S}, \end{cases} \tag{\textbf{CNS}}$$

where $\rho \geq 0$ is mass density, $U = (U^1, \ldots, U^n)$ is velocity, $P \geq 0$ is pressure given by,

$$P(\rho) = K\rho^\gamma \quad \text{where} \quad K > 0 \quad \text{and} \quad \gamma \geq 1.$$

$\mathbb{S} = \mathbb{S}(\nabla U)$ is the viscous stress tensor given by

$$\mathbb{S}(\nabla U) := \mu_1(\nabla U + \nabla U^\top) + \mu_2 \mathbb{I}_n \operatorname{div} U,$$

where $\mathbb{I}_n$ is $n \times n$ identity matrix, $\mu_1 > 0$ is the shear viscosity coefficient and quantity $\frac{2}{n}\mu_1 + \mu_2 \geq 0$ is the bulk viscosity coefficient. Suppose fluid occupies the exterior domain of a unit ball, which is given by $\Omega := \{x \in \mathbb{R}^n \mid |x| > 1\}$. Moreover, we consider the spherically symmetric solution:

$$\rho(t,x) = \rho(t,r), \qquad U(t,x) = u(t,r)\frac{x}{r}, \quad \text{where} \quad r := |x|.$$

Then the problem (**CNS**) can be reformulated in the variables $(t,r) \in (0,\infty) \times [1,\infty)$ as:

$$\begin{cases} \partial_t \rho + \dfrac{\partial_r(r^{n-1}\rho u)}{r^{n-1}} = 0, \\ \rho(\partial_t u + u \partial_r u) + \partial_r P(\rho) = \mu \partial_r \Big(\dfrac{\partial_r(r^{n-1}u)}{r^{n-1}} \Big), \end{cases} \tag{\textbf{SCNS}}$$

where $\mu := 2\mu_1 + \mu_2$. For the outflow problem, we consider the boundary and far-field conditions:

$$u(1,t) = u_b < 0, \qquad \lim_{r \to \infty} (\rho, u)(r,t) = (\rho_+, u_+), \tag{\textbf{B}}$$

where $\rho_+ > 0$ and $u_+ \in \mathbb{R}$ are the constant reference density and velocity.

2 Stationary Solution

The stationary solution to (**SCNS**)–(**B**), denoted as $(\tilde{\rho}, \tilde{u})(r)$, solves the time-independent problem:

$$\begin{cases} \partial_r(r^{n-1}\tilde{\rho}\tilde{u}) = 0, \\ \tilde{\rho}\tilde{u}\partial_r\tilde{u} + \partial_r P(\tilde{\rho}) = \mu\partial_r\Big(\dfrac{\partial_r(r^{n-1}\tilde{u})}{r^{n-1}}\Big), \\ \tilde{u}(1) = u_b, \qquad \lim\limits_{r\to\infty}(\tilde{\rho}, \tilde{u})(r) = (\rho_+, u_+). \end{cases} \tag{\textbf{ST}}$$

Suppose the solution exists, then integrating the first equation in $s \in [1, r]$, we obtain that

$$\tilde{u}(r) = u_b \frac{\tilde{\rho}(1)}{\tilde{\rho}(r)} r^{1-n} \qquad \text{for} \quad r \geq 1.$$

Since $\tilde{\rho}(r) \to \rho_+ < \infty$ and $r^{1-n} \to 0$ as $r \to \infty$, it follows that if $(\tilde{\rho}, \tilde{u})(r)$ is a solution to (**ST**), then it necessarily satisfies the condition:

$$u_+ = \lim_{r\to\infty} \tilde{u}(r) = 0.$$

I. Hashimoto and A. Matsumura in [1] obtained the existence and uniqueness of a $\mathcal{C}^2[1,\infty)$ solution to (**ST**) for both inflow and outflow problems, provided $|u_b| \leq \delta$ for some $\delta > 0$. For small initial perturbation, I. Hashimoto, S. Nishibata, and S. Sugizaki in [2] showed the long time stability of $(\tilde{\rho}, \tilde{u})$ for both inflow and outflow cases. However, it turns out that the small requirement on initial perturbation can be relaxed for outflow case $u_b < 0$, by employing careful a-priori estimates in the Lagrangian coordinate. The main aim of the present article is to give a survey on the long time stability of outflow stationary solution $(\tilde{\rho}, \tilde{u})$ with large initial perturbation.

3 Main Result

Theorem 1 (S. Nishibata and Y.H., To Appear)**.** *Assume $n \geq 2$ and $1 \leq \gamma \leq 2$. Let initial data (ρ_0, u_0) satisfy*

$$r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho}), \quad r^{\frac{n-1}{2}}\partial_r(\rho_0 - \tilde{\rho}), \quad r^{\frac{n-1}{2}}(u_0 - \tilde{u}), \quad r^{\frac{n-1}{2}}\partial_r(u_0 - \tilde{u}) \in L^2(1,\infty),$$
$$\rho_0 \in \mathcal{C}^1[1,\infty) \quad \text{and} \quad u_0 \in \mathcal{C}^2[1,\infty),$$

and the initial compatibility condition:

$$u_0(1) = u_b \qquad \text{and} \qquad \Big(\rho_0 u_0 \partial_r u_0 + \mu\partial_r\big(\frac{\partial_r(r^{n-1}u_0)}{r^{n-1}}\big) - \partial_r P(\rho_0)\Big)\Big|_{r=1} = 0.$$

*Then, there exists $\delta_0 = \delta_0(\rho_0, u_0, \rho_+, \mu, \gamma, K, n) > 0$ so that, if $|u_b| \leq \delta_0$, then initial boundary value problem (**SCNS**)–(**B**) has a unique classical solution (ρ, u) such that for any $T > 0$:*

$$r^{\frac{n-1}{2}}(\rho - \tilde{\rho}), \quad r^{\frac{n-1}{2}}\partial_r(\rho - \tilde{\rho}), \quad r^{\frac{n-1}{2}}(u - \tilde{u}), \quad r^{\frac{n-1}{2}}\partial_r(u - \tilde{u}) \in \mathcal{C}^0\big([0,T]; L^2(1,\infty)\big),$$
$$\rho \in \mathcal{C}^{1+\sigma,1+\sigma/2}, \qquad u \in \mathcal{C}^{2+\sigma,1+\sigma/2} \qquad \text{for some} \quad \sigma \in (0,1).$$

Furthermore, (ρ, u) satisfies the large time convergence:

$$\lim_{t\to\infty} \sup_{r\in[1,\infty)} |(\rho(r,t) - \tilde{\rho}(r), u(r,t) - \tilde{u}(r))| = 0.$$

It is worth noting that the weighted H^1 Sobolev norm of $(\rho - \tilde{\rho}, u - \tilde{u})$ are not required to be small, but merely finite (and possibly large). Before ending this section, we mention few previous results on the compressible Navier-Stokes equations:

Adhesion boundary problem $u_b = 0$. A. Matsumura and T. Nishida in 1983 [7] considered the heat-conducting flow, posed in a general 3D exterior domain Ω with adhesion boundary:

$$u|_{\partial\Omega} = 0, \quad \partial_n \theta|_{\partial\Omega} = 0 \quad \text{where } \partial_n \text{ denotes the normal derivative.}$$

And the flow is also under the influence of an external force f. They showed that any small perturbation in $H^3(\Omega)$ to the stationary solution $(\tilde{\rho}(x), 0, \bar{\theta})$ is stable in large time:

$$\lim_{t\to\infty} \sup_{x\in\Omega} \left|(\rho, u, \theta) - (\tilde{\rho}, 0, \bar{\theta})\right| = 0.$$

T. Nakamura, S. Nishibata, and S. Yanagi in 2004 [11] studied the spherically symmetric isentropic flow with adhesion boundary condition: $u_b = 0$, and potential force f. They showed the large time stability for any spherical large perturbation to the stationary solution $(\tilde{\rho}(r), 0)$, under the H^1 and Hölder norm. Then N. Nakamura and N. Nishibata in 2008 [9] extended this result to the spherically symmetric heat-conducting model.

Inflow and outflow problems. A. Matsumura in 2001 [6] classified the stationary states to the inflow and outflow problems in 1D half-space $x \in [0, \infty)$. Depending on (ρ_b, u_b) and (ρ_+, u_+), stationary solutions can either be viscosity shock, rarefaction wave, boundary layer solution, or certain superposition of the three.

A. Matsumura and K. Nishihara in 2001 [8] studied the isentropic inflow problem posed in 1D half-space. Suppose the stationary solution $(\tilde{\rho}, \tilde{u})$ is a superposition of boundary layer solution and weak rarefaction wave. Then they showed that any small perturbation to $(\tilde{\rho}, \tilde{u})$ is stable in large time $t \to \infty$ under the H^1 norm.

S. Kawashima, S. Nishibata, and P. Zhu in 2003 [4] considered the isentropic outflow problem posed in 1D half-space. They obtained the necessary and sufficient condition for the well-posedness of stationary solution. Moreover, they showed its time asymptotic stability under small perturbations.

T. Nakamura and S. Nishibata in 2011 [10] considered the heat-conducting inflow problem in 1D half-space. They have obtained a necessary and sufficient condition for the well-posedness of stationary solutions, and shown that they are stable under small perturbation in the H^1 and Hölder norm.

4 Strategy

The strategy for proving Theorem 1 is to reformulate equations (**SCNS**) in the Lagrangian coordinates, then obtain a set of a-priori estimates, exclusive to this Lagrangian formulation. For outflow problem, the transformation between Eulerian coordinate (t, r) and Lagrangian coordinate (t, x) is given by:

$$t = t \quad \text{and} \quad r = R(t, x), \qquad \text{where } R(t, x) \text{ is the solution to:}$$

$$x = B(t) + \int_1^{R(t,x)} \rho(t, r) r^{n-1} \,\mathrm{d}r \quad \text{and} \quad B(t) := -u_b \int_0^t \rho(1, s) \,\mathrm{d}s.$$

Note that $B(t)$ physically represents the total amount of mass flown out from the boundary $\{r = 1\}$ during the time period $[0, t]$. By Implicit Function theorem, and equations (**SCNS**), one can verify that

$$\partial_t R(t,x) = u(t, R(t,x)), \qquad \partial_x R(t,x) = \frac{R(t,x)^{1-n}}{\rho\big(t, R(t,x)\big)} \qquad \text{for } t \geq 0 \text{ and } x \in [B(t), \infty).$$

This is similar to the standard differential relations of Lagrangian coordinate for adhesion problem. However, in this case the Lagrangian domain is characterised by the free boundary $x = B(t)$. Set $v = 1/\rho$ to be the specific volume. Then the original Eulerian equations (**SCNS**) is reformulated as

$$\begin{cases} \partial_t v = \partial_x (R^{n-1} u) & \\ \partial_t u + R^{n-1}\partial_x p(v) = \mu R^{n-1} \partial_x\Big(\dfrac{\partial_x (R^{n-1}u)}{v}\Big) & \text{for } t \geq 0 \text{ and } x \in [B(t), \infty), \\ u(t, B(t)) = u_b & \text{for } \ t \geq 0, \end{cases} \tag{\textbf{L}}$$

where $p(v) = Kv^{-\gamma}$, and $R = R(t,x)$ is a function of (v, u) given by

$$R(t,x) = \Big(1 + n\int_{B(t)}^{x} v(t,y)\,\mathrm{d}y\Big)^{\frac{1}{n}}.$$

Subtracting (**L**) with the stationary equations (**ST**), one can derive the relative energy estimate for difference function $(\phi, \psi) := (v - 1/\tilde{\rho}, u - \tilde{u})$ as follows:

Lemma 1 (Relative Energy Estimate)**.** *Suppose* $1 \leq \gamma \leq 2$. *Then there exists a generic constant* $C > 0$ *and* $\delta = \delta(\rho_+, \mu, \gamma, K) > 0$ *such that if* $|u_b| \leq \delta$ *then for arbitrary* $T > 0$

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{B(t)}^{\infty} \mathcal{E}(t,x)\,\mathrm{d}x + |u_b| \int_0^T \frac{|\phi|^2}{v}\Big|_{x=B(t)} \mathrm{d}t \\ &+\mu \int_0^T \int_{B(t)}^{\infty} \Big(\frac{v\psi^2}{r^2} + \frac{r^{2(n-1)}|\partial_x \psi|^2}{v}\Big)\,\mathrm{d}x\mathrm{d}t \\ &+C^{-1} \int_0^T \int_{B(t)}^{\infty} \Big\{|u_b|^3 \frac{|\phi|^2}{r^{3n-2}} + |u_b|\frac{|\psi^2|}{r^n}\Big\}\,\mathrm{d}x\mathrm{d}t \ \leq \int_0^{\infty} \mathcal{E}(0,x)\,\mathrm{d}x. \end{aligned}$$

The main advantage for the Lagrangian formulation is the point-wise representation formula for $v(t,x)$, which was originally derived by A. V. Kazhikhov and V. V. Shelukhin in 1977 [5] for one dimensional adhesion problem in bounded interval with heat-conducting flow. Combining this formula for $v(t,x)$ with the relative energy estimate Lemma 1, one can obtain the upper and lower bound on specific volume $C_0^{-1} \leq v(t,x) \leq C_0$ for all $x \in [B(t), \infty)$, where $C_0 > 0$ is a constant that depends only on the initial data. Once the upper and lower bound on $v(t,x)$ is obtained, it can be used to obtain the H^1-estimates for (ϕ, ψ). For $\phi(t,x)$, one uses the following coercivity structure, which is exclusive to the Lagrangian formulation:

$$\partial_t\Big(\mu\frac{\partial_x\phi}{v} - \frac{\psi}{r^{n-1}}\Big) + \frac{\gamma K}{\mu} v^{-\gamma}\Big(\mu\frac{\partial_x\phi}{v} - \frac{\psi}{r^{n-1}}\Big) = \big(\text{lower order terms in } \phi \text{ and } \psi\big).$$

Note that the above is a modification of the argument used by Ya I Kanel' in 1968 [3], who originally derived the coercivity equation for one dimensional Cauchy problem for isentropic flow. The H^1 estimate for ψ can be attained using the dissipation structure of the momentum equation. Finally, applying the Parabolic Schauder estimates, one completes the desired a-priori estimate.

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