LOCALIZATION OF $\mathscr{C}_{w,v}$ VIA RIGHT BRAIDERS

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Contents

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1. Preliminaries

1.1. **Localization of a monoidal category via right braiders.** In this subsection we recall the localization of a monoidal category via a real commuting family of *right braiders* following [5, Section 2]. Since in [5, Section 2] the localization via the *left braiders* is studied in detail and the case for the right braiders is similar, we recall the right braiders case without proofs.

Let $\mathbf{k}$ be a commutative ring, Λ an abelian group, and $(\mathcal{T}, \otimes)$ a $\mathbf{k}$-linear monoidal category with a unit object $\mathbf{1}$ (see [6, Section 1.5] for the definition of monoidal categories and related notions). Assume that there is a direct sum decomposition of the category $\mathcal{T} = \bigoplus_{\lambda \in \Lambda} \mathcal{T}_\lambda$ such that $\otimes$ induces a bifunctor $\mathcal{T}_\lambda \times \mathcal{T}_\mu \to \mathcal{T}_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$ and $\mathbf{1} \in \mathcal{T}_0$. We call $\mathcal{T}$ a Λ-*graded monoidal category*. Let q be an invertible central object in $\mathcal{T}_0$. We write q^n $(n \in \mathbb{Z})$ for $q^{\otimes n}$ for the sake of simplicity.

Definition 1.1. A *graded right braider in* $\mathcal{T}$ is a triple $(C, \mathrm{R}^{\mathrm{r}}_C, \phi)$ of an object C, a $\mathbb{Z}$-linear map $\phi \colon \Lambda \to \mathbb{Z}$, and a morphism functorial in $X \in \mathcal{T}_\lambda$

M
$$\mathrm{R}^{\mathrm{r}}_C(X) \colon X \otimes C \to q^{-\phi(\lambda)} \otimes C \otimes X$$

M such that the following diagrams commute for any $X \in \mathcal{T}_\lambda$ and $Y \in \mathcal{T}_\mu$: (ϕ is the degree of R^{r}. Please check necessary changes according to this change.)

(1.1)

M
$$\begin{tikzcd}
X \otimes Y \otimes C \arrow[r, "X \otimes \mathrm{R}^{\mathrm{r}}_C(Y)"] \arrow[dr, "\mathrm{R}^{\mathrm{r}}_C(X \otimes Y)"'] & q^{-\phi(\mu)} \otimes X \otimes C \otimes Y \arrow[d, "\mathrm{R}^{\mathrm{r}}_C(X) \otimes Y"] \\
& q^{-\phi(\mu+\lambda)} \otimes C \otimes X \otimes Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \otimes C \arrow[r, "\mathrm{R}^{\mathrm{r}}_C(\mathbf{1})"] \arrow[dr, "\simeq"'] & C \otimes \mathbf{1} \arrow[d, "\wr"] \\
& C.
\end{tikzcd}$$

A graded right braider $(C, \mathrm{R}^{\mathrm{r}}_C, \phi)$ is called a *central object* if $\mathrm{R}^{\mathrm{r}}_C(X)$ is an isomorphism for any $X \in \mathcal{T}$.

Let us denote the category of graded right braiders by $\mathcal{T}^r_{br}$. Note that $\mathcal{T}^r_{br}$ is a monoidal category and there is a canonical faithful monoidal functor $\mathcal{T}^r_{br} \to \mathcal{T}$.

Definition 1.2. Let I be an index set. A family of graded right braiders $\{(C_i, \mathrm{R}^{\mathrm{r}}_{C_i}, \phi_i)\}_{i\in I}$ is called a *real commuting family of graded right braiders in $\mathcal{T}$* if

(a) $C_i \in \mathcal{T}_{\lambda_i}$ for some $\lambda_i \in \Lambda$, and $\phi_i(\lambda_j) + \phi_j(\lambda_i) = 0$ for any $i, j \in I$,
(b) $\mathrm{R}^{\mathrm{r}}_{C_i}(C_i) \in \mathbf{k}^\times \operatorname{\mathsf{id}}_{C_i\otimes C_i}$ for any $i \in I$,
(c) $\mathrm{R}^{\mathrm{r}}_{C_j}(C_i) \circ \mathrm{R}^{\mathrm{r}}_{C_i}(C_j) \in \mathbf{k}^\times \operatorname{\mathsf{id}}_{C_j\otimes C_i}$ for any $i, j \in I$.

Define a $\mathbb{Z}$-linear map

$$\phi\colon \mathbb{Z}^{\oplus I} \times \Lambda \to \mathbb{Z} \quad \text{given by} \quad (e_i, \lambda) \mapsto \phi_i(\lambda),$$

where $\{e_i\}_{i\in I}$ denotes the standard basis of $\mathbb{Z}^{\oplus I}$. We denote by ϕ_α the $\mathbb{Z}$-linear map

$$\phi_\alpha := \phi(\alpha, -)\colon \Lambda \to \mathbb{Z} \qquad \text{for each } \alpha \in \mathbb{Z}^{\oplus I}.$$

Note that one can choose a $\mathbb{Z}$-bilinear map $H\colon \mathbb{Z}^{\oplus I} \times \mathbb{Z}^{\oplus I} \to \mathbb{Z}$ such that

$$\phi_i(\lambda_j) = H(e_j, e_i) - H(e_i, e_j) \qquad \text{for any } i, j \in I.$$

Then we have

$$\phi(\alpha, L(\beta)) = H(\beta, \alpha) - H(\alpha, \beta) \quad \text{for any} \quad \alpha, \beta \in \mathbb{Z}^{\oplus I},$$

where $L\colon \mathbb{Z}^{\oplus I} \to \Lambda$ be the $\mathbb{Z}$-linear map given by $e_i \mapsto \lambda_i$ for $i \in I$.

Lemma 1.3 ([6, Lemma 2.3, Lemma 1.16])**.** *Let $\{(C_i, \mathrm{R}^{\mathrm{r}}_{C_i}, \phi_i)\}_{i\in I}$ be a real commuting family of right graded braiders in $\mathcal{T}$.*

(i) *There exists a family $\{\eta_{ij}\}_{i,j\in I}$ of elements in $\mathbf{k}^\times$ such that*

$$\mathrm{R}^{\mathrm{r}}_{C_i}(C_i) = \eta_{ii} \operatorname{\mathsf{id}}_{C_i\otimes C_i},$$
$$\mathrm{R}^{\mathrm{r}}_{C_j}(C_i) \circ \mathrm{R}^{\mathrm{r}}_{C_i}(C_j) = \eta_{ij}\eta_{ji} \operatorname{\mathsf{id}}_{C_j\otimes C_i}$$

for all $i, j \in I$.

(ii) *There exist a graded right braider $C^\alpha = (C^\alpha, \mathrm{R}^{\mathrm{r}}_{C^\alpha}, \phi_\alpha)$ for each $\alpha \in \mathbb{Z}^{\oplus I}_{\geqslant 0}$, and an isomorphism $\xi_{\alpha,\beta}\colon C^\alpha \otimes C^\beta \xrightarrow{\sim} q^{H(\alpha,\beta)} \otimes C^{\alpha+\beta}$ in $\mathcal{T}^r_{br}$ for $\alpha, \beta \in \mathbb{Z}^{\oplus I}_{\geqslant 0}$ such that*
(a) *$C^0 = \mathbf{1}$ and $C^{e_i} = C_i$ for $i \in I$,*

(b) *the diagram in* $\mathcal{T}^r_{br}$

$$
\begin{tikzcd}
C^\alpha \otimes C^\beta \otimes C^\gamma \arrow[r, "\xi_{\alpha,\beta}\otimes C^\gamma"] \arrow[d, "C^\alpha\otimes\xi_{\beta,\gamma}"'] & q^{H(\alpha,\beta)} \otimes C^{\alpha+\beta} \otimes C^\gamma \arrow[d, "\xi_{\alpha+\beta,\gamma}"] \\
q^{H(\beta,\gamma)} \otimes C^\alpha \otimes C^{\beta+\gamma} \arrow[r, "\xi_{\alpha,\beta+\gamma}"] & q^{H(\alpha,\beta)+H(\alpha,\gamma)+H(\beta,\gamma)} \otimes C^{\alpha+\beta+\gamma}
\end{tikzcd}
\tag{1.2}
$$

commutes for any $\alpha,\beta,\gamma \in \mathbb{Z}^{\oplus I}_{\geqslant 0}$,

(c) *the diagrams in* $\mathcal{T}^r_{br}$

(1.3)

$\boxed{\text{M}}$

$$
\begin{tikzcd}
C^0 \otimes C^0 \arrow[r, "\xi_{0,0}"] \arrow[d, "\wr"] & C^0 \arrow[d, "\wr"] \\
\mathbf{1}\otimes\mathbf{1} \arrow[r, "\simeq"] & \mathbf{1}
\end{tikzcd}
\quad , \quad \textit{and} \quad
\begin{tikzcd}
C^\alpha \otimes C^\beta \arrow[r, "\mathrm{R}^{\mathrm{r}}_{C^\beta}(C^\alpha)"] \arrow[d, "\xi_{\alpha,\beta}"'] & q^{-\phi(\alpha,L(\beta))} \otimes C^\beta \otimes C^\alpha \arrow[d, "\xi_{\beta,\alpha}"] \\
q^{\mathrm{H}(\alpha,\beta)} \otimes C^{\alpha+\beta} \arrow[r, "\eta(\alpha,\beta)\ \mathsf{id}_{C^{\alpha+\beta}}"] & q^{H(\alpha,\beta)} \otimes C^{\alpha+\beta}
\end{tikzcd}
$$

commute for any $i,j \in I$ *and* $\alpha,\beta,\gamma \in \mathbb{Z}^{\oplus I}_{\geqslant 0}$, *where*

$$
\eta(\alpha,\beta) := \prod_{i,j\in I} \eta_{i,j}^{a_i b_j} \in \mathbf{k}^\times \quad \textit{for } \alpha = \sum_{i\in I} a_i e_i \textit{ and } \beta = \sum_{j\in I} b_j e_j \textit{ in } \mathbb{Z}^{\oplus I}.
\tag{1.4}
$$

Define a partial order $\preceq$ on $\mathbb{Z}^{\oplus I}$ by

$$
\alpha \preceq \beta \quad \text{for } \alpha,\beta \in \mathbb{Z}^{\oplus I} \text{ with } \beta-\alpha \in \mathbb{Z}^{\oplus I}_{\geqslant 0},
$$

and set

$$
\mathcal{D}_{\alpha,\beta} := \{\delta \in \mathbb{Z}^{\oplus I} \mid \alpha+\delta, \beta+\delta \in \mathbb{Z}^{\oplus I}\}
$$

for $\alpha,\beta \in \mathbb{Z}^{\oplus I}$.

For $X \in \mathcal{T}_\lambda$, $Y \in \mathcal{T}_\mu$ and $\delta \in \mathcal{D}_{\alpha,\beta}$, we set

$$
\mathrm{H}^{\mathrm{gr}}_\delta((X,\alpha),(Y,\beta)) := \mathrm{Hom}_{\mathcal{T}}(X\otimes C^{\delta+\alpha}, q^{\mathrm{H}(\beta-\alpha,\delta)+\phi(\delta+\beta,\mu)} \otimes C^{\delta+\beta}\otimes Y).
\tag{1.5}
$$

For $\delta, \delta' \in \mathcal{D}_{\alpha,\beta}$ with $\delta \preceq \delta'$ and $f \in \mathrm{H}^{\mathrm{gr}}_{\delta}((X,\alpha),(Y,\beta))$, we define $\zeta^{\mathrm{gr}}_{\delta',\delta}(f) \in \mathrm{H}^{\mathrm{gr}}_{\delta'}((X,\alpha),(Y,\beta))$ to be the morphism such that the following diagram commutes:

$$\begin{tikzcd}
X \otimes C^{\delta+\alpha} \otimes C^{\delta'-\delta} \arrow[r, "f\otimes C^{\delta'-\delta}"] \arrow[dd, "\xi_{\delta+\alpha,\delta'-\delta}"', "\wr"] & q^{\mathrm{H}(\beta-\alpha,\delta)+\phi(\delta+\beta,\mu)} \otimes C^{\delta+\beta} \otimes Y \otimes C^{\delta'-\delta} \arrow[d, "\mathrm{R}^{\mathrm{r}}_{C^{\delta'-\delta}}(Y)"] \\
 & q^{\mathrm{H}(\beta-\alpha,\delta)+\phi(\delta'+\beta,\mu)} \otimes C^{\delta+\beta} \otimes C^{\delta'-\delta} \otimes Y \arrow[d, "\xi_{\delta+\beta,\delta'-\delta}", "\wr"'] \\
q^{\mathrm{H}(\delta+\alpha,\delta'-\delta)} \otimes X \otimes C^{\delta'+\alpha} \arrow[r, "q^{\mathrm{H}(\delta+\alpha,\delta'-\delta)}\otimes\zeta^{\mathrm{gr}}_{\delta',\delta}(f)"] & q^{\mathrm{H}(\beta-\alpha,\delta)+\phi(\delta'+\beta,\mu)+\mathrm{H}(\delta+\beta,\delta'-\delta)} \otimes C^{\delta'+\beta} \otimes Y.
\end{tikzcd}$$

Then, $\zeta^{\mathrm{gr}}_{\delta',\delta}$ is a map from $\mathrm{H}^{\mathrm{gr}}_{\delta}((X,\alpha),(Y,\beta))$ to $\mathrm{H}^{\mathrm{gr}}_{\delta'}((X,\alpha),(Y,\beta))$ and $\zeta^{\mathrm{gr}}_{\delta'',\delta'} \circ \zeta^{\mathrm{gr}}_{\delta',\delta} = \zeta^{\mathrm{gr}}_{\delta'',\delta}$ for $\delta \preceq \delta' \preceq \delta''$, so that $\{\zeta^{\mathrm{gr}}_{\delta',\delta}\}_{\delta,\delta' \in \mathcal{D}_{\alpha,\beta}}$ forms an inductive system indexed by the set $\mathcal{D}_{\alpha,\beta}$.

Define

$$\mathrm{Ob}(\widetilde{\mathcal{T}}) := \mathrm{Ob}(\mathcal{T}) \times \mathbb{Z}^{\oplus I}$$

and for $X \in \mathcal{T}_\lambda$ and $Y \in \mathcal{T}_\mu$ define

$$\mathrm{Hom}_{\widetilde{\mathcal{T}}}((X,\alpha),(Y,\beta)) := \varinjlim_{\substack{\delta \in \mathcal{D}_{\alpha,\beta}, \\ \lambda + \mathrm{L}(\alpha) = \mu + \mathrm{L}(\beta)}} \mathrm{H}^{\mathrm{gr}}_{\delta}((X,\alpha),(Y,\beta)).$$

Let $X \in \mathcal{T}_\lambda$, $Y \in \mathcal{T}_\mu$ and $Z \in \mathcal{T}_\nu$. For $f \in \mathrm{H}^{\mathrm{gr}}_{\delta}((X,\alpha),(Y,\beta))$ and $g \in \mathrm{H}^{\mathrm{gr}}_{\epsilon}((Y,\beta),(Z,\gamma))$, we define

$$\Psi^{\mathrm{gr}}_{\delta,\epsilon}(f,g) := \eta(\delta+\beta, \beta-\gamma) \cdot \widetilde{\Psi}^{\mathrm{gr}}_{\delta,\epsilon}(f,g),$$

where $\widetilde{\Psi}^{\mathrm{gr}}_{\delta,\epsilon}(f,g)$ is the morphism such that the following diagram commutes:

$$\begin{tikzcd}
X \otimes C^{\delta+\alpha} \otimes C^{\epsilon+\beta} \arrow[r, "f"] \arrow[dd, "\xi_{\delta+\alpha,\epsilon+\beta}"', "\wr"] & q^{a} \otimes C^{\delta+\beta} \otimes Y \otimes C^{\epsilon+\beta} \arrow[d, "g"] \\
 & q^{b} \otimes C^{\delta+\beta} \otimes C^{\epsilon+\gamma} \otimes Z \arrow[d, "\xi_{\delta+\beta,\epsilon+\gamma}", "\wr"'] \\
q^{\mathrm{H}(\epsilon+\beta,\delta+\alpha)} \otimes X \otimes C^{\delta+\epsilon+\beta+\alpha} \arrow[r, "\widetilde{\Psi}^{\mathrm{gr}}_{\delta,\epsilon}(f,g)"] & q^{c} \otimes C^{\delta+\epsilon+\beta+\gamma} \otimes Z,
\end{tikzcd}$$

where

$$\begin{aligned}
&a = \mathrm{H}(\beta - \alpha.\delta) + \phi(\delta + \beta, \mu), \quad b = a + \mathrm{H}(\gamma - \beta, \epsilon) + \phi(\epsilon + \gamma, \nu), \\
&c = b + \mathrm{H}(\delta + \beta, \epsilon + \gamma).
\end{aligned}$$

We have

$$c - \mathrm{H}(\delta + \alpha, \epsilon + \beta) = \mathrm{H}(\gamma - \alpha, \beta + \epsilon + \delta) + \phi(\delta + \epsilon + \beta + \gamma, \nu)$$

so that

$$\Psi^{\mathrm{gr}}_{\delta,\epsilon}(f, g) \in \mathrm{H}^{\mathrm{gr}}_{\delta+\epsilon+\beta}((X, \alpha), (Z, \gamma)).$$

It follows that

$$\Psi^{\mathrm{gr}}_{\delta',\epsilon'}(\zeta^{\mathrm{gr}}_{\delta',\delta}(f), \zeta^{\mathrm{gr}}_{\epsilon',\epsilon}(g)) = \zeta^{\mathrm{gr}}_{\delta'+\epsilon'+\beta,\delta+\epsilon+\beta}(\Psi_{\delta,\epsilon}(f, g)),$$

which yields the composition in $\widetilde{\mathcal{T}}$:

$$\mathrm{Hom}_{\widetilde{\mathcal{T}}}((X, \alpha), (Y, \beta)) \times \mathrm{Hom}_{\widetilde{\mathcal{T}}}((Y, \beta), (Z, \gamma)) \to \mathrm{Hom}_{\widetilde{\mathcal{T}}}((X, \alpha), (Z, \gamma)).$$

Because this composition in $\widetilde{\mathcal{T}}$ is associative, $\widetilde{\mathcal{T}}$ becomes a category. By the construction, we have the decomposition

$$\widetilde{\mathcal{T}} = \bigoplus_{\mu \in \Lambda} \widetilde{\mathcal{T}}_\mu, \qquad \text{where } \widetilde{\mathcal{T}}_\mu := \{(X, \alpha) \mid X \in \mathcal{T}_\lambda,\ \lambda + \mathrm{L}(\alpha) = \mu\}.$$

The category $\widetilde{\mathcal{T}}$ is a monoidal category with the following tensor product. For $\alpha, \alpha', \beta, \beta' \in \Gamma$, $X \in \mathcal{T}_\lambda$, $X' \in \mathcal{T}_{\lambda'}$, $Y \in \mathcal{T}_\mu$ and $Y' \in \mathcal{T}_{\mu'}$, we define

MH

$$(X, \alpha) \otimes (Y, \beta) := (q^{-\phi(\alpha,\mu)+\mathrm{H}(\alpha,\beta)} \otimes X \otimes Y, \alpha + \beta),$$

and, for $f \in \mathrm{H}^{\mathrm{gr}}_\delta((X, \alpha), (X', \alpha'))$ and $g \in \mathrm{H}^{\mathrm{gr}}_\epsilon((Y, \beta), (Y', \beta'))$, we define

MH

$$T^{\mathrm{gr}}_{\delta,\epsilon}(f, g) := \eta(\delta, \beta - \beta')\, \widetilde{T}^{\mathrm{gr}}_{\delta,\epsilon}(f, g),$$

where $\widetilde{T}^{\mathrm{gr}}_{\delta,\epsilon}(f,g)$ is the morphism such that the following diagram commutes:

$$\begin{tikzcd}
X \otimes C^{\delta+\alpha} \otimes Y \otimes C^{\epsilon+\beta} \arrow[r, "f\otimes g"] & q^b \otimes C^{\delta+\alpha'} \otimes X' \otimes C^{\epsilon+\beta'} \otimes Y' \arrow[d, "\mathrm{R}^{\mathrm{r}}_{C^{\epsilon+\beta'}}(X')"] \\
q^{-\phi(\delta+\alpha,\mu)} \otimes X \otimes Y \otimes C^{\delta+\alpha} \otimes C^{\epsilon+\beta} \arrow[u, "\mathrm{R}^{\mathrm{r}}_{C^{\delta+\alpha}}(Y)"] \arrow[d, "\xi_{\delta+\alpha,\epsilon+\beta}"', "\wr"] & q^c \otimes C^{\delta+\alpha'} \otimes C^{\epsilon+\beta'} \otimes X' \otimes Y' \arrow[d, "\wr"', "\xi_{\delta+\alpha',\epsilon+\beta'}"] \\
q^a \otimes X \otimes Y \otimes C^{\delta+\epsilon+\alpha+\beta} \arrow[r, "\widetilde{T}^{\mathrm{gr}}_{\delta,\epsilon}(f,g)"] & q^d \otimes C^{\delta+\epsilon+\alpha'+\beta'} \otimes X' \otimes Y',
\end{tikzcd}$$

for some $a, b, c, d \in \mathbb{Z}$ such that $\boxed{\text{MH}}$

$$\begin{aligned}d-a = &-\phi(\alpha',\mu') + \mathrm{H}(\alpha',\beta') - (-\phi(\alpha,\mu) + \mathrm{H}(\alpha,\beta)) \\ &+ \mathrm{H}(\alpha'+\beta'-\alpha-\beta, \delta+\epsilon) + \phi(\delta+\epsilon+\alpha'+\beta', \lambda'+\mu').\end{aligned}$$

Thus we have

$$T^{\mathrm{gr}}_{\delta,\epsilon}(f,g) \in \mathrm{H}^{\mathrm{gr}}_{\delta+\epsilon}((X,\alpha)\otimes(Y,\beta), (X',\alpha')\otimes(Y',\beta')).$$

Then we have

$$T^{\mathrm{gr}}_{\delta',\epsilon'}(\zeta^{\mathrm{gr}}_{\delta',\delta}(f), \zeta^{\mathrm{gr}}_{\epsilon',\epsilon}(g)) = \zeta^{\mathrm{gr}}_{\delta'+\epsilon',\delta+\epsilon}(T^{\mathrm{gr}}_{\delta,\epsilon}(f,g)) \quad \text{for } \delta' \succeq \delta \text{ and } \epsilon' \succeq \epsilon.$$

That is, the map $T^{\mathrm{gr}}_{\delta,\epsilon}$ is compatible with the inductive system and hence it induces a well-defined map

$$f \otimes g \in \mathrm{Hom}_{\widetilde{\mathcal{T}}}((X,\alpha)\otimes(Y,\beta), (X',\alpha')\otimes(Y',\beta')) \tag{1.6}$$

for $f \in \mathrm{Hom}_{\widetilde{\mathcal{T}}}((X,\alpha),(X',\alpha'))$ and $g \in \mathrm{Hom}_{\widetilde{\mathcal{T}}}((Y,\beta),(Y',\beta'))$.

Moreover, we have

$$\Psi_{\delta_1+\delta_2,\epsilon_1+\epsilon_2}(T_{\delta_1,\delta_2}(f_1,f_2), T_{\epsilon_1,\epsilon_2}(g_1,g_2)) = T_{\delta_1+\epsilon_1+\beta_1,\delta_2+\epsilon_2+\beta_2}(\Psi_{\delta_1,\epsilon_1}(f_1,g_1), \Psi_{\delta_2,\epsilon_2}(f_2,g_2))$$

where $f_k \in \mathrm{H}_{\delta_k}((X_k,\alpha_k),(Y_k,\beta_k))$ and $g_k \in \mathrm{H}_{\epsilon_k}((Y_k,\beta_k),(Z_k,\gamma_k))$ for $k=1,2$ (see [6, Proposition 2.5]).

It follows that the map $\otimes$ on $\widetilde{\mathcal{T}}$ defines a bifunctor $\otimes\colon \widetilde{\mathcal{T}} \times \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$.

Theorem 1.4. *Let $\{C_i = (C_i, \mathrm{R}^{\mathrm{r}}_{C_i}, \phi_i)\}_{i\in I}$ be a real commuting family of graded right braiders in $\mathcal{T}$. Then the category $\widetilde{\mathcal{T}}$ defined above becomes a monoidal category. There* *exists a monoidal functor* $\Upsilon\colon \mathcal{T} \to \widetilde{\mathcal{T}}$ *and a real commuting family of graded right braiders* $\{\widetilde{\mathscr{C}}_i = (\widetilde{\mathscr{C}}_i, \mathrm{R}^{\mathrm{r}}_{\widetilde{\mathscr{C}}_i}, \phi_i)\}_{i \in I}$ *in* $\widetilde{\mathcal{T}}$ *satisfy the following properties:*

(i) *for* $i \in I$, $\Upsilon(C_i)$ *is isomorphic to* $\widetilde{C}_i$ *and it is invertible in* $(\widetilde{\mathcal{T}})_{\mathrm{br}}$,

(ii) *for* $i \in I$ *and* $X \in \mathcal{T}_\lambda$, *the diagram*

$$\begin{tikzcd} \Upsilon(X \otimes C_i) \arrow[r, "\sim"] \arrow[d, "\Upsilon(\mathrm{R}^{\mathrm{r}}_{C_i}(X))"', "\wr"] & \Upsilon(X) \otimes \widetilde{C}_i \arrow[d, "\mathrm{R}^{\mathrm{r}}_{\widetilde{C}_i}(\Upsilon(X))"', "\wr"] \\ \Upsilon(q^{\phi_i(\lambda)} \otimes C_i \otimes X) \arrow[r, "\sim"] & q^{\phi_i(\lambda)} \otimes \widetilde{C}_i \otimes \Upsilon(X) \end{tikzcd}$$

commutes.

Moreover, the functor Υ *satisfies the following universal property:*

(iii) *If there are another* Λ*-graded monoidal category* $\mathcal{T}'$ *with an invertible central object* $q \in \mathcal{T}'_0$ *with and a* Λ*-graded monoidal functor* $\Upsilon'\colon \mathcal{T} \to \mathcal{T}'$ *such that*
 (a) Υ' *sends the central object* $q \in \mathcal{T}_0$ *to* $q \in \mathcal{T}'_0$,
 (b) $\Upsilon'(C_i)$ *is invertible in* $\mathcal{T}'$ *for any* $i \in I$ *and*
 (c) *for any* $i \in I$ *and* $X \in \mathcal{T}$, $\Upsilon(\mathrm{R}^{\mathrm{r}}_{C_i}(X))\colon \Upsilon'(X \otimes C_i) \to \Upsilon'(q^{\phi_i(\lambda)} \otimes C_i \otimes X)$ *is an isomorphism,*

 then there exists a monoidal functor $\mathcal{F}$, *which is unique up to a unique isomorphism, such that the diagram*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\Upsilon"] \arrow[dr, "\Upsilon'"'] & \widetilde{\mathcal{T}} \arrow[d, dotted, "\mathcal{F}"] \\ & \mathcal{T}' \end{tikzcd}$$

 commutes.

We denote by $\mathcal{T}[C_i^{\otimes -1} \mid i \in I]$ the localization $\widetilde{\mathcal{T}}$ in Theorem 1.4. If $\mathcal{T}$ is an abelian monoidal category with exact tensor product, then so is $\mathcal{T}[C_i^{\otimes -1} \mid i \in I]$, and the functor $\Upsilon\colon \mathcal{T} \to \mathcal{T}[C_i^{\otimes -1} \mid i \in I]$ is an exact monoidal functor.

MH Note that

MH
$$(X, \alpha+\beta) \simeq (q^{-\mathrm{H}(\alpha,\beta)} X \otimes C^{\alpha}, \beta), \qquad (\mathbf{1}, \beta) \otimes (\mathbf{1}, -\beta) \simeq q^{-\mathrm{H}(\beta,\beta)}(\mathbf{1}, 0)$$

for $\alpha \in \mathbb{Z}^{\oplus I}_{\geqslant 0}$ and $\beta \in \mathbb{Z}^{\oplus I}$.

Remark 1.5. Recall that a *graded left braider in* $\mathcal{T}$ is a triple $(C, \mathrm{R}^{\mathrm{l}}_C, \phi)$ of an object C, a $\mathbb{Z}$-linear map $\phi\colon \Lambda \to \mathbb{Z}$, and a morphism functorial in $X \in \mathcal{T}_\lambda$

$$\mathrm{R}^{\mathrm{l}}_C(X)\colon C \otimes X \to q^{-\phi(\lambda)} X \otimes C$$ M

with analogous conditions to (1.1). The materials in this subsection, containing the above theorem, are proved in [6, Section 2] for the localization via a real commuting family of graded *left braiders*.

1.2. **Quiver Hecke algebras.** A Cartan datum $\big(\mathsf{C}, \mathsf{P}, \Pi, \Pi^\vee, (\cdot,\cdot)\big)$ is a quintuple of a generalized Cartan matrix, C, a free abelian group P , the set of simple roots, $\Pi = \{\alpha_i \mid i \in I\} \subset \mathsf{P}$, set of simple coroots $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee := \mathrm{Hom}(\mathsf{P}, \mathbb{Z})$, and a $\mathbb{Q}$-valued symmetric bilinear form $(\cdot,\cdot)$ on P such that

(1) $\mathsf{C} = (\langle h_i, \alpha_j\rangle)_{i,j\in I}$,
(2) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ for any $i \in I$,
(3) $\langle h_i, \lambda\rangle = \dfrac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $i \in I$ and $\lambda \in \mathsf{P}$,
(4) for each $i \in I$, there exists $\Lambda_i \in \mathsf{P}$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for any $j \in I$.

Let $\mathsf{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $\mathsf{Q}_+ := \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$ be the root lattice and the positive root lattice of the symmetrizable Kac-Moody algebra $\mathfrak{g}(\mathsf{C})$, respectively.

Let W be the Weyl group of $\mathfrak{g}(\mathsf{C})$, the subgroup of $\mathrm{Aut}(\mathsf{P})$ generated by the simple reflections $\{s_i\}_{i\in I}$ where $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i$ for $\lambda \in \mathsf{P}$.

Let $(Q_{i,j}(u,v) \in \mathbf{k}[u,v])_{i,j\in I}$ be a family of polynomials such that

$$Q_{i,j}(u,v) = \begin{cases} \displaystyle\sum_{p(\alpha_i,\alpha_i)+q(\alpha_j,\alpha_j)=-2(\alpha_i,\alpha_j)} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases} \tag{1.7}$$

where $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$ and $Q_{i,j}(u,v) = Q_{j,i}(v,u)$ for all $i,j \in I$. We set

$$\overline{Q}_{i,j}(u,v,w) := \frac{Q_{i,j}(u,v) - Q_{i,j}(w,v)}{u-w} \in \mathbf{k}[u,v,w].$$

For $\beta \in \mathsf{Q}_+$, the set $I^\beta := \Big\{\nu = (\nu_1,\ldots,\nu_n) \in I^n \;\Big|\; \sum_{k=1}^n \alpha_{\nu_k} = \beta\Big\}$ is stable under the the symmetric group $\mathfrak{S}_n = \langle s_k \mid k = 1,\ldots,n-1\rangle$ action given by place permutations.

The height of $\beta = \sum_{i\in I} b_i\alpha_i \in \mathsf{Q}$ is given by $\mathrm{ht}(\beta) := \sum_{i\in I} |b_i|$.

Definition 1.6. *Let $\beta \in \mathsf{Q}_+$ with $\mathrm{ht}(\beta) = n$. The* quiver Hecke algebra $R(\beta)$ *associated with the Cartan datum $\big(\mathsf{C}, \Pi, \mathsf{P}, \Pi^\vee, (\cdot,\cdot)\big)$ and the family of polynomials $(Q_{i,j}(u,v))_{i,j\in I}$ is the* $\mathbf{k}$*-algebra generated by*

$$\{e(\nu) \mid \nu \in I^\beta\},\ \{x_k \mid 1 \leqslant k \leqslant n\},\ \text{and}\ \{\tau_l \mid 1 \leqslant l \leqslant n-1\}$$

subject to the defining relations:

$$\begin{aligned}
&e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu),\ \sum_{\nu \in I^\beta} e(\nu) = 1,\ x_k e(\nu) = e(\nu)x_k,\ x_k x_l = x_l x_k,\\
&\tau_l e(\nu) = e(s_l(\nu))\tau_l,\ \tau_k\tau_l = \tau_l\tau_k \ \text{if}\ |k-l| > 1,\\
&\tau_k^2 = \sum_{\nu \in I^\beta} Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),\\
&\tau_k x_l - x_{s_k(l)}\tau_k = \big(\delta(l = k+1) - \delta(l = k)\big) \sum_{\nu \in I^\beta,\ \nu_k = \nu_{k+1}} e(\nu),\\
&\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k = \sum_{\nu \in I^\beta,\ \nu_k = \nu_{k+2}} \overline{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu).
\end{aligned}$$

M The algebra $R(\beta)$ is equipped with the $\mathbb{Z}$-grading given by

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}). \tag{1.8}$$

We denote by $\mathrm{Modg}(R(\beta))$ the category of graded modules over $R(\beta)$. The full subcategory of $\mathrm{Modg}(R(\beta))$ consisting of the objects which are finite-dimensional over $\mathbf{k}$ is denoted by $R(\beta)$-gmod. For $M \in R$-gmod, the space $M^\star := \mathrm{H{\scriptstyle OM}}_{\mathbf{k}}(M, \mathbf{k})$ is an $R(\beta)$-module via the graded $\mathbf{k}$-algebra antiautomorphism of $R(\beta)$ which fixes the generators $e(\nu)$, x_k, and τ_k's. We say that M is *self-dual* if $M \simeq M^\star$ in R-gmod. For each simple module M in R-gmod, there exists $m \in \mathbb{Z}$ such that $q^m M$ is self-dual.

For $\alpha, \beta \in \mathsf{Q}_+$, we set

$$e(\alpha,\beta) = \sum_{\substack{\nu \in I^{\alpha+\beta} \\ \sum_{k=1}^{\mathrm{ht}(\alpha)} \alpha_{\nu_k} = \alpha,\ \sum_{k=1}^{\mathrm{ht}(\beta)} \alpha_{\nu_{k+\mathrm{ht}(\alpha)}} = \beta}} e(\nu) \ \in R(\alpha+\beta).$$

Then there is an injective $\mathbf{k}$-algebra homomorphism $R(\alpha)\otimes R(\gamma)\to e(\alpha,\beta)R(\alpha+\beta)e(\alpha,\beta)$ and hence we can define the *convolution product* by

$$M\circ N := R(\alpha+\beta)e(\alpha,\beta)\otimes_{R(\alpha)\otimes R(\beta)}(M\otimes N)$$

for $M\in \mathrm{Modg}(R(\alpha))$ and $N\in \mathrm{Mod}(R(\beta))$.

Then the categories

$$\mathrm{Modg}(R) := \bigoplus_{\beta\in \mathsf{Q}_+}\mathrm{Modg}\big(R(\beta)\big) \quad \text{and } R\text{-gmod} = \bigoplus_{\beta\in\mathsf{Q}_+} R(\beta)\text{-gmod}$$

are Q_+-graded monoidal categories.

For $M,N\in R$-gmod, we denote by $M\nabla N$ the head of $M\circ N$ and by $M\Delta N$ the socle of $M\circ N$. $\boxed{\text{M}}$

A simple module M is called *real* if $M\circ M$ is simple.

1.3. **R-matrices.** Let $\beta\in\mathsf{Q}_+$ with $m=\mathrm{ht}(\beta)$. For $k=1,\dots,m-1$ and $\nu\in I^\beta$, the *intertwiner* $\varphi_k\in R(\beta)$ is defined by

$$\varphi_k e(\nu) = \begin{cases} \big(\tau_k(x_k-x_{k+1})+1\big)e(\nu) & \text{if } \nu_k=\nu_{k+1},\\ \tau_k e(\nu) & \text{otherwise.}\end{cases} \tag{1.9}$$

For $m,n\in\mathbb{Z}_{\geqslant 0}$, we set $w[m,n]$ to be the element of $\mathfrak{S}_{m+n}$ such that

$$w[m,n](k) := \begin{cases} k+n & \text{if } 1\leqslant k\leqslant m,\\ k-m & \text{if } m<k\leqslant m+n.\end{cases}$$

Let $M\in\mathrm{Modg}(R(\beta)$ and $N\in\mathrm{Modg}(R(\gamma))$ and define the $R(\beta)\otimes R(\gamma)$-linear map $M\otimes N\to N\circ M$ by

$$u\otimes v\mapsto \varphi_{w[\mathrm{ht}(\gamma),\mathrm{ht}(\beta)]}(v\boxtimes u).$$

Then it extends to an $R(\beta+\gamma)$-module homomorphism (neglecting a grading shift)

$$\mathrm{R}^{\mathrm{univ}}_{M,N}\colon M\circ N\longrightarrow N\circ M.$$

For $\beta\in\mathsf{Q}_+$ and $i\in I$, let $\mathfrak{p}_{i,\beta}$ be an element in the center $Z(R(\beta))$ of $R(\beta)$

$$\mathfrak{p}_{i,\beta} := \sum_{\nu\in I^\beta}\Big(\prod_{a\in\{1,\dots,\mathrm{ht}(\beta)\},\ \nu_a=i} x_a\Big)e(\nu)\in Z(R(\beta)). \tag{1.10}$$

Assume that M is a simple module in $R(\beta)$-gmod, and there exists an $R(\beta)$-module M with an endomorphism z_{M} of M with degree $d_{\mathsf{M}} \in \mathbb{Z}_{>0}$ such that

$$\text{(1.11)}\quad \begin{aligned} &\text{(i)}\ \ \mathsf{M}/z_{\mathsf{M}}\mathsf{M} \simeq M,\\ &\text{(ii)}\ \ \mathsf{M} \text{ is a finitely generated free module over the polynomial ring } \mathbf{k}[z_{\mathsf{M}}],\\ &\text{(iii)}\ \ \mathfrak{p}_{i,\beta}\mathsf{M} \neq 0 \text{ for all } i \in I. \end{aligned}$$

We call $(\mathsf{M}, z_{\mathsf{M}})$ an *affinization of* M.

Let M be an affinization of a simple R-module M, and let N be a non-zero R-module. We define a homomorphism (up to a grading shift)

$$\mathrm{R}^{\mathrm{ren}}_{\mathsf{M},N} := {z_{\mathsf{M}}}^{-s}\mathrm{R}^{\mathrm{univ}}_{\mathsf{M},N} \colon \mathsf{M} \circ N \longrightarrow N \circ \mathsf{M}$$

where s is the largest integer such that $\mathrm{R}_{\mathsf{M},N}(\mathsf{M} \circ N) \subset {z_{\mathsf{M}}}^{s}(N \circ \mathsf{M})$. Then the homomorphism (up to a grading shift)

$$\mathbf{r}_{M,N} \colon M \circ N \longrightarrow N \circ M$$

induced from $\mathrm{R}^{\mathrm{norm}}_{\mathsf{M},N}$ by specializing at $z_{\mathsf{M}} = 0$ never vanishes. We call $\mathbf{r}_{M,N}$ the *r-matrix* between M and N. Let

$$\Lambda(M,N) := \deg(\mathbf{r}_{M,N}),$$

and define

$$\widetilde{\Lambda}(M,N) := \frac{1}{2}\big(\Lambda(M,N) + (\mathrm{wt}(M), \mathrm{wt}(N))\big), \qquad \mathfrak{d}(M,N) := \frac{1}{2}\big(\Lambda(M,N) + \Lambda(N,M)\big).$$

Note that $\mathfrak{d}(M,N)$ and $\widetilde{\Lambda}(M,N) \in \mathbb{Z}_{\geqslant 0}$ are non-negative integers ([6, Lemma 3.11]).

M A real simple module which admits an affinization is called *affreal*. The following result is used frequently throughout the paper.

Proposition 1.7 ([2, Theorem 3.2], [4, Proposition 3.2.9])**.** *Let M and N be simple modules in R-gmod. Assume that one of them is affreal. Then, the convolution $M \circ N$ has a simple head and a simple socle. Moreover, we have*

$$\dim \mathrm{H{\scriptsize OM}}_{R\text{-gmod}}(M \circ N, N \circ M) = 1$$

and

$$M \nabla N \simeq \mathrm{Im}(\mathbf{r}_{M,N}) \simeq N \Delta M \quad \text{up to grading shifts.}$$

1.4. **Partial order on the weight lattice.** We define the partial order $\preccurlyeq$ on P as follows: $\lambda \preccurlyeq \mu$ for $\lambda, \mu \in \mathsf{P}$ if there exists a sequence of positive real roots $\beta_1, \ldots, \beta_r$ such that $(\beta_k, s_{\beta_{k+1}} s_{\beta_{k+2}} \cdots s_{\beta_r} \mu) > 0$ for all $1 \leqslant k \leqslant r$ and $\lambda = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r} \mu$. We have $\mu - \lambda \in \mathsf{Q}_+$ if $\lambda \preccurlyeq \mu$. Hence $\preccurlyeq$ is a partial order on P.

Lemma 1.8. *Let $\lambda \preccurlyeq \mu$ and α be a simple root satisfying $(\alpha, \lambda) \leqslant 0$. Then we have*

(i) *if $(\alpha, \mu) \geqslant 0$, then we have $s_\alpha \lambda \preccurlyeq \mu$,*
(ii) *if $(\alpha, \mu) \leqslant 0$, then we have $s_\alpha \lambda \preccurlyeq s_\alpha \mu$.*

Proof. Let $\beta_1, \ldots, \beta_r$ be a sequence of positive real roots such that $(\beta_k, s_{\beta_{k+1}} s_{\beta_{k+2}} \cdots s_{\beta_r} \mu) > 0$ for all $1 \leqslant k \leqslant r$ and $\lambda = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r} \mu$.

If $r = 0$, then $\lambda = \mu$ and hence the assertion is trivial.

Let $r > 0$.

(a) Assume that $\alpha = \beta_1$. Then we have $s_\alpha \lambda \preccurlyeq \mu$. Hence we may assume that $(\alpha, \mu) \leqslant 0$. Then we have $s_\alpha \lambda \preccurlyeq \mu \preccurlyeq s_\alpha \mu$.

(b) Assume that $\alpha \neq \beta_1$. Set $\lambda' := s_{\beta_1} \lambda = s_{\beta_2} \cdots s_{\beta_r} \mu \preccurlyeq \mu$. Then $s_\alpha \lambda = s_{(s_\alpha \beta_1)}(s_\alpha \lambda')$. Since $0 < (\beta_1, \lambda') = (s_\alpha \beta_1, s_\alpha \lambda')$ we have $s_\alpha \lambda \preccurlyeq s_\alpha \lambda'$. If $(\alpha, \mu) \geqslant 0$, then we have $s_\alpha \preccurlyeq s_\alpha \lambda' \preccurlyeq \mu$, and if $(\alpha, \mu) \leqslant 0$, then we have $s_\alpha \lambda \preccurlyeq s_\alpha \lambda' \preccurlyeq s_\alpha \mu$. □

Lemma 1.9. *Let $w, v \in \mathsf{W}$. Then the following conditions are equivalent.*

(a) *$w \geqslant v$,*
(b) *$w\Lambda \preccurlyeq v\Lambda$ for any $\Lambda \in \mathsf{P}_+$,*
(c) *$w\Lambda_i \preccurlyeq v\Lambda_i$ for all $i \in I$.*

Proof. (a)$\Rightarrow$(b) If $w \geqslant v$, then there exists a sequence of positive real roots $\beta_1, \ldots, \beta_r$ such that $\big(s_{\beta_{k+1}} s_{\beta_{k+2}} \cdots s_{\beta_r} v\big)^{-1} \beta_k \in \Delta_+$ for all $1 \leqslant k \leqslant r$ and $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_r} v$. Then

$$\langle \beta_k,\ s_{\beta_{k+1}} s_{\beta_{k+2}} \cdots s_{\beta_r} v \Lambda \rangle = \langle \big(s_{\beta_{k+1}} s_{\beta_{k+2}} \cdots s_{\beta_r} v\big)^{-1} \beta_k,\ \Lambda \rangle \geqslant 0.$$

(b)$\Rightarrow$(c) is trivial. Let us show (c)$\Rightarrow$(a) by induction on $\ell(w)$. If $\ell(w) = 0$, then $w = \mathsf{id}$ so that $\Lambda_i \preccurlyeq v\Lambda_i$ for all $i \in I$. Since $v\Lambda_i \preccurlyeq \Lambda_i$, we have $v\Lambda_i = \Lambda_i$ for all $i \in I$ so that $v = \mathsf{id}$.

Assume that $\ell(w) > 0$. Take $a \in I$ such that $s_a w < w$. Then $w^{-1}\alpha_a \in \Delta_-$ and hence $(\alpha_a, w\Lambda_i) \leqslant 0$ for all $i \in I$. By the assumption and Lemma 1.8, we have either $s_a w\Lambda_i \preccurlyeq v\Lambda_i$, $(\alpha_a, v\Lambda_i) \geqslant 0$ or $s_a w\Lambda_i \preccurlyeq s_a v\Lambda_i$, $(\alpha_a, v\Lambda_i) < 0$.

If $s_a v > v$, then $v^{-1}\alpha_a \in \Delta_+$ so that $(\alpha_a, v\Lambda_i) \geqslant 0$ for all $i \in I$. Hence $s_a w\Lambda_i \preccurlyeq v\Lambda_i$ for all $i \in I$. By induction on $\ell(w)$, $s_a w \geqslant v$ so that $w > s_a w \geqslant v$.

If $s_a v < v$, then $v^{-1}\alpha_a \in \Delta_-$ so that $(\alpha_a, v\Lambda_i) \leqslant 0$ for all $i \in I$. Hence $s_a w\Lambda_i \preccurlyeq s_a v\Lambda_i$ for all $i \in I$. By induction on $\ell(w)$, $s_a w \geqslant s_a v$ and hence $w \geqslant v$, as desired. $\square$

Corollary 1.10. *Let $i \in I$. If $w \geqslant v$, $ws_i > w$, $vs_i > v$, and $ws_i\Lambda_i \preccurlyeq vs_i\Lambda_i$, then $ws_i \geqslant vs_i$.*

Proof. If $j \neq i$, then $ws_i\Lambda_j = w\Lambda_j$ and $vs_i\Lambda_j = v\Lambda_j$ so that $ws_i\Lambda_j \preccurlyeq vs_i\Lambda_j$. Hence the assertion follows from the lemma above. $\square$

1.5. **Categories $\mathscr{C}_w$ and $\mathscr{C}_{w,v}$.** In this subsection, we recall the categories $\mathscr{C}_w$, $\mathscr{C}_{*,v}$ and $\mathscr{C}_{w,v}$ defined in [5].

For $M \in \mathrm{Modg}\big(R(\beta)\big)$ we define

$$\mathsf{W}(M) := \{\gamma \in \mathsf{Q}_+ \cap (\beta - \mathsf{Q}_+) \mid e(\gamma, \beta-\gamma)M \neq 0\},$$
$$\mathsf{W}^*(M) := \{\gamma \in \mathsf{Q}_+ \cap (\beta - \mathsf{Q}_+) \mid e(\beta-\gamma, \gamma)M \neq 0\}.$$

For $w, v \in \mathsf{W}$, we define the full monoidal subcategories of R-gmod by

$$\begin{aligned} &\mathscr{C}_w := \{M \in R\text{-gmod} \bigm| \mathsf{W}(M) \subset \mathsf{Q}_+ \cap w\mathsf{Q}_-\}, \\ &\mathscr{C}_{*,v} := \{M \in R\text{-gmod} \bigm| \mathsf{W}^*(M) \subset \mathsf{Q}_+ \cap v\mathsf{Q}_+\}, \\ &\mathscr{C}_{w,v} := \mathscr{C}_w \cap \mathscr{C}_{*,v}. \end{aligned} \tag{1.12}$$

An ordered pair (M, N) of R-modules is called *unmixed* if

$$\mathsf{W}^*(M) \cap \mathsf{W}(N) \subset \{0\}.$$

Assume that $\lambda, \mu \in \mathsf{W}\Lambda$ for some $\Lambda \in \mathsf{P}_+$ and $\lambda \preccurlyeq \mu$. Then there exists an object $\mathsf{M}(\lambda, \mu)$ in $R(\mu - \lambda)$-gmod, called the *determinantial module*. (See [6, Section 3.3] for the precise definition and more properties of them.) Note that $\mathsf{M}(\lambda, \mu)$ is an affreal ([6, Theorem 3.26]). For $\Lambda \in \mathsf{P}_+$ and $w, v \in \mathsf{W}$ with $v \leqslant w$ we have

$$\mathsf{M}(w\Lambda, \Lambda) \in \mathscr{C}_w, \quad \mathsf{M}(w\Lambda, v\Lambda) \in \mathscr{C}_{w,v}.$$

1.6. **Localizations of $\mathscr{C}_w$ and $\mathscr{C}_{w,v}$ via left braiders.** In this subsection we recall the localizations of the categories $\mathscr{C}_w$, $\mathscr{C}_{w,v}$ via left braiders studied in [6, 7].

Let $L(i)$ denote the one-dimensional graded self-dual simple module of $R(\alpha_i)$. For any simple module $M \in R$-gmod, there exists a graded left braider $(M, \mathrm{R}^{\mathrm{l}}_M, \phi_M)$ in R-gmod which is *non-degenerate*, that is, $\mathrm{R}^{\mathrm{l}}_M(L(i)) \neq 0$ for all $i \in I$ ([6, Proposition 4.1]). Such a non-degenerate braider is unique up to a constant multiple ([6, Lemma 4.3]).

Let $w \in \mathsf{W}$ with $I_w = I$, where $\{i \in I \mid w\Lambda_i \neq \Lambda_i\}$. The family of graded left braiders in R-gmod

$$\big(\{\mathsf{M}(w\Lambda_i, \Lambda_i), \mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda_i,\Lambda_i)}, \phi_{\mathsf{M}(w\Lambda_i,\Lambda_i)}\big)\}_{i\in I}$$

is a real commuting family ([6, Proposition 5.1]). Moreover it is a family of central objects in the category $\mathscr{C}_w$. Note that

$$\phi_{\mathsf{M}(w\Lambda_i,\Lambda_i)} = -(w\Lambda_i + \Lambda_i, \beta) \quad \text{for any} \quad \beta \in \mathsf{Q}. \tag{1.13}$$

Hence there exist localizations of R-gmod and $\mathscr{C}_w$ via the above real commuting family of graded left braiders and we denote them by $\big(R\text{-gmod}\big)[\mathsf{M}(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]$ and $\widetilde{\mathscr{C}_w} = \mathscr{C}_w[\mathsf{M}(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]$, respectively. We have a commutative diagram of functors

$$\begin{tikzcd}
\mathscr{C}_w \arrow[r, tail, "\iota_w"'] \arrow[d, "\Phi_w"'] & R\text{-gmod} \arrow[d, "Q_w"] \\
\widetilde{\mathscr{C}_w} \arrow[r, tail, dashed, "\widetilde{\iota}_w"'] & \big(R\text{-gmod}\big)[\mathsf{M}(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]
\end{tikzcd}$$

where Φ_w and Q_w denote the localization functors, and $\widetilde{\iota}_w$ is the induced functor from the inclusion functor ι_w. M

Theorem 1.11 ([6, Theorem 5.9, Theorem 5.11], [7, Theorem 3.9])**.**

(a) *The functor* $\widetilde{\iota}_w \colon \widetilde{\mathscr{C}_w} \to \big(R\text{-gmod}\big)[\mathsf{M}(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]$ *is an equivalence of categories.*

(b) *The monoidal category* $\widetilde{\mathscr{C}_w}$ *is rigid, that is, every object of* $\widetilde{\mathscr{C}_w}$ *has a left dual and a right dual.*

Let $v \in \mathsf{W}$ such that $v \leqslant w$. In [7, Section 4], it is shown that there exists a real commuting family of graded left braiders

$$\{\big(\mathsf{M}(w\Lambda_i, w\Lambda_i), \mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda_i, v\Lambda_i)}, \phi^{\ell}_{w,v,\Lambda_i}\big)\}_{i\in I}.$$

in the category $\mathscr{C}_{*,v}$. Moreover it is a family of central objects in the category $\mathscr{C}_{w,v}$. If $v = \mathsf{id}$, then $\mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda_i, v\Lambda_i)}$ is the same with the non-degenerate braider $\mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda_i, \Lambda_i)}$ so that this abuse of notation is justified. Note that

$$\phi^{\ell}_{w,v,\Lambda_i} = -(w\Lambda_i + v\Lambda_i, \beta) \quad \text{for any} \quad \beta \in \mathsf{Q}. \tag{1.14}$$

Let us denote by $\mathscr{C}_{*,v}[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I]$ and $\widetilde{\mathscr{C}}_{w,v} = \mathscr{C}_{w,v}[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I]$, the localizations of $\mathscr{C}_{*,v}$ and $\mathscr{C}_{w,v}$, respectively. Then we have the commutative diagram of functors

$$\begin{tikzcd}
\mathscr{C}_{w,v} \arrow[r, tail, "\iota_{w,v}"'] \arrow[d, "\Phi_{w,v}"'] & \mathscr{C}_{*,v} \arrow[d, "\mathrm{Q}^{\mathrm{l}}_{w,v}"'] \\
\widetilde{\mathscr{C}}_{w,v} \arrow[r, tail, dashed, "\widetilde{\iota}_{w,v}"'] & \big(\mathscr{C}_{*,v}\big)[\mathsf{M}(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]
\end{tikzcd}$$

where $\Phi_{w,v}$ and $\mathrm{Q}^{\mathrm{l}}_{w,v}$ denote the localization functors, and $\widetilde{\iota}_w$ is the induced functor from $\iota_{w,v}$ which is the inclusion functor.

Theorem 1.12 ([7, Theorem 4.5])**.** *The functor $\widetilde{\iota}_{w,v} \colon \widetilde{\mathscr{C}}_{w,v} \to \big(\mathscr{C}_{*,v}\big)[\mathsf{M}(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]$ is an equivalence of categories.*

2. $\widetilde{\mathscr{C}}_{w,v}$ AS THE RIGHT LOCALIZATION OF $\mathscr{C}_w$

For $\eta, \beta \in \mathsf{Q}_+$, denote the functor $\mathrm{Res}_{\eta,\beta} \colon \mathrm{Modg}\big(R(\eta+\beta)\big) \to \mathrm{Modg}\big(R(\eta) \otimes R(\beta)\big)$ simply by $\mathrm{Res}_{*,\beta}$. Then for any $V, W \in R\text{-gmod}$, we have ([1, Theorem 2.1])

$$V \circ \mathrm{Res}_{*,\beta}(W) \rightarrowtail \mathrm{Res}_{*,\beta}(V \circ W) \quad \text{and} \quad \mathrm{Res}_{*,\beta}(V \circ W) \twoheadrightarrow q^{(\beta, \mathrm{wt}(W))}\mathrm{Res}_{*,\beta}(V) \circ W.$$

M The following lemma follows from ~~the description of the isomorphisms between graduations in~~ [1, Theorem 2.1].

Lemma 2.1. *For $X, Y, Z \in \mathrm{Modg}(R)$ and $\beta \in \mathsf{Q}_+$, the diagram below is commutative.*

$$\begin{tikzcd}
X \circ \mathrm{Res}_{*,\beta}(Y \circ Z) \arrow[r, tail] \arrow[d] & \mathrm{Res}_{*,\beta}(X \circ Y \circ Z) \arrow[d] \\
q^{(\beta,\mathrm{wt}(Z))} X \circ \mathrm{Res}_{*,\beta}(Y) \circ Z \arrow[r, tail] & q^{(\beta,\mathrm{wt}(Z))} \mathrm{Res}_{*,\beta}(X \circ Y) \circ Z.
\end{tikzcd}$$

Let $w \in \mathsf{W}$ with $I_w = I$ and $v \leqslant w$.

Proposition 2.2. *For any $\Lambda \in \mathsf{P}_+$, there exists a morphism in $\mathscr{C}_w$*

$$\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\Lambda,v\Lambda)}(X) \colon X \circ \mathsf{M}(w\Lambda, v\Lambda) \to q^{-(\mathrm{wt}(X), w\Lambda+v\Lambda)} \mathsf{M}(w\Lambda, v\Lambda) \circ X$$

functorial in $X \in \mathscr{C}_w$. Moreover, if X belongs to $\mathscr{C}_{w,v}$, then the morphism $\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\Lambda,v\Lambda)}(X)$ is an isomorphism.

Proof. Let $(\mathsf{M}(w\Lambda, \Lambda), \mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda,\Lambda)})$ be the non-degenerate left braider in R-gmod associated with $\mathsf{M}(w\Lambda, \Lambda)$. Then we have an isomorphism

$$q^A X \circ \mathsf{M}(w\Lambda, \Lambda) \xrightarrow[\mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda,\Lambda)}(X)^{-1}]{\sim} \mathsf{M}(w\Lambda, \Lambda) \circ X$$

functorial in $X \in \mathscr{C}_w$, where $A = (w\Lambda + \Lambda, \mathrm{wt}(X))$ (see (1.13)).

Let $\beta = \Lambda - v\Lambda$. Recall that $\mathrm{Res}_{*,\beta}\mathsf{M}(w\Lambda, \Lambda) \simeq \mathsf{M}(w\Lambda, v\Lambda) \otimes \mathsf{M}(v\Lambda, \Lambda)$ (including the grading shift). Let $\alpha = v\Lambda - w\Lambda$, and $\gamma = -\mathrm{wt}(X)$. Then we have a morphism in $\big(R(\gamma + \alpha) \otimes R(\beta)\big)$-gmod

$$\begin{aligned}
& q^A(X \circ \mathsf{M}(w\Lambda, v\Lambda)) \otimes \mathsf{M}(v\Lambda, \Lambda) \simeq q^A X \circ \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda, \Lambda)) \\
& \rightarrowtail q^A \mathrm{Res}_{*,\beta}(X \circ \mathsf{M}(w\Lambda, \Lambda)) \xrightarrow[\mathrm{Res}_{*,\beta}(\mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda,\Lambda)}(X)^{-1})]{\sim} \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda, \Lambda) \circ X) \\
& \twoheadrightarrow q^{-(\beta,\gamma)} \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda, \Lambda)) \circ X \simeq q^{-(\beta,\gamma)} (\mathsf{M}(w\Lambda, v\Lambda) \circ X) \otimes \mathsf{M}(v\Lambda, \Lambda).
\end{aligned} \tag{2.1}$$

By applying the functor $\mathrm{Hom}_{R(\beta)\text{-gmod}}(\mathsf{M}(v\Lambda, \Lambda), -)$ we obtain a morphism in $\mathscr{C}_w$

$$\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\Lambda,v\Lambda)}(X) \colon X \circ \mathsf{M}(w\Lambda, v\Lambda) \to q^{(-\mathrm{wt}(X), w\Lambda+v\Lambda)} \mathsf{M}(w\Lambda, v\Lambda) \circ X$$

which is functorial in $X \in \mathscr{C}_w$.

If an $R(\gamma)$-module X belongs to $\mathscr{C}_{*,v}$, then by [7, Lemma 4.1], we have isomorphisms

$$X \circ \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda)) \simeq \mathrm{Res}_{*,\beta}(X \circ \mathsf{M}(w\Lambda,\Lambda)) \quad \text{and} \tag{2.2}$$

$$\mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda) \circ X) \simeq q^{-(\beta,\gamma)}(\mathsf{M}(w\Lambda,v\Lambda) \circ X) \otimes \mathsf{M}(v\Lambda,\Lambda). \tag{2.3}$$

Hence the composition (2.1) is an isomorphism so that the morphism $\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\Lambda,v\Lambda)}(X)$ is an isomorphism for any $X \in \mathscr{C}_{w,v}$, as desired. $\square$

By [5, Theorem 4.12], for any $\Lambda, \Lambda' \in \mathsf{P}_+$ we have

$\boxed{\mathrm{M}}$ $$\Lambda(\mathsf{M}(w\Lambda',v\Lambda'),\mathsf{M}(w\Lambda,v\Lambda)) = (\mathrm{wt}(\mathsf{M}(w\Lambda',v\Lambda')), w\Lambda + v\Lambda) = (w\Lambda' - v\Lambda', w\Lambda + v\Lambda).$$

The following corollary is a direct consequence of this and Proposition 2.2.

Corollary 2.3. *Let $\phi^r_{w,v,\Lambda_i}(\gamma) := (\gamma, w\Lambda_i + v\Lambda_i)$ for $\gamma \in \mathsf{Q}$. Then the family*

$$\{\big(\mathsf{M}(w\Lambda_i,v\Lambda_i), \mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\Lambda_i,v\Lambda_i),}, \phi^r_{w,v,\Lambda_i}\big)\}_{i\in I}$$

is a real commuting family of right graded braiders in the category $\mathscr{C}_w$. It is also a family of central objects in $\mathscr{C}_{w,v}$.

Note that

$$\phi^r_{w,v,\Lambda_i} = -\phi^\ell_{w,v,\Lambda_i}.$$

The following theorem gives a characterization of $\mathscr{C}_{*,v}$.

Theorem 2.4. *A simple module M in R-gmod belongs to $\mathscr{C}_{*,v}$ if and only if*

$$\widetilde{\Lambda}(M,\mathsf{M}(v\Lambda,\Lambda)) = 0 \quad \text{for all } \Lambda \in \mathsf{P}_+.$$

Proof. If $M \in \mathscr{C}_{*,v}$, then $\big(M,\mathsf{M}(v\Lambda,\Lambda)\big)$ is unmixed and hence $\widetilde{\Lambda}(M,\mathsf{M}(v\Lambda,\Lambda)) = 0$.

Assume that $\widetilde{\Lambda}(M,\mathsf{M}(v\Lambda,\Lambda)) = 0$ for all $\Lambda \in \mathsf{P}_+$. By [5, Proposition 1.24] and [18, Theorem 2.19], there exist simple modules $X \in \mathscr{C}_{*,v}$ and $Y \in \mathscr{C}_v$ such that $M \simeq X \nabla Y$. Hence we have

$$0 = \widetilde{\Lambda}(M,\mathsf{M}(v\Lambda,\Lambda)) \geqslant \widetilde{\Lambda}(Y,\mathsf{M}(v\Lambda,\Lambda)) = (\mathrm{wt}(Y), v\Lambda)$$

$\boxed{\mathrm{M}}$ for any $\Lambda \in \mathsf{P}_+$, where the ~~second~~ inequality follows from [7, Theorem 2.11 (ii)] and
$\boxed{\mathrm{M}}$ the last equality from [6, Theorem 5.2]. (We need some comments.) It follows that $(v^{-1}\mathrm{wt}(Y),\Lambda) = 0$ for all $\Lambda \in \mathsf{P}_+$ so that $\mathrm{wt}(Y) = 0$. It follows that $M \simeq X \nabla \mathbf{1} = X$ and hence M belongs to $\mathscr{C}_{*,v}$ as desired. $\square$

Corollary 2.5. *A simple module X in $\mathscr{C}_w$ belongs to $\mathscr{C}_{w,v}$ if and only if X commutes with $\mathsf{M}(w\Lambda, v\Lambda)$ and $\Lambda(X, \mathsf{M}(w\Lambda, v\Lambda)) = (\mathrm{wt}(X), w\Lambda + v\Lambda)$ for all $\Lambda \in \mathsf{P}_+$.*

Proof. The "only if" part follows from Proposition 2.2. Let us prove the "if" part. M

Assume that X commutes with $\mathsf{M}(w\Lambda, v\Lambda)$ and $\Lambda(X, \mathsf{M}(w\Lambda, v\Lambda)) = (\mathrm{wt}(X), w\Lambda + v\Lambda)$ for all $\Lambda \in \mathsf{P}_+$.

Then, we have

$$\Lambda(X, \mathsf{M}(w\Lambda, \Lambda)) = -\Lambda(\mathsf{M}(w\Lambda, \Lambda), X) = (w\Lambda + \Lambda, \mathrm{wt}(X)),$$

where the first equality comes from the fact that $X \in \mathscr{C}_w$ so that X commutes with $\mathsf{M}(w\Lambda, \Lambda)$, and the second comes from [6, Corollary 5.10].

Since $\mathsf{M}(w\Lambda, \Lambda) = \mathsf{M}(w\Lambda, v\Lambda) \nabla \mathsf{M}(v\Lambda, \Lambda)$ and X commutes with $\mathsf{M}(w\Lambda, v\Lambda)$, we have $\Lambda(X, \mathsf{M}(w\Lambda, \Lambda)) = \Lambda(X, \mathsf{M}(w\Lambda, v\Lambda)) + \Lambda(X, \mathsf{M}(v\Lambda, \Lambda))$. Hence we have

$$\begin{aligned}\Lambda(X, \mathsf{M}(v\Lambda, \Lambda)) &= \Lambda(X, \mathsf{M}(w\Lambda, \Lambda)) - \Lambda(X, \mathsf{M}(w\Lambda, v\Lambda)) \\ &= (w\Lambda + \Lambda, \mathrm{wt}(X)) - (w\Lambda + v\Lambda, \mathrm{wt}(X)) = -(v\Lambda - \Lambda, \mathrm{wt}(X)).\end{aligned}$$

It follows that $\widetilde{\Lambda}(X, \mathsf{M}(v\Lambda, \Lambda)) = 0$ and hence X belongs to $\mathscr{C}_{w,v}$ by Theorem 2.4. □

Corollary 2.6. *A simple module X in $\mathscr{C}_{*,v}$ belongs to $\mathscr{C}_{w,v}$ if and only if X commutes with $\mathsf{M}(w\Lambda, v\Lambda)$ and $\Lambda(\mathsf{M}(w\Lambda, v\Lambda), X) = -(\mathrm{wt}(X), w\Lambda + v\Lambda)$ for all $\Lambda \in \mathsf{P}_+$.*

Proof. Simce the "only if" part is obvious, let us prove the "if" part.

Assume that a simple X in $\mathscr{C}_{*,v}$ commutes with $\mathsf{M}(w\Lambda, v\Lambda)$ and $\Lambda(\mathsf{M}(w\Lambda, v\Lambda), X) = -(\mathrm{wt}(X), w\Lambda + v\Lambda)$ for all $\Lambda \in \mathsf{P}_+$. Then we have M

$$\begin{aligned}\Lambda(X, \mathsf{M}(w\Lambda, \Lambda)) &= \Lambda(X, \mathsf{M}(w\Lambda, v\Lambda) \nabla \mathsf{M}(v\Lambda, \Lambda)) \\ &= \Lambda(X, \mathsf{M}(w\Lambda, v\Lambda)) + \Lambda(X, \mathsf{M}(v\Lambda, \Lambda)) \\ &= (\mathrm{wt}(X), w\Lambda + v\Lambda) - (\mathrm{wt}(X), v\Lambda - \Lambda) = (\mathrm{wt}(X), w\Lambda + \Lambda).\end{aligned}$$

Hence we have

$$\begin{aligned}\mathfrak{d}(X, \mathsf{M}(w\Lambda, \Lambda)) &= \Lambda(X, \mathsf{M}(w\Lambda, \Lambda)) + \Lambda(\mathsf{M}(w\Lambda, \Lambda), X) \\ &= (\mathrm{wt}(X), w\Lambda + \Lambda) + \Lambda(\mathsf{M}(w\Lambda, \Lambda), X) \\ &\leqslant (\mathrm{wt}(X), w\Lambda + \Lambda) - (\mathrm{wt}(X), w\Lambda + \Lambda) = 0,\end{aligned}$$

$\boxed{\text{M}}$ where the inequality is [6, Proposition 4.4]. Hence we have $\Lambda(\mathsf{M}(w\Lambda,\Lambda),X) = -(\mathrm{wt}(X), w\Lambda+\Lambda)$, and $\mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda,\Lambda)}(X)$ does not vanish by [6, Proposition 4.4]. Since X commutes with $\mathsf{M}(w\Lambda,\Lambda)$, $\mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda,\Lambda)}(X)$ is an isomorphism. Then [6, Corollary 5.10] implies that X belongs to $\mathscr{C}_w$. $\square$

By Corollary 2.3, we have localizations

$$\mathscr{C}_w \to \mathscr{C}_w[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I] \quad \text{and} \quad \mathscr{C}_{w,v} \to \mathscr{C}_{w,v}[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I].$$

Let us denote $\mathscr{C}_{w,v}[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I]$ by $\widetilde{\mathscr{C}}_{w,v}$. By the definition of localization, the embedding $\iota_{w,v}\colon \mathscr{C}_{w,v} \rightarrowtail \mathscr{C}_w$ induces a fully faithful functor

$$\tilde{\iota}_{w,v}\colon \widetilde{\mathscr{C}}_{w,v} := \mathscr{C}_{w,v}[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I] \rightarrowtail \mathscr{C}_w[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I].$$

Note that the subcategory $\widetilde{\mathscr{C}}_{w,v}$ is closed by taking subquotients and extensions in $\mathscr{C}_w[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I]$ ([6, Proposition 2.10]).

Theorem 2.7. *The functor* $\tilde{\iota}_{w,v}\colon \widetilde{\mathscr{C}}_{w,v} \to \mathscr{C}_w[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I]$ *is an equivalence of monoidal categories.*

Proof. In the course of the proof, we omit the grading shifts. Let Q denote the localization functor $\mathscr{C}_w \to \mathscr{C}_w[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I]$. It remains to show that for any $X \in \mathscr{C}_w$, the object $Q(X)$ belongs to $\widetilde{\mathscr{C}}_{w,v}$.

(a) Assume first that X is simple in $\mathscr{C}_w$ such that $Q(X) \not\simeq 0$ and X commutes with $\mathsf{M}(w\Lambda, v\Lambda)$ for all $\Lambda \in \mathsf{P}_+$. Since $Q(X) \not\simeq 0$, we have

$$\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\Lambda,v\Lambda)}(X) \neq 0 \quad \text{for all} \quad \Lambda \in \mathsf{P}_+.$$

Since $\mathsf{M}(w\Lambda, v\Lambda)$ is affreal, we have $\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\Lambda,v\Lambda)}(X) = \mathbf{r}_{X,\mathsf{M}(w\Lambda,v\Lambda)}$ up to a constant multiple and hence

$$\Lambda(X, \mathsf{M}(w\Lambda, v\Lambda)) = \phi^{r}_{w,v,\Lambda}(\mathrm{wt}(X)) = (\mathrm{wt}(X), w\Lambda + v\Lambda) \quad \text{for all} \quad \Lambda \in \mathsf{P}_+.$$

Hence X belongs to $\mathscr{C}_{w,v}$, by Corollary 2.5. Thus $Q(X)$ belongs to $\widetilde{\mathscr{C}}_{w,v}$.

(b) Assume that X is simple in $\mathscr{C}_w$ such that $Q(X) \not\simeq 0$. If $\mathfrak{d}(X, \mathsf{M}(w\Lambda, v\Lambda)) > 0$ for $\Lambda \in \mathsf{P}_+$, then we have $\mathfrak{d}(X \nabla \mathsf{M}(w\Lambda, v\Lambda), \mathsf{M}(w\Lambda, v\Lambda)) < \mathfrak{d}(X, \mathsf{M}(w\Lambda, v\Lambda))$ by [6, Corollary 3.18]. Hence by taking large enough $\lambda \in \mathsf{P}_+$, we may assume that $\mathfrak{d}(X \nabla \mathsf{M}(w\lambda, v\lambda), \mathsf{M}(w\Lambda, v\Lambda)) = 0$ for any $\Lambda \in \mathsf{P}_+$. Since $\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\lambda,v\lambda)}(X)$ is decomposed into

$$X \circ \mathsf{M}(w\lambda, v\lambda) \twoheadrightarrow X \nabla \mathsf{M}(w\lambda, v\lambda) \rightarrowtail \mathsf{M}(w\lambda, v\lambda) \circ X$$

and $Q(\mathrm{R}^{\mathrm{r}}_{\mathsf{M}(w\lambda,v\lambda)}(X))$ is an isomorphism, we have

$$Q(X \nabla \mathsf{M}(w\lambda, v\lambda)) \simeq Q(X) \circ \mathsf{M}(w\lambda, v\lambda).$$

Hence the object $Q(X) \simeq Q(X \nabla \mathsf{M}(w\lambda, v\lambda)) \circ \mathsf{M}(w\lambda, v\lambda)^{\circ -1}$ belongs to $\widetilde{\mathscr{C}}_{w,v}$ by (a).

(c) Since the subcategory $\widetilde{\mathscr{C}}_{w,v}$ of $\mathscr{C}_w[\mathsf{M}(\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]$ is closed under extension, every object $Q(X)$ for X in $\mathscr{C}_w$ belongs to $\widetilde{\mathscr{C}}_{w,v}$, as desired. □

Let $\mathrm{Q}^{\mathrm{r}}_{w,v}$ denote the composition of functors

$$\mathrm{Q}^{\mathrm{r}}_{w,v} \colon \mathscr{C}_w \to \mathscr{C}_w[\mathsf{M}(w\Lambda_i, v\Lambda_i)^{\circ -1}; i \in I] \xrightarrow{\sim} \widetilde{\mathscr{C}}_{w,v}. \tag{2.4}$$

In the following two propositions, we characterize the kernels of $\mathrm{Q}^{\mathrm{l}}_{w,v} \colon \mathscr{C}_{*,v} \to \widetilde{\mathscr{C}}_{w,v}$ and $\mathrm{Q}^{\mathrm{r}}_{w,v} \colon \mathscr{C}_w \to \widetilde{\mathscr{C}}_{w,v}$. M

M

Proposition 2.8. *Let X be a simple object of $\mathscr{C}_{*,v}$. Then, $\mathrm{Q}^{\mathrm{l}}_{w,v}(X) \not\simeq 0$ if and only if*

$$\Lambda(\mathsf{M}(w\lambda, v\lambda), X) = -(\mathrm{wt}(X), w\lambda + v\lambda)$$

for any $\lambda \in \mathsf{P}_+$.

Proof. "Only if" part is obvious. Let us show the "if" part.

There exists $\mu \in \mathsf{P}_+$ such that $\mathsf{M}(w\mu, v\mu) \nabla X$ commutes with $\mathsf{M}(w\Lambda, v\Lambda)$ for any $\Lambda \in \mathsf{P}_+$. Then we have

$$\begin{aligned}\Lambda(\mathsf{M}(w\lambda, v\lambda), \mathsf{M}(w\mu, v\mu) \nabla X) &= \Lambda\big(\mathsf{M}(w\lambda, v\lambda), \mathsf{M}(w\mu, v\mu)\big) + \Lambda\big(\mathsf{M}(w\lambda, v\lambda), X\big) \\ &= -\big(w\lambda + v\lambda, \mathrm{wt}(\mathsf{M}(w\mu, v\mu) \nabla X)\big)\end{aligned}$$

for any $\lambda \in \mathsf{P}_+$. Hence, Corollary 2.6 implies that $\mathsf{M}(w\mu, v\mu) \nabla X \in \mathscr{C}_{w,v}$.

Then

$$\mathrm{Q}^{\mathrm{l}}_{w,v}(\mathsf{M}(w\mu, v\mu)) \circ \mathrm{Q}^{\mathrm{l}}_{w,v}(X) \twoheadrightarrow \mathrm{Q}^{\mathrm{l}}_{w,v}(\mathsf{M}(w\mu, v\mu) \nabla X) \not\simeq 0,$$

implies that $\mathrm{Q}^{\mathrm{l}}_{w,v}(X) \not\simeq 0$. □

Proposition 2.9. *LetX be a simple object of $\mathscr{C}_w$. Then, $\mathrm{Q}^{\mathrm{r}}_{w,v}(X) \not\simeq 0$ if and only if*

$$\Lambda(X, \mathsf{M}(w\lambda, v\lambda)) = (\mathrm{wt}(X), w\lambda + v\lambda)$$

for any $\lambda \in \mathsf{P}_+$.

Proof. The proof is similar to the one of the preceding proposition by using Corollary 2.5 instead of Corollary 2.6. $\square$

3. Properties of $\mathscr{C}_{w,v}$

3.1. **Right rigidity.** As an application of Theorem 2.7, we will prove the right rigidity of $\widetilde{\mathscr{C}}_{w,v}$.

Theorem 3.1. *The category $\widetilde{\mathscr{C}}_{w,v}$ is right rigid, i.e., every object has a right dual.*

Proof. In the course of the proof, we omit the grading shifts. Let $X \in \mathscr{C}_{w,v}$. Since $\mathscr{C}_{w,v} \subset \mathscr{C}_w \subset \widetilde{\mathscr{C}}_w$ and the category $\widetilde{\mathscr{C}}_w$ is right rigid, there exists $Y \in \mathscr{C}_w$, $\Lambda \in \mathsf{P}_+$ and morphisms in $\mathscr{C}_w$

$$X \circ Y \xrightarrow{\varepsilon} \mathsf{M}(w\Lambda, \Lambda), \quad \mathsf{M}(w\Lambda, \Lambda) \xrightarrow{\eta} Y \circ X$$

such that the composition

$$X \circ \mathsf{M}(w\Lambda, \Lambda) \xrightarrow{X \otimes \eta} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes X} \mathsf{M}(w\Lambda, \Lambda) \circ X$$

is an isomorphism.

Recall that $\mathsf{M}(w\Lambda, \Lambda) \simeq \mathsf{M}(w\Lambda, v\Lambda) \nabla \mathsf{M}(v\Lambda, \Lambda)$. Let $\beta = \Lambda - v\Lambda \in \mathsf{Q}_+$ and $\gamma = -\mathrm{wt}(X) \in \mathsf{Q}_+$.

We have the following commutative diagram:

$$\begin{tikzcd}
X \circ (\mathrm{Res}_{*,\beta}\mathsf{M}(w\Lambda,\Lambda)) \arrow[r] \arrow[d, "\simeq\ (2.2)"] & X \circ \mathrm{Res}_{*,\beta}(Y \circ X) \arrow[d, tail] & \\
\mathrm{Res}_{*,\beta}(X \circ \mathsf{M}(w\Lambda,\Lambda)) \arrow[r, "X\circ\eta"] & \mathrm{Res}_{*,\beta}(X \circ Y \circ X) \arrow[r, "\varepsilon\circ X"] \arrow[d] & \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda) \circ X) \arrow[d, "\simeq\ (2.3)"] \\
& \mathrm{Res}_{*,\beta}(X \circ Y) \circ X \arrow[r] & \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda)) \circ X.
\end{tikzcd}$$

Since the composition of the arrows in the middle row is an isomorphism, we have an isomorphism

$$X \circ (\mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda))) \xrightarrow{\sim} \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda)) \circ X.$$

We claim that this isomorphism factors through $X \circ \mathrm{Res}_{*,\beta}(Y) \circ X$. Indeed in the diagram

$$\begin{tikzcd}
X \circ (\mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda))) \arrow[r] & X \circ \mathrm{Res}_{*,\beta}(Y \circ X) \arrow[r, tail] \arrow[d] & \mathrm{Res}_{*,\beta}(X \circ Y \circ X) \arrow[d] \\
& X \circ \mathrm{Res}_{*,\beta}(Y) \circ X \arrow[r, tail] & \mathrm{Res}_{*,\beta}(X \circ Y) \circ X \arrow[d] \\
& & \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda)) \circ X,
\end{tikzcd}$$

the square is commutative by Lemma 2.1.

Hence we have a sequence of morphisms

$$X \circ (\mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda))) \to X \circ \mathrm{Res}_{*,\beta}(Y) \circ X \to \mathrm{Res}_{*,\beta}(\mathsf{M}(w\Lambda,\Lambda)) \circ X$$

whose composition is an isomorphism.

Applying the functor $\mathrm{Hom}_{R\text{-gmod}}(\mathsf{M}(v\Lambda,\Lambda), -)$, we obtain a sequence of morphisms

$$X \circ \mathsf{M}(w\Lambda, v\Lambda) \xrightarrow{X \circ \eta'} X \circ Y' \circ X \xrightarrow{\varepsilon' \circ X} \mathsf{M}(w\Lambda, v\Lambda) \circ X$$

whose composition is an isomorphism where $Y' = \mathrm{Res}^{\mathsf{M}(v\Lambda,\Lambda)}(Y)$ and $\mathrm{Res}^{\mathsf{M}(v\Lambda,\Lambda)}(-)$ is the functor $\mathrm{Hom}_{R(\beta)\text{-gmod}}(\mathsf{M}(v\Lambda,\Lambda), \mathrm{Res}_{*,\beta}(-))$. Note that the morphisms ε' and η' are given by

$$\varepsilon' \colon X \circ Y' \to \mathrm{Res}^{\mathsf{M}(v\Lambda,\Lambda)}(X \circ Y) \to \mathrm{Res}^{\mathsf{M}(v\Lambda,\Lambda)}(\mathsf{M}(w\Lambda,\Lambda)) \simeq \mathsf{M}(w\Lambda, v\Lambda)$$

and

$$\eta' \colon \mathsf{M}(w\Lambda, v\Lambda) \simeq \mathrm{Res}^{\mathsf{M}(w\Lambda,\Lambda)}(\mathsf{M}(w\Lambda,\Lambda)) \to \mathrm{Res}^{\mathsf{M}(w\Lambda,\Lambda)}(Y \circ X) \to Y' \circ X.$$

Hence the assertion follows by Proposition 3.2 below. □

Proposition 3.2. *Let $\mathcal{C}$ be an idempotent complete additive monoidal category. If there exist morphisms*

$$X \otimes Y \xrightarrow{\varepsilon} \mathbf{1} \quad \textit{and} \quad \mathbf{1} \xrightarrow{\eta} Y \otimes X$$

such that the composition

$$X \xrightarrow{X \otimes \eta} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes X} X \quad (\textit{respectively, } Y \xrightarrow{\eta \otimes Y} Y \otimes X \otimes Y \xrightarrow{Y \otimes \varepsilon} Y)$$

is an isomorphism, then X has a right dual (respectively, left dual) in $\mathcal{C}$.

Proof. Assume that the composition

$$f \colon X \xrightarrow{X \otimes \eta} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes X} X$$

is an isomorphism. Let us show that X has a right dual.

Let g be the inverse of f and let η' be the composition

$$\eta' \colon \mathbf{1} \xrightarrow{\eta} Y \otimes X \xrightarrow{Y \otimes g} Y \otimes X.$$

Then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "X\otimes\eta'"] \arrow[dr, "X\otimes\eta"'] & X \otimes Y \otimes X \arrow[r, "\varepsilon\otimes X"] & X \\
& X \otimes Y \otimes X \arrow[u, "X\otimes Y\otimes g"'] \arrow[r, "\varepsilon\otimes X"'] & X \arrow[u, "g"']
\end{tikzcd}$$

so that the composition of morphisms in the top row is the identity. Hence, by replacing η with η', we may assume from the beginning that f is the identity. M

Let p be the composition

$$p\colon Y \xrightarrow{\eta\otimes Y} Y\otimes X\otimes Y \xrightarrow{Y\otimes\varepsilon} Y.$$

Then the following commutative diagram shows that $p\circ p = p$:

$$\begin{tikzcd}
Y \arrow[r, "\eta\otimes Y"] \arrow[d, "\eta\otimes Y"'] \arrow[rr, bend left, "p"] & Y\otimes X\otimes Y \arrow[r, "Y\otimes\varepsilon"] \arrow[d, "\eta\otimes Y\otimes X\otimes Y"'] & Y \arrow[d, "\eta\otimes Y"'] \arrow[dd, bend left=60, "p"] \\
Y\otimes X\otimes Y \arrow[r, "Y\otimes X\otimes\eta\otimes Y"] \arrow[dr, "Y\otimes f\otimes Y=\mathsf{id}"'] & Y\otimes X\otimes Y\otimes X\otimes Y \arrow[r, "Y\otimes X\otimes Y\otimes\varepsilon"] \arrow[d, "Y\otimes\varepsilon\otimes X\otimes Y"'] & Y\otimes X\otimes Y \arrow[d, "Y\otimes\varepsilon"'] \\
 & Y\otimes X\otimes Y \arrow[r, "Y\otimes\varepsilon"] & Y.
\end{tikzcd}$$

Let $\widetilde{Y} := \operatorname{Im} p$ so that p is factored as $Y \overset{r}{\twoheadrightarrow} \widetilde{Y} \overset{s}{\rightarrowtail} Y$ with $r\circ s = \mathsf{id}_{\widetilde{Y}}$. Let $\widetilde{\varepsilon}$ and $\widetilde{\eta}$ be the compositions M

$$\widetilde{\varepsilon}\colon X\otimes\widetilde{Y} \xrightarrow{X\otimes s} X\otimes Y \xrightarrow{\varepsilon} \mathbf{1} \quad\text{and}\quad \widetilde{\eta}\colon \mathbf{1} \xrightarrow{\eta} Y\otimes X \xrightarrow{r\otimes X} \widetilde{Y}\otimes X.$$

Then we have the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "X\otimes\widetilde{\eta}"] \arrow[dr, "X\otimes\eta"'] \arrow[dddrr, bend right, "X\otimes\eta"'] & & X\otimes\widetilde{Y}\otimes X \arrow[rr, "\widetilde{\varepsilon}\otimes X"] \arrow[dr, "X\otimes s\otimes X"] & & X \\
 & X\otimes Y\otimes X \arrow[ur, "X\otimes r\otimes X"] \arrow[rr, "X\otimes p\otimes X"] \arrow[dr, "X\otimes\eta\otimes Y\otimes X"'] & & X\otimes Y\otimes X \arrow[ur, "\varepsilon\otimes X"'] & \\
 & & X\otimes Y\otimes X\otimes Y\otimes X \arrow[ur, "X\otimes Y\otimes\varepsilon\otimes X"'] & & \\
 & & X\otimes Y\otimes X \arrow[u, "X\otimes Y\otimes X\otimes\eta"'] \arrow[uur, bend right, "X\otimes Y\otimes f"'] \arrow[uuurr, bend right, "\varepsilon\otimes X"'] & &
\end{tikzcd}$$

so that the composition in the top row is the identity id_X.

The composition of the middle column of the below commutative diagram

$$\begin{tikzcd}
Y \arrow[rr, "\eta\otimes Y"] \arrow[d, "r"'] & & Y\otimes X\otimes Y \arrow[dl, "Y\otimes X\otimes r"'] \arrow[d, "r\otimes X\otimes r"] \arrow[dr, "r\otimes X\otimes p"] & & \\
\widetilde{Y} \arrow[r, "\eta\otimes\widetilde{Y}"'] & Y\otimes X\otimes\widetilde{Y} \arrow[r, "r\otimes X\otimes\widetilde{Y}"'] \arrow[dr, "p\otimes X\otimes s"'] & \widetilde{Y}\otimes X\otimes\widetilde{Y} \arrow[r, "\widetilde{Y}\otimes X\otimes s"'] \arrow[d, "s\otimes X\otimes s"] & \widetilde{Y}\otimes X\otimes Y \arrow[r, "\widetilde{Y}\otimes\varepsilon"'] \arrow[dl, "s\otimes X\otimes Y"] & \widetilde{Y} \arrow[d, "s"] \\
 & & Y\otimes X\otimes Y \arrow[rr, "\widetilde{Y}\otimes\varepsilon"'] & & Y
\end{tikzcd}$$

is $p\otimes X\otimes p$ and hence we have

$$s\circ(\widetilde{Y}\otimes\widetilde{\varepsilon})\circ(\widetilde{\eta}\otimes\widetilde{Y})\circ r=(Y\otimes\varepsilon)\circ(p\otimes X\otimes p)\circ(\eta\otimes Y)=p,$$

where the last equality follows from the commutative diagram

$$\begin{tikzcd}
Y \arrow[rr, "\eta\otimes Y"'] \arrow[dr, "p"'] & & Y\otimes X\otimes Y \arrow[rr, "p\otimes X\otimes p"'] \arrow[dr, "Y\otimes X\otimes p"'] & & Y\otimes X\otimes Y \arrow[rr, "Y\otimes\varepsilon"'] & & Y \\
 & Y \arrow[rr, "\eta\otimes Y"'] & & Y\otimes X\otimes Y \arrow[ur, "p\otimes X\otimes Y"'] \arrow[rr, "Y\otimes\varepsilon"'] & & Y. \arrow[ur, "p"'] &
\end{tikzcd}$$

Hence we have

$$(\widetilde{Y}\otimes\widetilde{\varepsilon})\circ(\widetilde{\eta}\otimes\widetilde{Y})=r\circ p\circ s=r\circ s\circ r\circ s=\mathsf{id}_{\widetilde{Y}},$$

as desired. □

M We conjecture that $\widetilde{\mathscr{C}}_{w,v}$ is a rigid monoidal category.

3.2. Relations among $\widetilde{\mathscr{C}}_{w,v}$.

The following is known as *T-systems.*

Let $i\in I$ and $w,v\in\mathsf{W}$ satisfying $w<ws_i$, and $v<vs_i$. Then we have the following equalities in $A_q(\mathfrak{n})$.

$$\begin{aligned}
&q^{(w\Lambda_i-v\Lambda_i,ws_i\Lambda_i)}\mathsf{D}(w\Lambda_i,v\Lambda_i)\mathsf{D}(ws_i\Lambda_i,vs_i\Lambda_i)\\
&\qquad\qquad=q^{\mathsf{d}_i+(vs_i\Lambda_i,v\Lambda_i-ws_i\Lambda_i)}\mathsf{D}(w\Lambda_i,vs_i\Lambda_i)\mathsf{D}(ws_i\Lambda_i,v\Lambda_i)+\mathsf{D}(w\lambda,v\lambda)\\
&\qquad\qquad=q^{\mathsf{d}_i+(v\Lambda_i,vs_i\Lambda_i-w\Lambda_i)}\mathsf{D}(ws_i\Lambda_i,v\Lambda_i)\mathsf{D}(w\Lambda_i,vs_i\Lambda_i)+\mathsf{D}(w\lambda,v\lambda),
\end{aligned}$$

where $\lambda = s_i\Lambda_i + \Lambda_i$.

Proposition 3.4. *Let* $i \in I$ *and* $w, v \in \mathsf{W}$ *satisfying* $w \geqslant v$, $w < ws_i$, *and* $v < vs_i$.

(a) *If* $w \geqslant vs_i$, *then we have a short exact sequence in* R-gmod

$$0 \to q^{\mathsf{d}_i + (vs_i\Lambda_i, v\Lambda_i - ws_i\Lambda_i)} \mathsf{M}(w\Lambda_i, vs_i\Lambda_i) \circ \mathsf{M}(ws_i\Lambda_i, v\Lambda_i)$$
$$\to q^A \mathsf{M}(w\Lambda_i, v\Lambda_i) \circ \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i) \to \mathsf{M}(w\lambda, v\lambda) \to 0$$

where $A = (v\Lambda_i, vs_i\Lambda_i - ws_i\Lambda_i) = (ws_i\Lambda_i, w\Lambda_i - v\Lambda_i)$ *and* $\lambda = s_i\Lambda_i + \Lambda_i$.

(b) *if* $w \not\geqslant vs_i$, *then* $w\Lambda_i \not\preccurlyeq vs_i\Lambda_i$ *and we have*

$$q^A \mathsf{M}(w\Lambda_i, v\Lambda_i) \circ \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i) \simeq \mathsf{M}(w\lambda, v\lambda).$$

Hence in the both cases we have

$$q^{(ws_i\Lambda_i, w\Lambda_i - v\Lambda_i)} \mathsf{M}(w\Lambda_i, v\Lambda_i) \nabla \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i) \simeq \mathsf{M}(w(\Lambda_i + s_i\Lambda_i), v(\Lambda_i + s_i\Lambda_i)).$$

Proof. Since $w \geqslant v$, we have $w\Lambda_j \preccurlyeq v\Lambda_j = vs_i\Lambda_j$ for all $j \neq i$. If $w \not\geqslant vs_i$, then we get $w\Lambda_i \not\preccurlyeq vs_i\Lambda_i$ by Lemma 1.9. Then $\mathsf{D}(w\Lambda_i, vs_i\Lambda_i) = 0$ and hence Proposition 3.3 implies (b).

Assume that $w \geqslant vs_i$. Then $q^{(vs_i\Lambda_i, v\Lambda_i - ws_i\Lambda_i)} \mathsf{M}(w\Lambda_i, vs_i\Lambda_i) \circ \mathsf{M}(ws_i\Lambda_i, v\Lambda_i)$ is a simple module and it is self-dual. Thus Proposition 3.3 and Lemma [4, 3.2.18] implies (a), as desired. □

Theorem 3.5. *Let* $w \geqslant v$, $ws_i > w$, $vs_i > v$, *and* $\lambda, \mu \in \mathsf{P}_+$. *Then*

(i) *Either* $\xi := \lambda + s_i\mu \in \mathsf{P}_+$ *or* $s_i\xi = s_i\lambda + \mu \in \mathsf{P}_+$.

(ii) *We have*

$$q^{(w\lambda - v\lambda, ws_i\mu)} \mathsf{M}(w\lambda, v\lambda) \nabla \mathsf{M}(ws_i\mu, vs_i\mu) \simeq \mathsf{M}(w\xi, v\xi).$$

(iii) *We have*

$$\widetilde{\Lambda}\big(\mathsf{M}(w\lambda, v\lambda), \mathsf{M}(ws_i\mu, vs_i\mu)\big) = (w\lambda - v\lambda, ws_i\mu) = -(v\lambda, ws_i\mu - vs_i\mu),$$
$$\Lambda\big(\mathsf{M}(w\lambda, v\lambda), \mathsf{M}(ws_i\mu, vs_i\mu)\big) = (w\lambda - v\lambda, ws_i\mu + vs_i\mu) = -(w\lambda + v\lambda, ws_i\mu - vs_i\mu).$$

(iv) *If* $w \not\geqslant vs_i$, *then* $\mathsf{M}(w\lambda, v\lambda)$ *and* $\mathsf{M}(ws_i\mu, vs_i\mu)$ *commute.*

Proof. (i) For $j \in I$, we have $\langle h_j, \lambda + s_i\mu\rangle = \langle h_j, \lambda+\mu\rangle - \langle h_i,\mu\rangle\langle h_j,\alpha_i\rangle$ so that $\langle h_j, \lambda + s_i\mu\rangle \geqslant 0$ for $j \neq i$. Since $\langle h_i, \lambda + s_i\mu\rangle = \langle h_i,\lambda\rangle - \langle h_i,\mu\rangle$ and $\langle h_i, s_i\lambda + \mu\rangle = \langle h_i,\mu\rangle - \langle h_i,\lambda\rangle$ we have either $\xi \in \mathsf{P}_+$ or $s_i\xi \in \mathsf{P}_+$.

(ii) In the proof, we omit the grading shifts. Set $\mathsf{C}_\lambda := \mathsf{M}(w\lambda, v\lambda)$ and $\mathsf{C'}_\mu := \mathsf{M}(ws_i\mu, vs_i\mu)$. It is enough to show that there is an epimorphism $\mathsf{C}_\lambda \circ \mathsf{C'}_\mu \twoheadrightarrow \mathsf{M}(w\xi, v\xi)$.

(1) Assume that $\lambda = a\Lambda_i$ and $\mu = b\Lambda_i$ for some $a, b \in \mathbb{Z}_{\geqslant 0}$. We may assume that $a, b > 0$. We will proceed by induction on $a+b$. Set $\lambda' := (a-1)\Lambda_i$ and $\mu' := (b-1)\Lambda_i$. Note that $\eta := \Lambda_i + s_i\Lambda \in \mathsf{P}_+$ and $\mathsf{C}_\eta = \mathsf{C'}_\eta$. Hence we have

$$\begin{aligned}\mathsf{C}_\lambda \circ \mathsf{C'}_\mu &\simeq \mathsf{C}_{\lambda'} \circ \mathsf{C}_{\Lambda_i} \circ \mathsf{C'}_{\Lambda_i} \circ \mathsf{C'}_{\mu'} \twoheadrightarrow \mathsf{C}_{\lambda'} \circ \mathsf{C}_\eta \circ \mathsf{C'}_{\mu'} \\ &\simeq \mathsf{C}_\eta \circ \mathsf{C}_{\lambda'} \circ \mathsf{C'}_{\mu'} \twoheadrightarrow \mathsf{C}_\eta \circ \mathsf{C}_{\lambda'+\mu'} \twoheadrightarrow \mathsf{C}_{\eta+\lambda'+\mu'},\end{aligned}$$

M where the first epimorphism follows from Proposition 3.4 and the second last epimorphism follows from the induction hypothesis.

(2) Set $\lambda = \lambda' + a\Lambda_i$, $\mu = \mu' + b\Lambda_i$, and $\eta' := a\Lambda_i + bs_i\Lambda_i$, where $a = \langle h_i, \lambda\rangle$ and $b = \langle h_i, \mu\rangle$. Then we have

$$\mathsf{C}_\lambda \circ \mathsf{C'}_\mu \simeq \mathsf{C}_{\lambda'} \circ \mathsf{C}_{a\Lambda_i} \circ \mathsf{C'}_{b\Lambda_i} \circ \mathsf{C'}_{\mu'} \twoheadrightarrow \mathsf{C}_{\lambda'} \circ \mathsf{C}_{\eta'} \circ \mathsf{C'}_{\mu'}.$$

Since $\mathsf{C}_{\lambda'} = \mathsf{C'}_{\lambda'}$ and $\mathsf{C'}_{\mu'} = \mathsf{C}_{\mu'}$, we have

$$\mathsf{C}_{\lambda'} \circ \mathsf{C}_{\eta'} \circ \mathsf{C'}_{\mu'} \simeq \begin{cases} \mathsf{C}_{\lambda'+\eta'+\mu'} \simeq \mathsf{C}_{\lambda+s_i\mu} = \mathsf{M}(w\xi, v\xi) & \text{if} \quad \eta' \in \mathsf{P}_+, \\ \mathsf{C'}_{\lambda'+\eta'+\mu'} \simeq \mathsf{C'}_{s_i\lambda+\mu} = \mathsf{M}(w\xi, v\xi) & \text{if} \quad s_i\eta' \in \mathsf{P}_+, \end{cases}$$

as desired.

(iii) follows from (b) and [4, Lemma 3.1.4].

(iv) Since $\mathsf{M}(w\lambda, v\lambda)$ is a product of $\mathsf{M}(w\Lambda_j, v\Lambda_j)$'s, and $\mathsf{M}(ws_i\mu, vs_i\mu)$ is a product
M of $\mathsf{M}(w\Lambda_k, v\Lambda_k)$ $(k \neq i)$ together with $\mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i)$, the assertion follows from Corollary 3.4 (b). □

Recall the functors

$$\mathrm{Q}^{\mathrm{l}}_{w,v} \colon \mathscr{C}_{*,v} \to \widetilde{\mathscr{C}}_{w,v} \quad \text{and} \quad \mathrm{Q}^{\mathrm{r}}_{w,v} \colon \mathscr{C}_{w} \to \widetilde{\mathscr{C}}_{w,v}.$$

Corollary 3.6. *Let $w \geqslant v$, $ws_i > w$, $vs_i > v$, and $\lambda, \mu \in \mathsf{P}_+$.*

(i) *If* $\xi := \lambda + s_i\mu \in \mathsf{P}_+$, *then we have*
$$\mathrm{Q}^{\mathrm{l}}_{w,v}\big(\mathsf{M}(ws_i\mu, vs_i\mu)\big) \simeq q^{-(w\lambda - v\lambda, wsI - i\mu)}\mathsf{M}(w\lambda, v\lambda)^{\circ -1} \circ \mathsf{M}(w\xi, v\xi).$$

(ii) *If* $\eta := s_i\lambda + \mu \in \mathsf{P}_+$, *then we have*
$$\mathrm{Q}^{\mathrm{r}}_{ws_i,vs_i}\big(\mathsf{M}(w\lambda, v\lambda)\big) \simeq q^{-(w\lambda - v\lambda, wsI - i\mu)}\mathsf{M}(ws_i\eta\, vs_i\eta) \circ \mathsf{M}(ws_i\mu, vs_i\mu)^{\circ -1}.$$

Proof. Since the proof is similar, we only prove (i).

By Theorem 3.5 (iii) and Proposition 2.8, we have $\mathrm{Q}^{\mathrm{l}}_{w,v}\big(\mathsf{M}(ws_i\mu, vs_i\mu)\big) \not\simeq 0$. Hence
$$\mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\lambda,v\lambda)}\big(\mathsf{M}(ws_i\mu, vs_i\mu)\big) \colon \mathsf{M}(w\lambda, v\lambda) \circ \mathsf{M}(ws_i\mu, vs_i\mu) \longrightarrow \mathsf{M}(ws_i\mu, vs_i\mu) \circ \mathsf{M}(w\lambda, v\lambda)$$
does not vanish. Since it is an isomorphism in $\widetilde{\mathscr{C}}_{w,v}$, its image $\mathsf{M}(w\lambda, v\lambda)\nabla\mathsf{M}(ws_i\mu, vs_i\mu) \simeq \mathsf{M}(w\xi, v\xi)$ (in R-gmod) is isomorphic to $\mathsf{M}(w\lambda, v\lambda) \circ \mathrm{Q}^{\mathrm{l}}_{w,v}\big(\mathsf{M}(ws_i\mu, vs_i\mu)\big)$ in $\widetilde{\mathscr{C}}_{w,v}$. $\square$

Theorem 3.7. *Let* $i \in I$ *and* $w, v \in \mathsf{W}$ *satisfying* $v < w$, $w < ws_i$, *and* $v < vs_i$. *If* $w \not\geq vs_i$, *then we have* $\mathscr{C}_{w,v} = \mathscr{C}_{ws_i,vs_i}$.

Proof. Set $\lambda = s_i\Lambda_i + \Lambda_i \in \mathsf{P}_+$. Note that $\mathsf{M}(w\Lambda, v\Lambda)$ and $\mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i)$ commute and $\mathsf{M}(w\Lambda, v\Lambda) \circ \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i) \simeq \mathsf{M}(w\lambda, v\lambda)$ by Proposition 3.4 (b).

Assume that a simple module X belongs to $\mathscr{C}_{w,v}$. Then we have $X \in \mathscr{C}_{ws_i}$ and hence
$$\begin{aligned}(\mathrm{wt}(X), w\lambda + v\lambda) = \Lambda(X, \mathsf{M}(w\lambda, v\lambda)) &= \Lambda(X, \mathsf{M}(w\Lambda_i, v\Lambda_i)) + \Lambda(X, \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i)) \\ &= (\mathrm{wt}(X), w\Lambda_i + v\Lambda_i) + \Lambda(X, \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i))\end{aligned}$$
so that
$$\Lambda(X, \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i)) = (\mathrm{wt}(X), w(\lambda - \Lambda_i) + v(\lambda - \Lambda_i)) = (\mathrm{wt}(X), ws_i\Lambda_i + vs_i\Lambda_i).$$
Because $X \in \mathscr{C}_{w,v}$, we have
$$0 = \mathfrak{d}\big(X, \mathsf{M}(w\lambda, v\lambda)\big) = \mathfrak{d}\big(X, \mathsf{M}(w\Lambda_i, v\Lambda_i)\big) + \mathfrak{d}\big(X, \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i)\big) = \mathfrak{d}\big(X, \mathsf{M}(ws_i\Lambda_i, vs_i\Lambda_i)\big).$$
Hence by Corollary 2.5, X belongs to $\mathscr{C}_{ws_i,vs_i}$.

If X is a simple module in $\mathscr{C}_{ws_i,vs_i}$, then X belongs to $\mathscr{C}_{w,v}$ by the same argument as the above using Corollary 2.6.

Since the categories $\mathscr{C}_{w,v}$ and $\mathscr{C}_{ws_i,vs_i}$ are closed under extensions, we obtain that $\mathscr{C}_{w,v} = \mathscr{C}_{ws_i,vs_i}$, as desired. $\square$

Theorem 3.8. *If $w \geqslant v$, $ws_i > w$ and $vs_i > v$, then there is an equivalence of monoidal categories*

$$\widetilde{\mathscr{C}}_{w,v} \simeq \widetilde{\mathscr{C}}_{ws_i,vs_i}.$$

$\boxed{\text{M}}$

Proof. Set $\mathcal{C} = \mathscr{C}_{ws_i,v}$. Then, $\mathscr{C}_{w,v} \subset \mathcal{C}$ and $\mathscr{C}_{ws_i,vs_i} \subset \mathcal{C}$. In $\mathcal{C}$, there exist a real commuting family of left braiders $\{\mathsf{M}(w\Lambda_j, v\Lambda_j), \mathrm{R}^{\mathrm{l}}_{\mathsf{M}(w\Lambda_j,v\Lambda_j),}, \phi^{l}_{w,v,\Lambda_j}\}_{j\in I}$ and a real commuting family of right braiders $\{\mathsf{M}(ws_i\Lambda_j, vs_i\Lambda_j), \mathrm{R}^{\mathrm{r}}_{\mathsf{M}(ws_i\Lambda_j,vs_i\Lambda_j),}, \phi^{r}_{ws_i,vs_i,\Lambda_j}\}_{j\in I}$.

Let us denote by

$$\begin{aligned} \mathcal{C}^{\mathrm{l}} &:= \mathscr{C}_{ws_i,v}[\mathsf{M}(w\Lambda_j, v\Lambda_j)^{\circ -1}; j \in I], \\ \mathcal{C}^{\mathrm{r}} &:= \mathscr{C}_{ws_i,v}[\mathsf{M}(ws_i\Lambda_j, vs_i\Lambda_j)^{\circ -1}; j \in I] \end{aligned}$$

their localizations. Since the composition of the fully faithful functors

$$\widetilde{\mathscr{C}}_{w,v} \longrightarrow \mathcal{C}^{\mathrm{l}} \longrightarrow \mathscr{C}_{*,v}[\mathsf{M}(w\Lambda_j, v\Lambda_j)^{\circ -1}; j \in I]$$

is an equivalence, $\mathscr{L} \colon \widetilde{\mathscr{C}}_{w,v} \longrightarrow \mathcal{C}^{\mathrm{l}}$ is an equivalence of monoidal categories. Similarly, since the composition of the fully faithful functors

$$\mathscr{R} \colon \widetilde{\mathscr{C}}_{ws_i,vs_i} \longrightarrow \mathcal{C}^{\mathrm{r}} \longrightarrow \mathscr{C}_{ws_i}[\mathsf{M}(ws_i\Lambda_j, vs_i\Lambda_j)^{\circ -1}; j \in I]$$

is an equivalence, $\widetilde{\mathscr{C}}_{ws_i,vs_i} \longrightarrow \mathcal{C}^{\mathrm{r}}$ is an equivalence of monoidal categories. Hence it is enough to show that $\mathcal{C}^{\mathrm{l}}$ and $\mathcal{C}^{\mathrm{r}}$ are equivalent as monoidal categories.

In order to see this, we shall prove that

$$\mathscr{L} \quad \text{factors as } \begin{tikzcd} \mathcal{C} \arrow[r, "\mathscr{R}"'] \arrow[rr, bend left, "\mathscr{L}"] & \mathcal{C}^{\mathrm{r}} \arrow[r, dotted, "\Phi"'] & \mathcal{C}^{\mathrm{l}} \end{tikzcd}, \tag{3.1}$$

$$\mathscr{R} \quad \text{factors as } \begin{tikzcd} \mathcal{C} \arrow[r, "\mathscr{L}"'] \arrow[rr, bend left, "\mathscr{R}"] & \mathcal{C}^{\mathrm{l}} \arrow[r, dotted, "\Psi"'] & \mathcal{C}^{\mathrm{r}} \end{tikzcd}. \tag{3.2}$$

Since the proof of (3.2) is similar, we shall prove only (3.1). Set $\mathsf{C}'_\mu = \mathsf{M}(ws_i\lambda, vs_i\lambda)$ for any $\mu \in \mathsf{P}_+$. By Theorem 1.4, it is enough to show that

(a) $\mathscr{L}(\mathsf{C}'_\mu)$ is invertible in $\mathcal{C}^{\mathrm{l}}$ for any $\mu \in \mathsf{P}_+$,

(b) $\mathscr{L}\big(M \circ \mathsf{C}'_\mu\big) \xrightarrow{\mathscr{L}(\mathrm{R}^{\mathrm{r}}_{\mathsf{C}'_\mu})} \mathscr{L}\big(\mathsf{C}'_\mu \circ M\big)$ is an isomorphism in $\mathcal{C}^{\mathrm{l}}$ for any $\mu \in \mathsf{P}_+$ and $M \in \mathcal{C}$.

(a) follows from Corollary 3.6.

Let us show (b). Let $0 \to Z \to M \circ \mathsf{C}'_\mu \to \mathsf{C}'_\mu \circ M \to Z' \to 0$ be an exact sequence. Since $\mathscr{R}\big(\mathrm{R}^{\mathrm{r}}_{\mathsf{C}'_\mu}(M)\big)$ is an isomorphism, we have $\mathscr{R}(Z) \simeq \mathscr{R}(Z') \simeq 0$.

Then Lemma 3.9 below implies that $\mathscr{L}(Z) \simeq \mathscr{L}(Z') \simeq 0$ and hence $\mathscr{L}\big(\mathrm{R}^{\mathrm{r}}_{\mathsf{C}'_\mu}(M)\big)$ is an isomorphism.

Thus there exist functors $\Phi\colon \mathcal{C}^{\mathrm{r}} \longrightarrow \mathcal{C}^{\mathrm{l}}$ and $\Psi\colon \mathcal{C}^{\mathrm{l}} \longrightarrow \mathcal{C}^{\mathrm{l}}$, and it is obvious that they are quasi-inverse to each other. $\square$

Lemma 3.9. *Assume that $w \geqslant v$, $ws_i > w$, and $vs_i > v$. Let $Z \in \mathscr{C}_{ws_i,v}$. If $\mathrm{Q}^{\mathrm{r}}_{ws_i,vs_i}(Z) \simeq 0$, then $\mathrm{Q}^{\mathrm{l}}_{w,v}(Z) \simeq 0$.*

Proof. We may assume that Z is simple.

Assuming that $\mathrm{Q}^{\mathrm{r}}_{ws_i,vs_i}(Z) \simeq 0$ and $\mathrm{Q}^{\mathrm{l}}_{w,v}(Z) \not\simeq 0$, we shall derive a contradiction.

Let us denote $\mathsf{C}_\lambda := \mathsf{M}(w\lambda, v\lambda)$ and $\mathsf{C}'_\mu := \mathsf{M}(ws_i\mu, vs_i\mu)$ for $\lambda, \mu \in \mathsf{P}_+$. Then, $\mathrm{R}^{\mathrm{l}}_{\mathsf{C}_\lambda}(Z)$ does not vanish for any $\lambda \in \mathsf{P}_+$. Hence $\mathsf{C}_\lambda \nabla Z \simeq \mathrm{Im}\big(\mathrm{R}^{\mathrm{l}}_{\mathsf{C}_\lambda}(Z)\big)$ is isomorphic to $\mathsf{C}_\lambda \circ Z$ is $\widetilde{\mathscr{C}_{w,v}}$, and hence $\mathrm{Q}^{\mathrm{l}}_{w,v}(\mathsf{C}_\lambda \nabla Z) \not\simeq 0$.

There exists $\lambda_0 \in \mathsf{P}_+$ such that $Z' := \mathsf{C}_{\lambda_0} \nabla Z$ commutes with C_Λ for any $\Lambda \in \mathsf{P}_+$. We have $\mathrm{Q}^{\mathrm{r}}_{ws_i,vs_i}(Z') \simeq 0$ and $\mathrm{Q}^{\mathrm{l}}_{w,v}(Z') \not\simeq 0$. By replacing Z with Z', we may assume from the beginning that Z commutes with all C_Λ for $\Lambda \in \mathsf{P}_+$. By Proposition 2.8 and $\mathrm{Q}^{\mathrm{l}}_{w,v}(Z') \not\simeq 0$, we have $\Lambda(\mathsf{C}_\lambda, Z) = -(w\lambda + v\lambda, \mathrm{wt}(Z))$ for any $\lambda \in \mathsf{P}_+$. Hence Corollary 2.6 implies that $Z \in \mathscr{C}_{w,v}$. Hence we have

$$\Lambda(Z, \mathsf{C}_\lambda) = (w\lambda + v\lambda, \mathrm{wt}(Z)) \text{ for any } \lambda \in \mathsf{P}_+.$$

Since $\mathrm{Q}^{\mathrm{r}}_{ws_i,vs_i}(Z) \simeq 0$, Proposition 2.9 implies that there exists $\mu \in \mathsf{P}_+$ such that

$$\Lambda(Z, \mathsf{C}'_\mu) \neq \big(ws_i\mu + vs_i\mu, \mathrm{wt}(Z)\big).$$

Let us take $\lambda \in \mathsf{P}_+$ such that $\xi := \lambda + s_i\mu \in \mathsf{P}_+$. Then we have a contradiction

$$\begin{aligned}(w\xi + v\xi, \mathrm{wt}(Z)) &= \Lambda(Z, \mathsf{C}_\xi) = \Lambda(Z, \mathsf{C}_\lambda \nabla \mathsf{C}'_\mu)\\ &= \Lambda(Z, \mathsf{C}_\lambda) + \Lambda(Z, \mathsf{C}'_\mu) = (w\lambda + v\lambda, \mathrm{wt}(Z)) + \Lambda(Z, \mathsf{C}'_\mu)\\ &\neq (w\lambda + v\lambda, \mathrm{wt}(Z)) + (ws_i\mu + vs_i\mu, \mathrm{wt}(Z))\\ &= (w\xi + v\xi, \mathrm{wt}(Z)).\end{aligned}$$

Here the third equality follows from the commutativity of Z and C_λ. □

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