MONOIDAL CATEGORIFICATION AND QUANTUM AFFINE ALGEBRAS II

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Abstract. We introduce a new family of real simple modules over the quantum affine algebras, called the affine determinantial modules, which contains the Kirillov-Reshetikhin (KR)-modules as a special subfamily, and then prove T-systems among them which generalize the T-systems among KR-modules and unipotent quantum minors in the quantum unipotent coordinate algebras simultaneously. We develop new combinatorial tools: admissible chains of i-boxes which produce commuting families of affine determinantial modules, and box moves which describe the T-system in a combinatorial way. Using these results, we prove that various module categories over the quantum affine algebras provide monoidal categorifications of cluster algebras. As special cases, Hernandez-Leclerc categories $\mathscr{C}_{\mathfrak{g}}^0$ and $\mathscr{C}_{\mathfrak{g}}^-$ provide monoidal categorifications of the cluster algebras for an arbitrary quantum affine algebra.

Contents

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1. Introduction

The notion of *monoidal categorification* is proposed by Hernandez-Leclerc in [20]. This notion offers a framework for proving Laurent positivity and linear independence for a cluster algebra $\mathscr{A}$, when $\mathscr{A}$ is isomorphic to the Grothendieck ring of a monoidal category $\mathscr{C}$. Conversely, it provides quite interesting information on the monoidal category $\mathscr{C}$: once we show that $\mathscr{C}$ is a monoidal categorification of a cluster algebra, we acquire a family $\mathscr{F}$ of *real prime simple objects* in $\mathscr{C}$ (identified with the *cluster variables* in $\mathscr{A}$) whose certain groupings in $\mathscr{F}$ (identified with the *clusters* in $\mathscr{A}$) consist of *mutually commuting* real prime simple objects.

The purpose of this paper is to prove that various module categories over quantum affine algebras provide monoidal categorifications of cluster algebras.

A cluster algebra $\mathscr{A}$, introduced by Fomin and Zelevinsky in [9], is a commutative $\mathbb{Z}$-subalgebra of $\mathbb{Z}[X_k^{\pm 1} \mid k \in \mathsf{K}]$ with a distinguished set of generators, called the cluster variables, which is grouped into overlapping subsets, called the clusters. The clusters are defined inductively by a procedure called *mutation* from the initial cluster $\{X_k\}_{k\in\mathsf{K}}$, which is performed via an exchange matrix $\widetilde{B}$. The notion of cluster algebras is extended to a quantum version, *quantum cluster algebras* $\mathscr{A}_q$ in $\mathbb{Z}[q^{\pm 1/2}][X_k^{\pm 1}]_{k\in\mathsf{K}}$ by Berenstein and Zelevinsky in [3], which are not commutative any more, but their cluster variables in each cluster are q-commutative. The q-commutativity is controlled by a $\mathbb{Z}$-valued $\mathsf{K}\times\mathsf{K}$-matrix L. From their introductions, numerous connections and applications have been discovered in various fields of mathematics (see [9, 3, 47, 48, 64, 17] and references therein).

For each Kac-Moody algebra $\mathfrak{g}$ of affine type, let $U_q'(\mathfrak{g})$ be the corresponding quantum affine algebra and let $\mathscr{C}_\mathfrak{g}$ be the category of finite-dimensional integrable modules over $U_q'(\mathfrak{g})$. Since the category $\mathscr{C}_\mathfrak{g}$ has interesting properties including monoidality and rigidity, it has been intensively studied since 1990's, in various aspects of point of view. To name a few, the complete classification of simple modules in $\mathscr{C}_\mathfrak{g}$ is obtained in terms of Drinfel'd polynomials ([4, 5, 6] for the untwisted cases and [7] for the twisted cases) and it is proved in [1, 34, 62] that every simple module in $\mathscr{C}_\mathfrak{g}$ can be obtained as the head of a tensor product of *fundamental modules*. Also, it is proved in [49, 50] (for simply-laced untwisted affine types) and [18, 19] (for general types) that the *q-characters* of *Kirillov-Reshetikhin* (KR) *modules*, a special class of modules in $\mathscr{C}_\mathfrak{g}$, are solutions of the *T-system* which is closely related to discrete dynamical systems arising from the thermodynamic Bethe-ansatz, Y-system (see [46, 26]).

The first result on monoidal categorifications was established in [20, 52] for relatively small monoidal subcategories $\mathscr{C}_\mathfrak{g}^1$ of $\mathscr{C}_\mathfrak{g}$ for untwisted simply-laced affine type $\mathfrak{g}$. One of the main ideas of those papers is interpreting the T-system among KR-modules as exchange relations of a cluster algebra by mutations. (See [61] for the monoidal categorifications of related categories $\mathscr{C}_{\mathfrak{g}}^N$ ($N \in \mathbb{Z}_{\geqslant 1}$) and see [24, 25] for the relation between cluster algebras and T-systems.)

One of other successful instances on the monoidal categorification is given in [33] by using the monoidal subcategories $\mathcal{C}_w$ in R-gmod which *categorify* the quantum unipotent coordinate algebras. Here R denotes a $\mathbb{Z}$-graded algebra, called *quiver Hecke algebra*, introduced independently by Khovanov-Lauda [45] and Rouquier [59, 60], and R-gmod denotes the category of finite-dimensional graded modules over R. We shall explain the result in a more precise way. In [14, 15], Geiß, Leclerc and Schröer showed that the quantum unipotent coordinate algebra $A_q(\mathfrak{n}(w))$, associated with a symmetric quantum group $U_q(\mathsf{g})$ and its Weyl group element w, has a skew-symmetric quantum cluster algebra structure. To see this, they (i) used a system of quantum determinantial identities among *unipotent quantum minors* $D_{\underline{w}}[a, b]$ in $A_q(\mathfrak{n}(w))$, called also T-system, (ii) constructed an initial quiver $\mathrm{Q}_{\mathrm{GLS}}(\underline{w})$ arising from a choice of a *reduced expression* $\underline{w}$ of w and (iii) employed the representation theory of preprojective algebras related to $A_q(\mathfrak{n}(w))$. Here, the T-system also plays the role of exchange relation of the quantum cluster algebra by mutations. In [33], it is proved that $\mathcal{C}_w$ provides a monoidal categorification of the quantum cluster algebra $A_q(\mathfrak{n}(w))$ by showing

(i) $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C}_w) \simeq A_{q^{1/2}}(\mathfrak{n}(w)) := \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathfrak{n}(w))$,

(ii) there exists a *quantum monoidal seed* $\mathscr{S} = (\{M_i\}_{i \in \mathsf{K}}, L, \widetilde{B}, D)$ in $\mathcal{C}_w$ consisting of
   - (a) a commuting family $\{M_i\}_{i \in \mathsf{K}}$ of real simple modules in $\mathcal{C}_w$, called *determinantial modules*, corresponding to the q-commuting family of quantum *flag* minors $D_{\underline{w}}[0, b]$ via the isomorphism in (i),
   - (b) the matrix $L = (-\Lambda(M_i, M_j))_{i,j \in \mathsf{K}}$ where $\Lambda(M_i, M_j)$ denotes the homogeneous degree of the *R-matrix* between M_i and M_j,
   - (c) the incident matrix $\widetilde{B}$ of the quiver $\mathrm{Q}_{\mathrm{GLS}}(\underline{w})$, and
   - (d) the set D of *weights* of M_i's in the root lattice of g

   such that $[\mathscr{S}] := (\{q^{m_i}[M_i]\}_{i \in \mathsf{K}}, L, \widetilde{B})$ is a quantum seed of $A_{q^{1/2}}(\mathfrak{n}(w))$ for some $m_i \in \frac{1}{2}\mathbb{Z}$,

(iii) $\mathscr{S}$ admits successive mutations in all directions in K^{ex}, which is *revealed* as a consequence of the Λ-*admissibility* of $\mathscr{S}$ and the isomorphism $A_{q^{1/2}}(\mathfrak{n}(w)) \simeq \mathscr{A}([\mathscr{S}])$ [33, Theorem 7.1.3].

Here the Λ-admissibility of $\mathscr{S}$ means that, for every $k \in \mathsf{K}^{\mathrm{ex}}$,

$$
\text{(1.1)} \qquad
\begin{aligned}
&\text{there exists a simple module } M'_k \text{ satisfying the following properties:}\\
&\quad \text{(a) } M'_k \text{ corresponds to the mutated cluster variable } X'_k \text{ in the direction } k,\\
&\quad \text{(b) } 1 = \mathfrak{d}(M_k, M'_k) := \big(\Lambda(M_k, M'_k) + \Lambda(M'_k, M_k)\big)/2,\\
&\quad \text{(c) } M'_k \text{ is real and commutes with } M_i \text{ for all } i \in \mathsf{K} \setminus \{k\}.
\end{aligned}
$$

The $\mathbb{Z}$-grading structure of R, the quantum cluster algebra structure of $A_q(\mathfrak{n}(w))$ and the primeness of cluster variables ([16]) play important role in the proof of the monoidal categorification by $\mathcal{C}_w$ in [33].

In [28], Kang-Kashiwara-Kim constructed a functor $\mathcal{F}$, called *quantum affine Schur-Weyl duality functor*, from R-gmod to $\mathscr{C}_\mathfrak{g}$ by observing the singularities of the $U'_q(\mathfrak{g})$-module homomorphism $\mathrm{R}^{\mathrm{norm}}_{M,N_z}$, called the *normalized R-matrix*, for modules M, N in $\mathscr{C}_\mathfrak{g}$. The quantum affine Schur-Weyl duality functor thus makes a bridge between the representation theory of quiver Hecke algebras and the one of quantum affine algebras.

On the other hand, a monoidal full subcategory $\mathscr{C}_Q$ of $\mathscr{C}_\mathfrak{g}$ is introduced in [21] for simply-laced untwisted affine $\mathfrak{g}$, which is defined by using the combinatorics of the Auslander-Reiten quiver of a Dynkin quiver Q. Then the notion of $\mathscr{C}_Q$ is extended to all affine types by generalizing Dynkin quivers Q to Q-data $\mathscr{Q}$ in [32, 57, 55, 13] (see § 6.1). In [29, 32, 42, 55], the Schur-Weyl duality functor $\mathcal{F}_\mathscr{Q} \colon R_{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod} \to \mathscr{C}_\mathscr{Q}$ is constructed for a Q-datum $\mathscr{Q}$, and $\mathcal{F}_\mathscr{Q}$ sends simple modules in $R_{\mathfrak{g}_{\mathrm{fin}}}$-gmod to simple modules in the monoidal full subcategory $\mathscr{C}_\mathscr{Q} \subset \mathscr{C}_\mathfrak{g}$ bijectively. Here, we associate a simply-laced finite-dimensional simple Lie algebra $\mathfrak{g}_{\mathrm{fin}}$ to each affine Lie algebra $\mathfrak{g}$ (see (6.1) and (6.11)), and $R_{\mathfrak{g}_{\mathrm{fin}}}$ is the quiver Hecke algebra associated with $\mathfrak{g}_{\mathrm{fin}}$.

Thus one can conclude that $\mathscr{C}_\mathscr{Q}$ provides a monoidal categorification of the coordinate ring $\mathbb{C}[N]$ of the maximal unipotent group N associated with $\mathfrak{g}_{\mathrm{fin}}$ (see also [21, 23, 12]).

In $\mathscr{C}_\mathfrak{g}$, there are other interesting and important monoidal full subcategories (see § 6.3 for more details).

(A) The subcategory $\mathscr{C}^0_\mathfrak{g}$ is defined in [20] for a simply-laced untwisted affine $\mathfrak{g}$ satisfying the following property: for any simple module V in $\mathscr{C}_\mathfrak{g}$, it decomposes as a tensor product of parameter shifts of simple modules in $\mathscr{C}^0_\mathfrak{g}$. Thus sometimes $\mathscr{C}^0_\mathfrak{g}$ is referred to as the *skeleton* subcategory.

(B) The subcategory $\mathscr{C}^-_\mathfrak{g}$ is defined in [22] for an untwisted affine $\mathfrak{g}$ which contains all simple modules in $\mathscr{C}^0_\mathfrak{g}$ up to parameter shifts. It is proved in [22] that the Grothendieck ring $K(\mathscr{C}^-_\mathfrak{g})$ has a cluster algebra structure with an initial cluster consisting of KR-modules, by using T-systems among KR-modules. Note that the definition of the subcategories $\mathscr{C}_{\mathfrak{g}}^0$ and $\mathscr{C}_{\mathfrak{g}}^-$ are also extended to all affine types.

(C) The subcategory $\mathscr{C}_{\mathscr{Q}}$ has a remarkable property as a subcategory of $\mathscr{C}_{\mathfrak{g}}^0$: for any fundamental module V in $\mathscr{C}_{\mathfrak{g}}^0$, there exists a unique fundamental module U in $\mathscr{C}_{\mathscr{Q}}$ and $k \in \mathbb{Z}$ such that $V \simeq \mathscr{D}^k(U)$. Here $\mathscr{D}^k(U)$ denotes the k-th repeated dual of U. Thus $\mathscr{C}_{\mathscr{Q}}$ is understood as a *heart* subcategory of $\mathscr{C}_{\mathfrak{g}}^0$.

(D) For $N \in \mathbb{Z}_{\geqslant 1}$, the subcategory $\mathscr{C}_{\mathfrak{g}}^N$ is defined in [20] (denoted by $\mathscr{C}_{N-1}$ in [20]) for a simply-laced untwisted affine $\mathfrak{g}$, which is generated by $(|I_{\mathrm{fin}}| \times N)$-many fundamental modules. Here I_{fin} is the index set of simple roots of $\mathfrak{g}_{\mathrm{fin}}$.

The subcategories in (A)–(D) are also referred to as *Hernandez-Leclerc categories*.

On the other hand, the authors of this paper have recently developed interesting results on $\mathscr{C}_{\mathfrak{g}}$ which are briefly summarized as follows.

(I) In [38, 39], the $\mathbb{Z}$-valued invariants $\Lambda(M,N)$, $\mathfrak{d}(M,N)$ and $\Lambda^\infty(M,N)$ for a pair of modules M and N in $\mathscr{C}_{\mathfrak{g}}$ are introduced by analyzing the R-matrices associated with $M \otimes N_z$. By using them, we associate a simply-laced finite root system of type $\mathfrak{g}_{\mathrm{fin}}$ to each $\mathscr{C}_{\mathfrak{g}}$. They can be understood as quantum affine analogues of $\mathbb{Z}$-graded structure and weights of modules in R-gmod, respectively. Contrary to R-gmod, the category $\mathscr{C}_{\mathfrak{g}}$ is a rigid monoidal category and these invariants enjoy interesting properties that result from rigidity. (See § 2.3.) Also in [38], a *criterion* for a subcategory $\mathcal{C}$ in $\mathscr{C}_{\mathfrak{g}}$ to provide a monoidal categorification of a cluster algebra is established (see Theorem 7.8).

(II) In [40], the authors generalize the notion of Q-datum $\mathscr{Q}$ one step further to the notion of (*complete*) *duality datum* $\mathcal{D}$ (see [40, Definition 4.7] and Definition 4.8). It induces a Schur-Weyl duality functor $\mathcal{F}_{\mathcal{D}} \colon R_{\mathsf{g}}\text{-gmod} \to \mathscr{C}_{\mathfrak{g}}^0$ which sends simple modules in R_{g}-gmod to simple modules in $\mathscr{C}_{\mathfrak{g}}^0$ injectively, where g is the simply-laced finite-dimensional simple Lie algebra determined by $\mathcal{D}$ (see § 4.1). Also it is proved that the category $\mathscr{C}_{\mathcal{D}}$, the image of $\mathcal{F}_{\mathcal{D}}$, also enjoys the similar properties to those of $\mathscr{C}_{\mathscr{Q}}$ in the following sense: for each complete duality datum $\mathcal{D}$ and a reduced expression $\underline{w_0}$ of the longest element $w_0 \in W_{\mathsf{g}}$, $\mathscr{C}_{\mathfrak{g}}^0$ is generated by the images $\{\mathsf{S}_k\}_{1\leqslant k\leqslant \ell(w_0)}$ of $\{\mathsf{D}_{\underline{w_0}}[k]\}_{1\leqslant k\leqslant \ell(w_0)}$ in R_{g}-gmod by $\mathcal{F}_{\mathcal{D}}$ (Definition 2.27), and their repeated duals $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$. More precisely, every simple module V in $\mathscr{C}_{\mathfrak{g}}^0$ can be obtained as the head of an ordered tensor product of S_k's. Thus $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ plays the same role as fundamental modules in $\mathscr{C}_{\mathfrak{g}}^0$, and we call them the *affine cuspidal modules* (Definition 4.5) associated with the complete *PBW-pair* $(\mathcal{D}, \widehat{\underline{w}}_0)$. Here, a PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ is a pair of a duality datum $\mathcal{D}$ and $\widehat{\underline{w}}_0 := (\imath_k)_{k\in\mathbb{Z}} \in {I_{\mathsf{g}}}^{\mathbb{Z}}$ which is a sequence in the index set I_{g} of simple roots of g such that $s_{\imath_{k+1}} \cdots s_{\imath_{k+\ell}}$ is a reduced expression of the longest element w_0 of the Weyl group for every k ($\ell := \ell(w_0)$) (see § 4.3).

Furthermore, it is proved that the invariants defined in both categories, such as $\Lambda(M, N)$ and $\mathfrak{d}(M, N)$, are preserved under the functor $\mathcal{F}_{\mathcal{D}}$ (see § 4 for notations and details).

With these recently developed results at hand, we show in this paper that various subcategories of $\mathscr{C}_{\mathfrak{g}}$ provide monoidal categorifications of cluster algebras. In our results, there is *no* restriction on the affine type of $\mathfrak{g}$; i.e., our results hold for an *arbitrary* $U_q'(\mathfrak{g})$.

The main results of this paper can be summarized in the following three main theorems:

(**MT1**) We give a *vast generalization of T-system* which implies the determinantial identities among quantum unipotent minors and the functional relations among q-characters of KR-modules simultaneously.

(**MT2**) We develop two combinatorial notions: *admissible chains of i-boxes* which provide commuting families of affine determinantial modules, and *box moves* which describe T-system in a combinatorial way.

(**MT3**) We study a family of subcategories $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{\widehat{w}}_0}$ of $\mathscr{C}_{\mathfrak{g}}^0$, introduced in [40], which contains Hernandez-Leclerc categories, and prove that $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{\widehat{w}}_0}$ provides a monoidal categorification of a cluster algebra, when $\mathcal{D}$ is the duality datum arising from a Q-datum $\mathscr{Q}$.

Here is a remark on the main theorems. Although a criterion providing a monoidal categorification of a cluster algebra has been established in [38], applying the criterion to subcategories of $\mathscr{C}_{\mathfrak{g}}$ is a quite different problem. More precisely, it is not easy to find a monoidal seed that satisfies the conditions of the criterion. To overcome this difficulty, we use the new invariants and the PBW theory to study affine determinantial modules (see (1.2) below) and their T-systems in a general setting, and then obtain (**MT1**) and (**MT2**). These theorems allow us to find desired monoidal seeds, which provides (**MT3**). Some of these results have been already announced in [41].

We first define *affine determinantial modules* as follows. Let $(\mathcal{D}, \underline{\widehat{w}}_0)$ be a PBW-pair. Then we define the *affine determinantial module*

$$M[a, b] := \mathrm{hd}(\mathsf{S}_b \otimes \mathsf{S}_{b^-} \otimes \cdots \otimes \mathsf{S}_{a^+} \otimes \mathsf{S}_a) \tag{1.2}$$

for any interval $[a, b]$ such that $a \leqslant b$ and $\imath_a = \imath_b$, called an *i-box*. Here, we set $s^+ := \min\{t \mid s < t,\ \imath_t = \imath_s\}$ and $s^- := \max\{t \mid t < s,\ \imath_t = \imath_s\}$ for $s \in \mathbb{Z}$. It is proved in Theorem 4.21 that each affine determinantial module is real simple.

Main Theorem 1 ([Theorem 4.25]). *For an* arbitrary *PBW-pair* $(\mathcal{D}, \widehat{\underline{w}}_0)$ *and an* i*-box* $[a,b]$*, we have an exact sequence*

$$0 \to \bigotimes_{\jmath \in I_{\mathfrak{g}};\ d(\imath_a,\jmath)=1} M[a(\jmath)^+, b(\jmath)^-] \to M[a^+,b] \otimes M[a,b^-] \to M[a,b] \otimes M[a^+,b^-] \to 0, \tag{1.3}$$

where $d(\imath,\jmath)$ *denotes the distance between* $\imath$ *and* $\jmath \in I_{\mathfrak{g}}$ *in the Dynkin diagram* $\triangle$ *of* $\mathfrak{g}$*,* $s(\jmath)^+ := \min\{t \mid s \leqslant t,\ \imath_t = \jmath\}$ *and* $s(\jmath)^- := \max\{t \mid t \leqslant s,\ \imath_t = \jmath\}$ *for* $s \in \mathbb{Z}$ *and* $\jmath \in I_{\mathfrak{g}}$*.*

The exact sequence (1.3) *is also called* T-system*.*

When an i-box $[a,b]$ is contained in $[1,\ell]$, $M[a,b]$ is isomorphic to $\mathcal{F}_{\mathcal{D}}(\mathsf{D}_{\underline{w}_0}[a,b])$ and hence (1.3) can be interpreted as a T-system among quantum unipotent minors. When a PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ is *associated with a* Q*-datum*, the affine determinantial module $M[a,b]$ is a KR-module for an *arbitrary* i-box $[a,b]$ (Theorem 6.14), and hence we can interpret (1.3) as a well-known T-system among KR-modules.

However, when the i-box $[a,b]$ is *not* contained in $[1,\ell]$ nor is the PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ associated with any Q-datum, the affine determinantial modules and exact sequences among those modules were not investigated before as far as the authors know.

Thus Main Theorem 1 can be understood as a vast generalization of the T-system in R-gmod and the T-system among KR-modules simultaneously.

We also develop the combinatorics to construct a commuting family $\mathsf{M}(\mathfrak{C})$ of affine determinantial modules and to describe T-system among affine determinantial modules as exchange relations in the cluster algebra by mutations. More precisely, we define the notion of an *admissible chain of* i*-boxes* $\mathfrak{C} = (\mathfrak{c}_1, \ldots, \mathfrak{c}_l)$ as a sequence of i-boxes $\mathfrak{c}_k$ (see Definition 5.1), so that $\widetilde{\mathfrak{c}}_k := \bigcup_{1 \leqslant j \leqslant k} \mathfrak{c}_j$ is an interval with $|\widetilde{\mathfrak{c}}_k| = k$ for $1 \leqslant k \leqslant l$. We say that $\widetilde{\mathfrak{c}}_l = [a,b]$ is the *range* of $\mathfrak{C}$

Main Theorem 2 ([Theorem 5.5, Lemma 5.10 and Proposition 7.13]). *Let* $(\mathcal{D}, \widehat{\underline{w}}_0)$ *be an* arbitrary *PBW-pair and let* $[a,b]$ *be any interval such that* $a \leqslant b \in \mathbb{Z} \sqcup \{\pm\infty\}$*.*

(a) *For any admissible chain* $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leqslant k \leqslant b-a+1}$ *of* i*-boxes with the range* $[a,b]$*,* $\mathsf{M}(\mathfrak{C}) := \{M(\mathfrak{c}_k)\}_{1 \leqslant k \leqslant b-a+1}$ *forms a commuting family of affine determinantial modules.*
(b) *For any admissible chains* $\mathfrak{C}$ *and* $\mathfrak{C}'$ *with the same range, we can* obtain $\mathsf{M}(\mathfrak{C}')$ *from* $\mathsf{M}(\mathfrak{C})$ *by applying a sequence of T-systems described in terms of newly introduced notion, called* box moves*.*

Next we construct an initial exchange matrix (equivalently, an initial quiver) and the initial cluster variable modules which are expected to give a cluster algebra structure on $K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$ for $-\infty \leqslant a \leqslant b < +\infty$. Here, $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ denotes the monoidal subcategory of $\mathscr{C}_{\mathfrak{g}}^0$ generated by $\{\mathsf{S}_k\}_{a\leqslant k\leqslant b}$, and the categories explained in (A)–(D) are special cases of $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$. We present such data by adapting the combinatorics given in [14]: for the subsequence $(\imath_k)_{a\leqslant k\leqslant b}$ of $\widehat{\underline{w}}_0$, we take the initial cluster variable modules $\{\mathsf{M}_s\}_{a\leqslant s\leqslant b}$ and the initial quiver $\mathrm{Q}_{\mathrm{GLS}} := \mathrm{Q}_{\mathrm{GLS}}((\imath_k)_{a\leqslant k\leqslant b})$ with the vertices $[a,b]$ as follows:

$$\mathsf{M}_s := M[s, b(\imath_s)^-] \quad \text{and} \quad s \to t \quad \begin{cases} \text{if } s^- < t^- < s < t \text{ and } d(\imath_s,\imath_t) = 1, \\ \text{or} \\ \text{if } t = s^-, \end{cases}$$

for $a \leqslant s \neq t \leqslant b$ (see (7.5)).

Since the sequence $\big([b-s+1, b(\imath_{b-s+1})^-]\big)_{1\leqslant s\leqslant b-a+1}$ is an admissible chain $\mathfrak{C}_-$ of i-boxes, $\mathsf{M}(\mathfrak{C}_-) := \{\mathsf{M}_s\}_{a\leqslant s\leqslant b}$ is a commuting family of affine determinantial modules.

Now, for any PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ and any i-box $[a,b]$, we obtain a monoidal seed

$$\mathscr{S} = (\{\mathsf{M}_s\}_{a\leqslant s\leqslant b}, \widetilde{B}_{\mathrm{Q}_{\mathrm{GLS}}}) \tag{1.4}$$

where $\widetilde{B}_{\mathrm{Q}_{\mathrm{GLS}}}$ denotes the incident matrix of $\mathrm{Q}_{\mathrm{GLS}}$. Then we prove in Theorem 7.20 that

$$\mathscr{S} \text{ is } \Lambda\text{-}\textit{admissible} \text{ (see (1.1))}$$

~~To show this assertion, we employ the similar framework of~~ [33, Section 11].

Do we need to mention this?
— Se-jin

Recall that the subcategory $\mathscr{C}_{\mathfrak{g}}^-$ is a special case of $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ and its Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}}^-)$ has a cluster algebra structure. Using the cluster algebra structure, we have the following theorem:

Main Theorem 3 ([Theorem 8.1], [Proposition 8.4])**.** *Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair with $\mathcal{D} = \mathcal{D}_{\mathscr{Q}}$ for some* Q*-datum $\mathscr{Q}$. For any admissible chain $\mathfrak{C} = (\mathfrak{c}_k)_{1\leqslant k\leqslant l}$ for $l := b-a+1 \in \mathbb{Z}_{\geqslant 1} \sqcup \{\infty\}$ with the range $\widetilde{\mathfrak{c}}_l = [a,b]$ $(a \leqslant b \in \mathbb{Z} \sqcup \{\pm\infty\})$, there exists an Λ-admissible monoidal seed $\mathscr{S}$ such that*

(a) *its set of cluster variable modules is $\mathsf{M}(\mathfrak{C}) := \{M(\mathfrak{c}_k)\}_{1\leqslant k\leqslant l}$,*
(b) *its set of frozen variable modules is*

$$\{M[a(\imath)^+, b(\imath)^-] \mid \imath \in I_{\mathrm{fin}},\ -\infty < a \leqslant a(\imath)^+ \leqslant b(\imath)^- \leqslant b < +\infty\},$$

(c) *$K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$ has a cluster algebra structure with the initial seed $[\mathscr{S}] := \{\ [M(\mathfrak{c}_k)]\ \}_{1\leqslant k\leqslant l}$, and $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ provides a monoidal categorification of the cluster algebra $\mathscr{A}([\mathscr{S}])$.*

In particular, the categories $\mathscr{C}_{\mathfrak{g}}^0$ and $\mathscr{C}_{\mathfrak{g}}^-$ provide monoidal categorifications of the cluster algebras.

For a pair $\mathfrak{C}$ and $\mathfrak{C}'$ of admissible chains of i-boxes, the monoidal seed $\mathsf{M}(\mathfrak{C}')$ is obtained from $\mathsf{M}(\mathfrak{C})$ by successive mutations. Note that $\mathsf{M}(\mathfrak{C})$ consists of KR-modules when $\underline{w}_0$ in Main Theorem 3 is *adapted* to $\mathscr{Q}$.

To prove Main Theorem 3, we first prove that the quiver $\mathrm{Q}_{\mathrm{GLS}}$ for $[-\infty, \infty]$ and $\mathscr{Q}$-adapted $\underline{w}_0$ coincides with the quiver Q_{HL} in [22] by analyzing a sequence $\big((\imath_k, p_k)\big)_{k\in\mathbb{Z}}$ in $I_{\mathrm{fin}} \times \mathbb{Z}$ associated with a Q-datum, called an *admissible sequence* (Proposition 7.27). Then we prove Main Theorem 3 for $\mathscr{C}_{\mathfrak{g}}^{-}$ by using the criterion established in [38]. Finally, we deduce the result on $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ for any interval $[a, b]$ and $\underline{w}_0$ from the one on $\mathscr{C}_{\mathfrak{g}}^{-}$ by developing various methods including mutation equivalence of Λ-admissible seeds.

We conjecture that Main Theorem 3 still holds when we weaken the complete PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ with $\mathcal{D} = \mathcal{D}_{\mathscr{Q}}$ in the statement to any PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ (see Conjecture 8.13).

Organization This paper is organized as follows. In Section 2, we give the necessary background on the quantum affine algebras, their representations, the invariants related to R-matrices and the root system associated with $\mathscr{C}_{\mathfrak{g}}$. In Section 3, we give the necessary background on the quiver Hecke algebras, the quantum unipotent coordinate rings and T-systems among the determinantial modules. In Section 4, we recall the quantum affine Schur-Weyl duality functor $\mathcal{F}$ and the results in [40]. Then we define the affine determinantial modules, study their commuting condition and prove T-system among affine determinantial modules. In Section 5, we develop the combinatorics by introducing the notions of admissible chain of i-boxes and box moves. In Section 6, we review the notion of Q-datum and introduce the notion of admissible sequence in $I_{\mathrm{fin}} \times \mathbb{Z}$. Investigating them, we show that Hernandez-Leclerc categories (A)–(D) and KR-modules are special cases of $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ and affine determinantial modules, respectively. In Section 7, we first review the cluster algebras and the criterion on monoidal categorification by a monoidal subcategory $\mathcal{C}$ of $\mathscr{C}_{\mathfrak{g}}^{0}$ established in [38]. Then we study the properties of monoidal seeds of several kinds and prove that the monoidal seed $\mathscr{S}$ in (1.4) is Λ-admissible. In the last part, we prove the $\mathrm{Q}_{\mathrm{GLS}}$ coincides with the quiver Q_{HL} under certain condition. In Section 8, we prove Main theorem 3.

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Convention. Throughout this paper, we keep the following conventions.

(1) For a statement $\mathtt{P}$, $\delta(\mathtt{P})$ is 1 or 0 according that $\mathtt{P}$ is true or not. In particular, we set $\delta_{i,j} := \delta(i = j)$ (Kronecker's delta).

(2) For a field $\mathbf{k}$, $a \in \mathbf{k}$ and $f(z) \in \mathbf{k}(z)$, we denote by $\mathrm{zero}_{z=a} f(z)$ the order of zero of $f(z)$ at $z = a$.
(3) For $k, l \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geqslant 1}$, we write $k \equiv_s l$ if s divides $k - l$ and $k \not\equiv_s l$, otherwise.
(4) For an object M of an abelian category with finite length, we denote by $\mathrm{hd}(M)$ the head of M and by $\mathrm{soc}(M)$ the socle of M.
(5) For a finite set A, we denote by $|A|$ the number of elements in A.
(6) $\mathrm{ord}(\sigma)$ denotes the order of σ for an element σ of a finite group.
(7) For vertices i, j in a simply-laced Dynkin diagram, $d(i, j)$ denotes the number of edges between i and j.
(8) $\mathfrak{S}_n$ stands for the symmetric group of degree n.

2. Review on Quantum affine algebras

In this section, we will briefly review the definition of quantum affine algebras and their representation theory. Then, we will recall the invariants related to R-matrices which were recently introduced in [38]. We refer to [38] for more details.

2.1. **Quantum affine algebras.** Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be an *affine Cartan datum* consisting of an *affine Cartan matrix* $A = (\mathrm{a}_{ij})_{i,j \in I}$ with a finite index set I, a *weight lattice* P, a set of *simple roots* $\Pi = \{\alpha_i \mid i \in I\} \subset P$, a *coweight lattice* $P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ and a set of *simple coroots* $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$. We have $\langle h_i, \alpha_j \rangle = \mathrm{a}_{ij}$ for all $i, j \in I$ where $\langle\ ,\ \rangle \colon P^\vee \times P \to \mathbb{Z}$ is the canonical pairing. We choose $\{\Lambda_i\}_{i \in I}$ such that $\langle h_j, \Lambda_i \rangle = \delta_{i,j}$ for $i, j \in I$ and call them the *fundamental weights*.

We also take the *imaginary root* $\delta = \sum_{i \in I} \mathsf{u}_i \alpha_i$ and the *center* $c = \sum_{i \in I} \mathsf{c}_i h_i$ such that $\{\lambda \in \bigoplus_{i \in I} \mathbb{Z}\alpha_i \mid \langle h_i, \lambda \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}\delta$ and $\{h \in \bigoplus_{i \in I} \mathbb{Z}h_i \mid \langle h, \alpha_i \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}c$ (see [27, Chapter 4]). We set $P_{\mathrm{cl}} := P/(P \cap \mathbb{Q}\delta)$ and call it the *classical weight lattice*. We choose $\rho \in P$ (resp. $\rho^\vee \in P^\vee$) such that $\langle h_i, \rho \rangle = 1$ (resp. $\langle \rho^\vee, \alpha_i \rangle = 1$) for all $i \in I$. Set $\mathfrak{h} := \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$. Then there exists a non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ satisfying

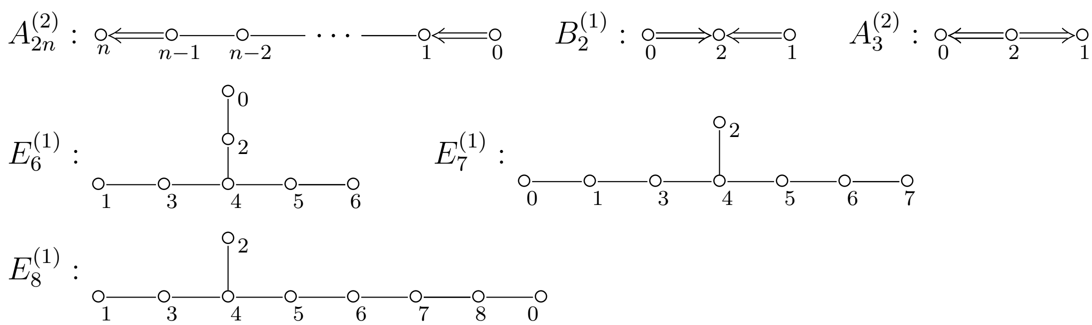
$$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad \langle c, \lambda \rangle = (\delta, \lambda) \quad \text{for any } i \in I \text{ and } \lambda \in \mathfrak{h}^*.$$

We denote by $\mathfrak{g}$ the *affine Kac-Moody algebra* associated with $(A, P, \Pi, P^\vee, \Pi^\vee)$ and by $W := \langle s_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$ the *Weyl group* of $\mathfrak{g}$ where

$$s_i \lambda = \lambda - \langle h_i, \lambda \rangle \alpha_i \quad \text{for } \lambda \in P.$$

We will use the convention in [27] to choose $0 \in I$ except $A^{(2)}_{2n}$-type, in which case we take the longest simple root as α_0, and $B^{(1)}_2$, $A^{(2)}_3$, and $E^{(1)}_k$ ($k = 6, 7, 8$) types, in which we take the following Dynkin diagrams $\triangle_{\mathfrak{g}}$:

(2.1)



We define $\mathfrak{g}_0$ to be the subalgebra of $\mathfrak{g}$ generated by the *Chevalley generators* e_i, f_i and h_i for $i \in I_0 := I \setminus \{0\}$ and W_0 to be the subgroup of W generated by s_i for $i \in I_0$. Note that $\mathfrak{g}_0$ is a finite-dimensional simple Lie algebra and W_0 contains the longest element w_0.

Let q be an indeterminate and let $\mathbf{k}$ be the algebraic closure of the subfield $\mathbb{C}(q)$ in the algebraically closed field $\widehat{\mathbf{k}} := \bigcup_{m>0} \mathbb{C}((q^{1/m}))$. For $i \in I$, we set $q_i = q^{(\alpha_i,\alpha_i)/2}$.

Let us denote by $U_q'(\mathfrak{g})$ the *quantum affine algebra* associated with an affine Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ generated by $e_i, f_i, q_i^{\pm h_i}$ ($i \in I$) over $\mathbf{k}$.

2.2. **Finite-dimensional representations.** We say that a module M over $U_q'(\mathfrak{g})$ is *integrable* if (i) M decomposes into $P_{\rm cl}$-weight spaces; i.e., $M = \bigoplus_{\lambda \in P_{\rm cl}} M_\lambda$ where $M_\lambda = \{u \in M \mid q_i^{h_i} u = q_i^{\langle h_i,\lambda\rangle}\}$, and (ii) the action of e_i and f_i on M is nilpotent ($i \in I$). We denote by $\mathscr{C}_{\mathfrak{g}}$ the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules.

A simple module M in $\mathscr{C}_{\mathfrak{g}}$ contains a non-zero vector u of weight $\lambda \in P_{\rm cl}$ such that (i) $\langle h_i, \lambda\rangle \geqslant 0$ for all $i \in I_0$, (ii) all the weights of M are contained in $\lambda - \sum_{i \in I_0} \mathbb{Z}_{\geqslant 0} {\rm cl}(\alpha_i)$, where ${\rm cl}\colon P \to P_{\rm cl}$ denotes the canonical projection. Such a λ is unique and u is unique up to a constant multiple. We call λ the *dominant extremal weight* of M and u a *dominant extremal weight vector* of M.

For an indeterminate z and a $U_q'(\mathfrak{g})$-module M, let us denote by M_z the $U_q'(\mathfrak{g})$-module $\mathbf{k}[z^{\pm1}] \otimes M$ defined by

$$e_i(u_z) = z^{\delta_{i,0}}(e_i u)_z, \quad f_i(u_z) = z^{-\delta_{i,0}}(f_i u)_z, \quad q_i^{h_i}(u_z) = (q_i^{h_i} u)_z,$$

where u_z denotes $1 \otimes z$ for $u \in M$.

For $x \in \mathbf{k}^\times$, we define

$$M_x := M_z/(z - x)M_z.$$

We call x a *spectral parameter*. Note that, for a module M in $\mathscr{C}_{\mathfrak{g}}$ and $x \in \mathbf{k}^\times$, M_x is also contained in $\mathscr{C}_{\mathfrak{g}}$. The functor T_x defined by $\mathsf{T}_x(M) = M_x$ is an autofunctor of $\mathscr{C}_{\mathfrak{g}}$ which commutes with tensor products.

For each $i \in I_0$, we set

$$\varpi_i := \gcd(\mathsf{c}_0, \mathsf{c}_i)^{-1} \mathrm{cl}(\mathsf{c}_0 \Lambda_i - \mathsf{c}_i \Lambda_0) \in P_{\mathrm{cl}}.$$

Then there exists a unique simple module $V(\varpi_i)$ in $\mathscr{C}_{\mathfrak{g}}$, called the *fundamental module* of weight ϖ_i, satisfying certain conditions (see, e.g, [34, §5.2]).

For a $U_q'(\mathfrak{g})$-module M, we denote by $\overline{M} = \{\bar{u} \mid u \in M\}$ the $U_q'(\mathfrak{g})$-module whose module structure is given as $x\bar{u} := \overline{\overline{x}u}$ for $x \in U_q'(\mathfrak{g})$. Here $^-$ is the ring automorphism of $U_q'(\mathfrak{g})$ such that $\overline{q} = q^{-1}$, $\overline{e_i} = e_i$, $\overline{f_i} = f_i$ and $\overline{q^h} = q^{-h}$.

Then we have

$$\overline{M_a} \simeq (\overline{M})_{\overline{a}}, \qquad \overline{M \otimes N} \simeq \overline{N} \otimes \overline{M}.$$

Set $\pi_i = \gcd(\mathsf{c}_0, \mathsf{c}_i)^{-1}(\mathsf{c}_0 \Lambda_i - \mathsf{c}_i \Lambda_0) \in P$. Let m_i be a positive integer such that

$$W\pi_i \cap \left(\pi_i + \mathbb{Z}\delta\right) = \pi_i + \mathbb{Z}m_i\delta.$$

Note that $m_i = (\alpha_i, \alpha_i)/2$ in the case where $\mathfrak{g}$ is the dual of an untwisted affine algebra, and $m_i = 1$ otherwise. Then, for $x, y \in \mathbf{k}^\times$, we have $V(\varpi_i)_x \simeq V(\varpi_i)_y$ if and only if $x^{m_i} = y^{m_i}$ ([1, §1.3]).

For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we say that M and N *commute* if $M \otimes N \simeq N \otimes M$. We say that they *strongly commute* (or M *strongly commutes with* N) if $M \otimes N$ is simple. Note that M and N commute as soon as they strongly commute. We say that a simple module L in $\mathscr{C}_{\mathfrak{g}}$ is *real* if L strongly commutes with itself, *i.e.*, if $L \otimes L$ is simple.

Recall that the simple objects in $\mathscr{C}_{\mathfrak{g}}$ are parameterized by I_0-tuples of polynomials $\mathcal{P} = \{\mathcal{P}_i(u)\}_{i \in I_0}$, where $\mathcal{P}_i(u) \in \mathbf{k}[u]$ and $\mathcal{P}_i(0) = 1$ ([6, 7], see also [29, (1.5)] for the choice of a function $I_0 \to \{\pm 1\}$). We denote by $\mathcal{P}_i^V(u)$ the polynomials corresponding to a simple module V, and call $\mathcal{P}_i^V(u)$ the *Drinfel'd polynomials* of V. The Drinfel'd polynomials $\mathcal{P}_i^V(u)$ are determined by the eigenvalues of the simultaneously commuting actions of some *Drinfel'd generators* of $U_q'(\mathfrak{g})$ on a subspace of V (see, e.g., [6] for more details). Note that $\mathcal{P}_i^{V_x}(u) = \mathcal{P}_i^V(x^{m_i}u)$ for any $x \in \mathbf{k}^\times$.

The *Kirillov-Reshetikhin* (KR) *module*, usually denoted by $W^{(k)}_{m,a}$ for $k \in I_0$, $a \in \mathbf{k}^\times$ and $m \geqslant 1$, is a simple module of dominant extremal weight $m\varpi_k$ in $\mathscr{C}_{\mathfrak{g}}$ whose Drinfel'd polynomials $\mathcal{P} = \{\mathcal{P}_i\}_{i \in I_0}$ are (see [19, 51, 58] and [53, Remark 3.3.1] for more detail)

$$\mathcal{P}_i(u) = \delta_{i,k}(1 - au)(1 - a\check{q}_k{}^2 u)(1 - a\check{q}_k{}^4 u) \cdots (1 - a\check{q}_k{}^{2m-2} u) + (1 - \delta_{i,k}) \quad (i \in I_0),$$

where

$$\check{q}_k = \begin{cases} q_k & \text{unless } \mathfrak{g} = A_{2n}^{(2)} \text{ and } k = n, \\ q & \text{if } \mathfrak{g} = A_{2n}^{(2)} \text{ and } k = n. \end{cases}$$

Then the Kirillov-Reshetikhin module $V(i^m) := W^{(i)}_{m,(-\check{q}_i)^{1-m}}$ is simple and bar-invariant. With the terminology of $V(\varpi_i)_z$, the module $V(i^m)$ can be described as follows:

$$V(i^m) \simeq \begin{cases} \operatorname{hd}\big(V(\varpi_i)_{(-q_i)^{m-1}} \otimes V(\varpi_i)_{(-q_i)^{m-3}} \otimes \cdots \otimes V(\varpi_i)_{(-q_i)^{1-m}}\big) & \text{if } \mathfrak{g} \text{ is untwisted,} \\ \operatorname{hd}\big(V(\varpi_i)_{(-q)^{m-1}} \otimes V(\varpi_i)_{(-q)^{m-3}} \otimes \cdots \otimes V(\varpi_i)_{(-q)^{1-m}}\big) & \text{otherwise,} \end{cases} \tag{2.2}$$

for $i \in I_0$ and $m \in \mathbb{Z}_{\geqslant 1}$ (see [19]).

Note that the category $\mathscr{C}_\mathfrak{g}$ is *rigid*, i.e., every module M in $\mathscr{C}_\mathfrak{g}$ has the right dual $\mathscr{D}M$ and the left dual $\mathscr{D}^{-1}M$. Hence we have the evaluation morphisms

$$M \otimes \mathscr{D}M \to \mathbf{1} \quad \text{and} \quad \mathscr{D}^{-1}M \otimes M \to \mathbf{1}.$$

We extend this to $\mathscr{D}^k$ for $k \in \mathbb{Z}$. In particular, the duals of $V(i^m)_x$ ($x \in \mathbf{k}^\times$) are given as follows:

$$\mathscr{D}\big(V(i^m)_x\big) \simeq V((i^*)^m)_{p^*x} \quad \text{and} \quad \mathscr{D}^{-1}\big(V(i^m)_x\big) \simeq V((i^*)^m)_{(p^*)^{-1}x}.$$

Here $p^* := (-1)^{\langle \rho^\vee, \delta \rangle} q^{\langle c, \rho \rangle}$ and $i^* \in I_0$ is defined by $\alpha_{i^*} = -w_0\, \alpha_i$.

We set

$$\sigma(\mathfrak{g}) := I_0 \times \mathbf{k}^\times / \sim \tag{2.3}$$

where the equivalence relation is given by

$$(i,x) \sim (j,y) \iff V(\varpi_i)_x \simeq V(\varpi_i)_y.$$

We denote by $[(i,a)]$ the equivalence class of (i,a) in $\sigma(\mathfrak{g})$. When no confusion arises, we simply write (i,a) for the equivalence class $[(i,a)]$.

The set $\sigma(\mathfrak{g})$ has a graph structure: we join (i,a) and (j,b) when $V(i)_a$ and $V(j)_b$ do not commute. We choose a connected component $\sigma_0(\mathfrak{g})$ of $\sigma(\mathfrak{g})$.

We denote by $\mathscr{C}^0_\mathfrak{g}$ the full smallest full subcategory of $\mathscr{C}_\mathfrak{g}$ such that it contains $V(\varpi_i)_a$ ($(i,a) \in \sigma_0(\mathfrak{g})$) and is stable under taking subquotients, extensions and tensor products (see [20, 36] and Definition 6.16 below).

2.3. **R-matrices and invariants.** For modules M and N in $\mathscr{C}_\mathfrak{g}$, there exists a $\mathbf{k}((z)) \otimes U_q'(\mathfrak{g})$-module isomorphism, denoted by $\mathrm{R}^{\mathrm{univ}}_{M \otimes N_z}$ and called the *universal R-matrix* of M and N:

$$\mathrm{R}^{\mathrm{univ}}_{M,N_z} \colon \mathbf{k}((z)) \underset{\mathbf{k}[z^{\pm1}]}{\otimes} (M \otimes N_z) \longrightarrow \mathbf{k}((z)) \underset{\mathbf{k}[z^{\pm1}]}{\otimes} (N_z \otimes M)$$

satisfying certain properties (see [34] and [1, Appendices A and B]).

For non-zero modules M and N in $\mathscr{C}_\mathfrak{g}$, if there exists $f(z) \in \mathbf{k}((z))^\times$ such that

$$f(z)\mathrm{R}^{\mathrm{univ}}_{M,N_z}(M \otimes N_z) \subset N_z \otimes M,$$

then we say that $\mathrm{R}^{\mathrm{univ}}_{M,N_z}$ is *rationally renormalizable*. In the rationally renormalizable case, one can choose $c_{M,N}(z) \in \mathbf{k}((z))^\times$ as $f(z)$ such that, for any $x \in \mathbf{k}^\times$, the specialization of $\mathrm{R}^{\mathrm{ren}}_{M,N_z} := c_{M,N}(z)\mathrm{R}^{\mathrm{univ}}_{M,N_z} \colon M \otimes N_z \to N_z \otimes M$ at $z = x$

$$\mathrm{R}^{\mathrm{ren}}_{M,N_z}\Big|_{z=x} \colon M \otimes N_x \to N_x \otimes M$$

does not vanish. Such $\mathrm{R}^{\mathrm{ren}}_{M,N_z}$ and $c_{M,N}(z)$ are unique up to a multiple of $\mathbf{k}[z^{\pm1}]^\times = \bigsqcup_{n\in\mathbb{Z}} \mathbf{k}^\times z^n$. We call $c_{M,N}(z)$ the *renormalizing coefficient*. We write $\mathbf{r}_{M,N} := \mathrm{R}^{\mathrm{ren}}_{M,N_z}\Big|_{z=1}$ and call it *R-matrix*. The R-matrix $\mathbf{r}_{M,N}$ is well-defined up to a constant multiple when $\mathrm{R}^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. By the definition, $\mathbf{r}_{M,N}$ never vanishes.

For simple modules M and N in $\mathscr{C}_\mathfrak{g}$, let u and v be dominant extremal weight vectors of M and N, respectively. Then there exists $a_{M,N}(z) \in \mathbf{k}((z))^\times$ such that

$$\mathrm{R}^{\mathrm{univ}}_{M,N_z}(u \otimes v_z) = a_{M,N}(z)(v_z \otimes u).$$

Then $\mathrm{R}^{\mathrm{norm}}_{M,N_z} := a_{M,N}(z)^{-1}\mathrm{R}^{\mathrm{univ}}_{M,N_z}\Big|_{\mathbf{k}(z)\otimes_{\mathbf{k}[z^{\pm1}]}(M\otimes N_z)}$ induces a unique $\mathbf{k}(z) \otimes U_q'(\mathfrak{g})$-module isomorphism

$$\mathrm{R}^{\mathrm{norm}}_{M,N_z} \colon \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm1}]} \big(M \otimes N_z\big) \xrightarrow{\ \sim\ } \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm1}]} \big(N_z \otimes M\big)$$

sending $u \otimes v_z$ to $v_z \otimes u$. Hence, the universal R-matrix $\mathrm{R}^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. We call $a_{M,N}(z)$ the *universal coefficient* of M and N, and $\mathrm{R}^{\mathrm{norm}}_{M,N_z}$ the *normalized R-matrix*. Note that $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm1}]} (M \otimes N_z)$ is a simple $\mathbf{k}(z) \otimes U_q'(\mathfrak{g})$-module by [34, Proposition 9.5].

Let $d_{M,N}(z) \in \mathbf{k}[z]$ be a monic polynomial of the smallest degree such that the image of $d_{M,N}(z)\mathrm{R}^{\mathrm{norm}}_{M,N_z}(M \otimes N_z)$ is contained in $N_z \otimes M$. We call $d_{M,N}(z)$ the *denominator of* $\mathrm{R}^{\mathrm{norm}}_{M,N_z}$. Then we have

$$\mathrm{R}^{\mathrm{ren}}_{M,N_z} = d_{M,N}(z)\mathrm{R}^{\mathrm{norm}}_{M,N_z} \colon M \otimes N_z \longrightarrow N_z \otimes M \quad \text{up to a multiple of } \mathbf{k}[z^{\pm1}]^\times.$$

Hence, we have

$$\mathrm{R}^{\mathrm{ren}}_{M,N_z} = a_{M,N}(z)^{-1} d_{M,N}(z) \mathrm{R}^{\mathrm{univ}}_{M,N_z} \quad \text{and} \quad c_{M,N}(z) = \frac{d_{M,N}(z)}{a_{M,N}(z)}$$

up to a multiple of $\mathbf{k}[z^{\pm 1}]^\times$.

Note that for simple modules M, N, we have

$$\mathrm{Hom}_{\mathbf{k}[z^{\pm1}]\otimes U_q'(\mathfrak{g})}(M \otimes N_z, N_z \otimes M) = \mathbf{k}[z^{\pm1}]\mathrm{R}^{\mathrm{ren}}_{M,N_z}.$$

Similarly, there exists a $\mathbf{k}[z^{\pm1}] \otimes U_q'(\mathfrak{g})$-linear homomorphism $\mathrm{R}^{\mathrm{ren}}_{M_z,N} \colon M_z \otimes N \longrightarrow N \otimes M_z$ such that

$$\mathrm{Hom}_{\mathbf{k}[z^{\pm1}]\otimes U_q'(\mathfrak{g})}(M_z \otimes N, N \otimes M_z) = \mathbf{k}[z^{\pm1}]\mathrm{R}^{\mathrm{ren}}_{M_z,N}. \tag{2.4}$$

Remark 2.1.

(a) The denominator formulas and universal coefficients were studied and computed between fundamental modules in [1, 8, 11, 29, 54, 55], and between KR-modules in [56].

(b) For (i,x) and $(j,y) \in \sigma(\mathfrak{g})$, we put d many arrows from (i,x) to (j,y), where d is the order of zeros of $d_{V(\varpi_i),V(\varpi_j)}(z_{V(\varpi_j)}/z_{V(\varpi_i)})$ at $z_{V(\varpi_j)}/z_{V(\varpi_i)} = y/x$. Thus, $\sigma(\mathfrak{g})$ has a quiver structure.

We set

$$\widetilde{p} := p^{*2} = q^{2\langle c,\rho\rangle} \quad \text{and} \quad \varphi(z) := \prod_{s\in\mathbb{Z}_{\geqslant 0}} (1-\widetilde{p}^{\,s}z) = \sum_{n=0}^{\infty} \frac{(-1)^n \widetilde{p}^{\,n(n-1)/2}}{\prod_{k=1}^{n}(1-\widetilde{p}^{\,k})}\, z^n \in \mathbf{k}[[z]].$$

Definition 2.2. We define the subset $\mathcal{G}$ of $\mathbf{k}((z))^\times$ as follows:

$$\mathcal{G} := \left\{ cz^m \prod_{a\in\mathbf{k}^\times} \varphi(az)^{\eta_a} \;\middle|\; \begin{array}{l} c\in\mathbf{k}^\times,\ m\in\mathbb{Z}\ , \\ \eta_a\in\mathbb{Z} \text{ vanishes except finitely many } a\text{'s} \end{array} \right\}.$$

Note that $\mathcal{G}$ forms a group with respect to the multiplication and $\mathbf{k}(z)^\times \subset \mathcal{G}$.

Proposition 2.3 ([38, Proposition 3.2])**.** *Let M and N be non-zero modules in $\mathscr{C}_\mathfrak{g}$.*

(i) *If $\mathrm{R}^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable, then $c_{M,N}(z)$ belongs to $\mathcal{G}$.*

(ii) *If M and N are simple, then $a_{M,N}(z)$ as well as $c_{M,N}(z)$ is contained in $\mathcal{G}$.*

For a subset S of $\mathbb{Z}$, we set

$$\widetilde{p}^{\,S} := \{\widetilde{p}^{\,k} \mid k \in S\}.$$

By taking $S = \mathbb{Z}$ or $\mathbb{Z}_{\leqslant 0}$, the following group homomorphisms from $\mathcal{G}$ to $\mathbb{Z}$ were introduced in [38, Section 3]:

$$\mathrm{Deg}\colon \mathcal{G} \to \mathbb{Z} \quad \text{and} \quad \mathrm{Deg}^\infty\colon \mathcal{G} \to \mathbb{Z},$$

which are defined by

$$\mathrm{Deg}(f(z)) = \sum_{a \in \widetilde{p}^{\,\mathbb{Z}_{\leqslant 0}}} \eta_a - \sum_{a \in \widetilde{p}^{\,\mathbb{Z}_{>0}}} \eta_a \quad \text{and} \quad \mathrm{Deg}^\infty(f(z)) = \sum_{a \in \widetilde{p}^{\,\mathbb{Z}}} \eta_a$$

for $f(z) = cz^m \prod \varphi(az)^{\eta_a} \in \mathcal{G}$.

Note that

$$\mathrm{Deg}(f(z)) = 2\,\mathrm{zero}_{z=1} f(z) \quad \text{for } f(z) \in \mathbf{k}(z)^\times \subset \mathcal{G} \tag{2.5}$$

(see [38, Lemma 3.4]).

Definition 2.4 ([38, Definition 3.6, Definition 3.14])**.** Let $M, N \in \mathscr{C}_\mathfrak{g}$.

(1) If $\mathrm{R}^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable, we define the integers $\Lambda(M,N)$ and $\Lambda^\infty(M,N)$ as follows:

$$\Lambda(M,N) = \mathrm{Deg}(c_{M,N}(z)) \quad \text{and} \quad \Lambda^\infty(M,N) = \mathrm{Deg}^\infty(c_{M,N}(z)).$$

(2) For simple modules M and N in $\mathscr{C}_\mathfrak{g}$, we define $\mathfrak{d}(M,N)$ by

$$\mathfrak{d}(M,N) = \frac{1}{2}\big(\Lambda(M,N) + \Lambda(\mathscr{D}^{-1}M, N)\big).$$

Proposition 2.5 ([38, Proposition 3.16, Corollary 3.19])**.** *For simple modules M and N in $\mathscr{C}_\mathfrak{g}$, we have*

$$\mathfrak{d}(M,N) = \mathrm{zero}_{z=1}\big(d_{M,N}(z) d_{N,M}(z^{-1})\big). \tag{2.6}$$

In particular, we have

$$\mathfrak{d}(M,N) \in \mathbb{Z}_{\geqslant 0} \quad \textit{and} \quad \mathfrak{d}(M,N) = \frac{1}{2}\big(\Lambda(M,N) + \Lambda(N,M)\big) = \mathfrak{d}(N,M).$$

Corollary 2.6 ([38, Corollary 3.17])**.** Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$. Assume that one of them is real. Then M and N strongly commute if and only if $\mathfrak{d}(M,N) = 0$.

Interestingly, the invariants Λ and Λ^∞ are calculated by $\mathfrak{d}$ as follows:

Proposition 2.7 ([38, Proposition 3. 22], [40, Proposition 2. 16])**.** *For simple modules M and N in $\mathscr{C}_\mathfrak{g}$, we have the followings:*

(i) $\Lambda(M,N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k<0)}\, \mathfrak{d}(M, \mathscr{D}^k N)$,

(ii) $\Lambda^\infty(M,N) = \sum_{k\in\mathbb{Z}} (-1)^k \,\mathfrak{d}(M, \mathscr{D}^k N)$,

(iii) $\operatorname{zero}_{z=1} c_{M,N}(z) = \sum_{k=0}^{\infty} (-1)^k \,\mathfrak{d}(M, \mathscr{D}^k N)$.

Proposition 2.8 ([38, Proposition 3.18, Corollary 3.20])**.** *For simple modules M and N in $\mathscr{C}_\mathfrak{g}$, we have the followings:*

(i) $\Lambda(M,N) = \Lambda(\mathscr{D}^{-1}N, M) = \Lambda(N, \mathscr{D}M)$.

(ii) *If M is real, then we have $\Lambda(M,M) = 0$.*

We conjecture that $\Lambda(M,M) = 0$ holds for an arbitrary simple module M.

Proposition 2.9 ([30, Corollary 3.11], [38, Proposition 2.11])**.**

(i) *Let M_k be a module in $\mathscr{C}_\mathfrak{g}$ $(k = 1,2,3)$, and let $\varphi_1\colon L \to M_2 \otimes M_3$ and $\varphi_2\colon M_1 \otimes M_2 \to L'$ be* non-zero *morphisms. Assume further that M_2 is a simple module. Then the composition*

$$M_1 \otimes L \xrightarrow{M_1 \otimes \varphi_1} M_1 \otimes M_2 \otimes M_3 \xrightarrow{\varphi_2 \otimes M_3} L' \otimes M_3$$

does not vanish.

(ii) *Let M, N_1 and N_2 be non-zero modules in $\mathscr{C}_\mathfrak{g}$, and assume that $\mathrm{R}^{\mathrm{univ}}_{N_k, M_z}$ is rationally renormalizable for $k = 1, 2$. Then $\mathrm{R}^{\mathrm{univ}}_{N_1 \otimes N_2, M_z}$ is rationally renormalizable, and we have*

$$\frac{c_{N_1,M}(z) c_{N_2,M}(z)}{c_{N_1 \otimes N_2, M}(z)} \in \mathbf{k}[z^{\pm 1}].$$

If we assume further that M is simple, then we have

$$c_{N_1 \otimes N_2, M}(z) \equiv c_{N_1,M}(z) c_{N_2,M}(z) \mod \mathbf{k}[z^{\pm1}]^\times$$

and the following diagram commutes up to a constant multiple:

$$\begin{tikzcd} N_1 \otimes N_2 \otimes M \arrow[r, "N_1 \otimes \mathbf{r}_{N_2,M}"'] \arrow[rr, bend left, "\mathbf{r}_{N_1 \otimes N_2,\, M}"] & N_1 \otimes M \otimes N_2 \arrow[r, "\mathbf{r}_{N_1,M} \otimes N_2"'] & M \otimes N_1 \otimes N_2. \end{tikzcd}$$

Theorem 2.10 ([30])**.** *Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$ and assume that one of them is real. Then*

(a) $\operatorname{Hom}(M \otimes N, N \otimes M) = \mathbf{k}\, \mathbf{r}_{M,N}$.

(b) *$M \otimes N$ and $N \otimes M$ have simple socles and simple heads.*

(c) *Moreover, $\operatorname{Im}(\mathbf{r}_{M,N})$ is isomorphic to the head of $M \otimes N$ and the socle of $N \otimes M$.*

(d) *$M \otimes N$ is simple whenever its head and its socle are isomorphic.*

For modules M and N in $\mathscr{C}_\mathfrak{g}$, we denote by $M \nabla N$ and $M \Delta N$ the head and the socle of $M \otimes N$, respectively.

Proposition 2.11 ([33, Proposition 3.2.17] (see also [42, Lemma 7.3]))**.** *Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$. Assume that one of them is real and $\mathfrak{d}(M,N) = 1$. Then we have an exact sequence*

$$0 \to M \Delta N \to M \otimes N \to M \nabla N \to 0.$$

In particular, $M \otimes N$ has composition length 2.

Definition 2.12 (cf. [35, Definition 2.5])**.** A sequence $(L_1, \ldots, L_r)$ of real simple modules in $\mathscr{C}_\mathfrak{g}$ is called a *normal sequence* if the composition of the R-matrices

$$\mathbf{r}_{L_1,\ldots,L_r} := \prod_{1\leqslant i<k\leqslant r} \mathbf{r}_{L_i,L_k} = (\mathbf{r}_{L_{r-1},L_r}) \circ \cdots \circ (\mathbf{r}_{L_2,L_r} \circ \cdots \circ \mathbf{r}_{L_2,L_3}) \circ (\mathbf{r}_{L_1,L_r} \circ \cdots \circ \mathbf{r}_{L_1,L_2})$$
$$: L_1 \otimes \cdots \otimes L_r \longrightarrow L_r \otimes \cdots \otimes L_1$$

does not vanish.

Lemma 2.13 ([38, Lemma 4.15, Lemma 4.16])**.** *Let $(L_1, \ldots, L_r)$ be a normal sequence of real simple modules in $\mathscr{C}_\mathfrak{g}$. Then the image of $\mathbf{r}_{L_1,\ldots,L_r}$ is simple and coincides with the head of $L_1 \otimes \cdots \otimes L_r$ and also with the socle of $L_r \otimes \cdots \otimes L_1$. Moreover we have*

(i) *$(L_2, \ldots, L_r)$ is a normal sequence and $\Lambda\big(L_1, \mathrm{Im}(\mathbf{r}_{L_2,\ldots,L_r})\big) = \sum_{k=2}^{r} \Lambda(L_1, L_k)$,*

(ii) *$(L_1, \ldots, L_{r-1})$ is a normal sequence and $\Lambda\big(\mathrm{Im}(\mathbf{r}_{L_1,\ldots,L_{r-1}}), L_r\big) = \sum_{k=1}^{r-1} \Lambda(L_k, L_r)$.*

Lemma 2.14 ([38, Lemma 4.17])**.** *For real simple modules L, M and N in $\mathscr{C}_\mathfrak{g}$, the triple (L, M, N) is a normal sequence if one of the following three conditions holds:*

(i) *L and M strongly commute,*
(ii) *M and N strongly commute,*
(iii) *L and $\mathscr{D}^{-1}N$ strongly commute.*

Definition 2.15 ([40, Definition 5.1])**.** Let (M, N) be an ordered pair of simple modules in $\mathscr{C}_\mathfrak{g}$. We call it *unmixed* if

$$\mathfrak{d}(\mathscr{D}M, N) = 0,$$

and *strongly unmixed* if

$$\mathfrak{d}(\mathscr{D}^k M, N) = 0 \quad \text{for any } k \in \mathbb{Z}_{\geqslant 1}.$$

Definition 2.16. A sequence $(M_s, M_{s-1}, \dots, M_1)$ of real simple modules over $U_q'(\mathfrak{g})$ is *(strongly) unmixed* if (M_k, M_i) is (strongly) unmixed for all $s \geqslant k > i \geqslant 1$.

Proposition 2.17 ([40, Lemma 5.2])**.** *For a strongly unmixed pair (M, N), we have*

$$\Lambda^\infty(M, N) = \Lambda(M, N) = \Lambda(N, \mathscr{D}M).$$

Proposition 2.18 ([40, Lemma 5.3])**.** *Any unmixed sequence of real simple modules is a normal sequence.*

Proposition 2.19 ([38, Proposition 4.2])**.** *For simple modules L, M and N, we have*

$$\mathfrak{d}(S, L) \leqslant \mathfrak{d}(M, L) + \mathfrak{d}(N, L)$$

for any simple subquotient S of $M \otimes N$.

Proposition 2.20. *Let $(M_k, M_{k-1}, \dots, M_1)$ be a strongly unmixed sequence. Then*

$$(\mathrm{hd}(M_k \otimes M_{k-1} \otimes \dots M_s), \mathrm{hd}(M_{s-1} \otimes \dots M_2 \otimes M_1))$$

is strongly unmixed for any $1 < s \leqslant k$.

We recall the following criteria of reality of simple modules.

Proposition 2.21 ([33, Proposition 3.2.20, Corollary 3.2.21], [38, Proposition 4.9])**.** *Let X, Y, M and N be simple modules in $\mathscr{C}_\mathfrak{g}$. Assume that there is an exact sequence*

$$0 \to X \to M \otimes N \to Y \to 0,$$

and $X \otimes N$ and $Y \otimes N$ are simple.

(i) *If $X \otimes N \not\simeq Y \otimes N$, then N is a real simple module.*
(ii) *If M is real, then N is a real simple module.*

Lemma 2.22 ([40, Lemma 2.27])**.** *Let M, N be a real simple module such that $\mathfrak{d}(M, N) \leqslant 1$. Then $M \nabla N$ is real.*

Remark that only the case $\mathfrak{d}(M, N) = 1$ is proved in [40, Lemma 2.27]. However, the assertion is obvious when M and N commute.

Lemma 2.23. *Let M and N be real simple modules. We assume that $M \nabla N$ commutes with M. Then $M \nabla N$ is real simple.*

Proof. We have a commutative diagram (up to constant multiples):

$$
\begin{tikzcd}
M\otimes N\otimes M\otimes N \arrow[d, two heads] & M\otimes M\otimes N\otimes N \arrow[l, "\mathbf{r}_{M,N}"'] \arrow[r, "\mathbf{r}_{M,N}"] \arrow[d, two heads] \arrow[dl, phantom, "\textcircled{A}"] & M\otimes N\otimes M\otimes N \arrow[d, two heads] \\
(M\nabla N)\otimes M\otimes N \arrow[r, "\sim", "\mathbf{r}_{M\nabla N,M}"'] \arrow[d, two heads] & M\otimes(M\nabla N)\otimes N \arrow[r, "\mathbf{r}_{M\nabla N,N}"'] \arrow[ur, tail] \arrow[d, phantom, "\textcircled{C}"] & M\otimes N\otimes(M\nabla N) \arrow[d, two heads] \arrow[u, phantom, "\textcircled{B}"] \\
(M\nabla N)\otimes(M\nabla N) \arrow[rr, "\mathbf{r}_{M\nabla N,M\nabla N}"'] & & (M\nabla N)\otimes(M\nabla N).
\end{tikzcd}
\tag{2.7}
$$

Here, the commutativity (up to a constant multiple) of Ⓐ follows from the fact that $M\otimes M\otimes N$ has a simple head $M\otimes(M\nabla N)$ and that the composition $M\otimes M\otimes N \xrightarrow{\mathbf{r}_{M,N}} M\otimes N\otimes M \longrightarrow (M\nabla N)\otimes M$ does not vanish by Proposition 2.9 (i). The commutativity (up to a constant multiple) of Ⓑ follows from the fact that the composition $(M\nabla N)\otimes N \rightarrowtail N\otimes M\otimes N \twoheadrightarrow N\otimes(M\nabla N)$ does not vanish by Proposition 2.9 (i) and that $\dim\mathrm{Hom}\big((M\nabla N)\otimes N, N\otimes(M\nabla N)\big)=1$ by Theorem 2.10 (a).

The commutativity (up to a constant multiple) of Ⓒ follows from the fact that the composition $(M\nabla N)\otimes M\otimes N \to M\otimes N\otimes(M\nabla N) \to (M\nabla N)\otimes(M\nabla N)$ does not vanish by Proposition 2.9 (i).

Thus we obtain the commutativity of the diagram (2.7).

The composition $M\otimes M\otimes N\otimes N \xrightarrow{\mathbf{r}_{M,N}} M\otimes N\otimes M\otimes N \longrightarrow (M\nabla N)\otimes(M\nabla N)$ is an epimorphism since it is the composition of the epimorphisms $M\otimes M\otimes N\otimes N \twoheadrightarrow (M\nabla N)\otimes M\otimes N$ and $(M\nabla N)\otimes M\otimes N \twoheadrightarrow (M\nabla N)\otimes(M\nabla N)$. It implies that $\mathbf{r}_{M\nabla N,M\nabla N} = \mathrm{id}_{(M\nabla N)\otimes(M\nabla N)}$ up to a constant multiple. □

Lemma 2.24. *Let L_j and M_j be real simple modules $(j=1,2)$. Assume that*

(i) *$L_j\nabla M_j$ commutes with L_k for $j,k=1,2$,*
(ii) *L_1 and L_2 commute.*

Then we have the followings:

(a) *$L_j\nabla M_j$ is real for $j=1,2$.*
(b) *If $\mathfrak{d}(\mathscr{D}L_j, M_2)=0$ for $j=1,2$, then*

$$(L_1\otimes L_2)\nabla(M_1\nabla M_2) \simeq \big((L_1\otimes L_2)\nabla M_1\big)\nabla M_2 \simeq (L_1\nabla M_1)\nabla(L_2\nabla M_2).$$

(c) *Assume that* $\mathfrak{d}(\mathscr{D}L_j, M_k) = 0$ *for* $j, k = 1, 2$. *Then* M_1 *and* M_2 *commute if and only if* $L_1 \nabla M_1$ *and* $L_2 \nabla M_2$ *commute.*

Proof. (a) follows from the preceding lemma.

(b) The first isomorphism follows from the fact that $(L_1 \otimes L_2, M_1, M_2)$ is normal.

On the other hand, $(L_2, L_1 \nabla M_1, M_2)$ is normal, and hence $L_2 \otimes (L_1 \nabla M_1) \otimes M_2$ has a simple head. Since we have epimorphisms

$$L_2 \otimes (L_1 \nabla M_1) \otimes M_2 \simeq (L_1 \nabla M_1) \otimes L_2 \otimes M_2 \twoheadrightarrow (L_1 \nabla M_1) \nabla (L_2 \nabla M_2) \quad \text{and}$$
$$L_2 \otimes (L_1 \nabla M_1) \otimes M_2 \simeq \big((L_2 \otimes L_1) \nabla M_1\big) \otimes M_2 \twoheadrightarrow \big((L_1 \otimes L_2) \nabla M_1\big) \nabla M_2,$$

we obtain

$$(L_1 \nabla M_1) \nabla (L_2 \nabla M_2) \simeq \big((L_1 \otimes L_2) \nabla M_1\big) \nabla M_2.$$

(c) Assume first that M_1 and M_2 commute. Then we have by (ii)

$$\begin{aligned}(L_1 \nabla M_1) \nabla (L_2 \nabla M_2) &\simeq (L_1 \otimes L_2) \nabla (M_1 \nabla M_2) \\ &\simeq (L_1 \otimes L_2) \nabla (M_2 \nabla M_1) \simeq (L_2 \nabla M_2) \nabla (L_1 \nabla M_1).\end{aligned}$$

Hence $L_1 \nabla M_1$ and $L_2 \nabla M_2$ commute.

Conversely, assume that $L_1 \nabla M_1$ and $L_2 \nabla M_2$ commute. Then we have

$$\begin{aligned}(L_1 \otimes L_2) \nabla (M_1 \nabla M_2) &\simeq (L_1 \nabla M_1) \nabla (L_2 \nabla M_2) \\ &\simeq (L_2 \nabla M_2) \nabla (L_1 \nabla M_1) \simeq (L_1 \otimes L_2) \nabla (M_2 \nabla M_1).\end{aligned}$$

Hence we have $M_1 \nabla M_2 \simeq M_2 \nabla M_1$, which implies that M_1 and M_2 commute. □

Proposition 2.25. *Let* M, N *and* L *be simple* $U_q'(\mathfrak{g})$*-modules such that* L *is real. Assume that*

(i) $\mathfrak{d}(\mathscr{D}M, L) = 0$, $\mathfrak{d}(\mathscr{D}L, N) = 0$, *and*
(ii) $M \otimes L \otimes N$ *has a simple head.*

Then we have

$$\mathfrak{d}(L, \operatorname{hd}(M \otimes L \otimes N)) = \mathfrak{d}(L, M \nabla L) + \mathfrak{d}(L, L \nabla N).$$

Proof. The condition $\mathfrak{d}(\mathscr{D}L, N) = 0$ implies that $(L, M \nabla L, N)$ is normal by Lemma 2.14. Thus we have

$$\begin{aligned}\Lambda(L, \operatorname{hd}(M \otimes L \otimes N)) &= \Lambda(L, (M \nabla L) \nabla N) \\ &= \Lambda(L, M \nabla L) + \Lambda(L, N) = \Lambda(L, M \nabla L) + \Lambda(L, L \nabla N)\end{aligned}$$

by Lemma 2.13. Similarly, $(M, L \nabla N, L)$ is normal and hence we have

$$\begin{aligned}\Lambda(\mathrm{hd}(M \otimes L \otimes N), L) &= \Lambda(M \nabla (L \nabla N), L) \\ &= \Lambda(M, L) + \Lambda(L \nabla N, L) = \Lambda(M \nabla L, L) + \Lambda(L \nabla N, L)\end{aligned}$$

by the same reason. Summing up two above equations, we have

$$\mathfrak{d}(L, \mathrm{hd}(M \otimes L \otimes N)) = \mathfrak{d}(L, M \nabla L) + \mathfrak{d}(L, L \nabla N). \qquad \square$$

Lemma 2.26 ([38, Proposition 3.11, Proposition 4.5]). *Let L, M, N be simple modules in $\mathscr{C}_\mathfrak{g}$ and let S be a simple quotient of $M \otimes N$. Then, $m := \Lambda(L, M) + \Lambda(L, N) - \Lambda(L, S)$ belongs to $\mathbb{Z}_{\geqslant 0}$ and the diagram*

$$\begin{tikzcd} L_z \otimes M \otimes N \arrow[r, "\mathrm{R}^{\mathrm{ren}}_{L_z, M}"] \arrow[d] & M \otimes L_z \otimes N \arrow[r, "\mathrm{R}^{\mathrm{ren}}_{L_z, N}"] & M \otimes N \otimes L_z \arrow[d] \\ L_z \otimes S \arrow[rr, "a(z)(z-1)^m \mathrm{R}^{\mathrm{ren}}_{L_z, S}"] & & S \otimes L_z \end{tikzcd}$$

commutes for some $a(z) \in \mathbf{k}(z)$ which has neither pole nor zero at $z = 1$.

Definition 2.27. A *root module* is a real simple module L such that

$$\mathfrak{d}(L, \mathscr{D}^k(L)) = \delta(k = \pm 1) \quad \text{for any } k \in \mathbb{Z}.$$

The following proposition is obtained by the explicit denominator formulas between fundamental representations (e.g., see [54, Appendix], [55] and [11]).

Proposition 2.28. *Every fundamental representation is a root module.*

2.4. **Simply-laced finite root system associated with $\mathscr{C}^0_\mathfrak{g}$.** In [39], we associate to the category $\mathscr{C}^0_\mathfrak{g}$ a simply-laced finite type root system in a canonical way: for a simple module $M \in \mathscr{C}_\mathfrak{g}$, set $\mathsf{E}(M) \in \mathrm{Hom}_{\mathrm{Set}}(\sigma(\mathfrak{g}), \mathbb{Z})$ (see (2.3)) by

$$\mathsf{E}(M)(i, a) = \Lambda^\infty(M, V(\varpi_i)_a) \qquad \text{for } (i, a) \in \sigma(\mathfrak{g}).$$

Let

$$\mathcal{W}_0 := \{\mathsf{E}(M) \mid M \text{ is simple in } \mathscr{C}^0_\mathfrak{g}\} \quad \text{and} \quad \Delta_0 := \{\mathsf{E}(V(\varpi_i)_a) \mid (i, a) \in \sigma_0(\mathfrak{g})\} \subset \mathcal{W}_0.$$

Then, $\mathcal{W}_0$ is a $\mathbb{Z}$-submodule of $\mathrm{Hom}_{\mathrm{Set}}(\sigma(\mathfrak{g}), \mathbb{Z})$ and endowed with a symmetric bilinear form $(\ ,\ )$ which satisfies

$$(\mathsf{E}(M), \mathsf{E}(N)) = -\Lambda^\infty(M, N) \text{ for any simple modules } M,\ N \text{ in } \mathscr{C}^0_\mathfrak{g}.$$

Theorem 2.29 ([39]). *The pair $(\mathbb{R} \otimes \mathcal{W}_0, \Delta_0)$ is a root system, the bilinear form $(\ ,\ )$ is positive definite and invariant by the Weyl group action, and we have $\Delta_0 = \{\alpha \in \mathcal{W}_0 \mid (\alpha, \alpha) = 2\}$.*

3. Review on Quiver Hecke algebras

In this section, we briefly recall the definition of quiver Hecke algebras R and the categorification of the quantum unipotent coordinate rings by R. Then we will review T-system among the determinantial modules.

3.1. **Symmetric quiver Hecke algebras.** In this subsection, we recall basic notions of symmetric quiver Hecke algebras associated to a finite simple Lie algebra g of simply-laced type. We denote by $\Pi_{\mathsf{g}} = \{\alpha_\imath \mid \imath \in I_{\mathsf{g}}\}$ the set of simple roots of g and by $(\ ,\ )$ the symmetric bilinear form on the root lattice Q of g invariant by the Weyl group action and normalized by $(\alpha_\imath, \alpha_\imath) = 2$. Set $\mathsf{Q}^+ = \sum_{\imath \in I_{\mathsf{g}}} \mathbb{Z}_{\geqslant 0}\alpha_\imath$.

Take a family of polynomials $(Q_{\imath\jmath})_{\imath,\jmath\in I_{\mathsf{g}}}$ in $\mathbf{k}[u,v]$ which are of the form

$$Q_{\imath\jmath}(u,v) = \pm\delta(\imath \neq \jmath)(u-v)^{-(\alpha_\imath,\alpha_\jmath)}$$

such that $Q_{\imath\jmath}(u,v) = Q_{\jmath\imath}(v,u)$.

For each $\beta \in \mathsf{Q}^+$ with $|\beta| = n$, we set $I^\beta_{\mathsf{g}} = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n_{\mathsf{g}} \mid \sum_{k=1}^n \alpha_{\nu_k} = \beta\}$. Note that the symmetric group $\mathfrak{S}_n = \langle \mathfrak{s}_k \mid 1 \leqslant k \leqslant n-1\rangle$ acts on I^β_{g} by place permutations.

The *symmetric quiver Hecke algebra* (also called *Khovanov-Lauda-Rouquier algebra*) $R(\beta)$ at $\beta \in \mathsf{Q}^+$ associated to g and $(Q_{\imath\jmath})_{\imath,\jmath\in I_{\mathsf{g}}}$, is the algebra over $\mathbf{k}$ generated by the elements $\{e(\nu)\}_{\nu\in I^\beta_{\mathsf{g}}}$, $\{x_k\}_{1\leqslant k\leqslant n}$ and $\{\tau_m\}_{1\leqslant m\leqslant n-1}$ satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^\beta_{\mathsf{g}}} e(\nu) = 1, \quad x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu)x_k,$$

$$\tau_m e(\nu) = e(\mathfrak{s}_m(\nu))\tau_m, \quad \tau_k\tau_m = \tau_m\tau_k \text{ if } |k-m|>1, \quad \tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_m - x_{\mathfrak{s}_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } m=k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } m = k+1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \delta_{\nu_k,\nu_{k+2}} \frac{Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k - x_{k+2}} e(\nu).$$

The algebra $R(\beta)$ is $\mathbb{Z}$-graded with

$$\deg e(\nu) = 0, \quad \deg x_k e(\nu) = 2, \quad \text{and} \quad \deg \tau_m e(\nu) = -(\alpha_{\nu_m}, \alpha_{\nu_{m+1}}).$$

Let us denote by $R(\beta)$-gMod the category of graded $R(\beta)$-modules with homomorphisms of degree 0, by $R(\beta)$-gmod the full subcategory of $R(\beta)$-gMod consisting of finite-dimensional graded $R(\beta)$-modules. For an $R(\beta)$-module M, we set $\mathrm{wt}(M) := -\beta \in \mathsf{Q}^-$.

For the sake of simplicity, we say that M is an R-module instead of saying that M is a graded $R(\beta)$-module. Let us denote by q the grading shift functor: for $M \in R\text{-gmod} := \bigoplus_{\beta \in \mathsf{Q}^+} R(\beta)\text{-gmod}$. We have $(qM)_n = M_{n-1}$ by definition.

For $\imath \in I_{\mathsf{g}}$, $L(\imath)$ denotes the 1-dimensional simple graded $R(\alpha_\imath)$-modules $\mathbf{k}u(\imath)$ with the action

$$x_1 u(\imath) = 0.$$

We also sometimes ignore grading shifts if there is no risk of confusion. Hence, for R-modules M and N, we sometimes say that $f\colon M \to N$ is a homomorphism if $f\colon q^a M \to N$ is a morphism in R-gmod for some $a \in \mathbb{Z}$. If we want to emphasize that $f\colon q^a M \to N$ is a morphism in R-gmod, we say so. We set

$$\mathrm{H{\scriptstyle OM}}_{R(\beta)}(M,N) := \bigoplus_{a \in \mathbb{Z}} \mathrm{Hom}_{R(\beta)\text{-gmod}}(q^a M, N).$$

For $\beta, \gamma \in \mathsf{Q}^+$, set $e(\beta,\gamma) := \sum_{\nu \in I_{\mathsf{g}}^{\beta},\, \nu' \in I_{\mathsf{g}}^{\gamma}} e(\nu * \nu') \in R(\beta+\gamma)$, where $\nu * \nu'$ denotes the concatenation of ν and ν'. Then there is a $\mathbf{k}$-algebra homomorphism $R(\beta) \otimes R(\gamma) \to e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma)$. The *convolution product* $- \circ -\colon R(\beta)\text{-gmod} \times R(\gamma)\text{-gmod} \to R(\beta+\gamma)\text{-gmod}$ is a bifunctor given by

$$M \circ N := R(\beta+\gamma)e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M \otimes N).$$

Set $R\text{-gmod} := \bigoplus_{\beta \in \mathsf{Q}^+} R(\beta)\text{-gmod}$. Then the category R-gmod is a monoidal category with the tensor product $\circ$ and the unit object $\mathbf{1} := \mathbf{k} \in R(0)$-gmod.

For $1 \leqslant a < |\beta|$, we define the *intertwiner* $\varphi_a \in R(\beta)$ by

$$\varphi_a e(\nu) = \begin{cases} \big(\tau_a x_a - x_a \tau_a\big) e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ \tau_a e(\nu) & \text{otherwise.} \end{cases}$$

For $m, n \in \mathbb{Z}_{\geqslant 0}$, let us denote by $w[m,n]$ the element of $\mathfrak{S}_{m+n}$ defined by

$$w[m,n](k) = \begin{cases} k+n & \text{if } 1 \leqslant k \leqslant m, \\ k-m & \text{if } m < k \leqslant m+n, \end{cases}$$

and set $\varphi_{[m,n]} := \varphi_{\imath_1} \cdots \varphi_{\imath_t}$, where $s_{\imath_1} \cdots s_{\imath_t}$ is a reduced expression of $w[m,n]$. It does not depend on the choice of a reduced expression.

Let M be an $R(\beta)$-module and N an $R(\gamma)$-module. Then the map $M \otimes N \to N \circ M$ given by $u \otimes v \longmapsto \varphi_{[|\gamma|,|\beta|]}(v \otimes u)$ induces an $R(\beta+\gamma)$-module homomorphism

$$R_{M,N} \colon M \circ N \longrightarrow N \circ M.$$

Let z be an indeterminate homogeneous of degree 2, and let ψ_z be the graded algebra homomorphism

$$\psi_z \colon R(\beta) \to \mathbf{k}[z] \otimes R(\beta)$$

given by

$$\psi_z(x_k) = x_k + z, \quad \psi_z(\tau_k) = \tau_k, \quad \psi_z(e(\nu)) = e(\nu).$$

For an $R(\beta)$-module M, we denote by M_z the $\big(\mathbf{k}[z] \otimes R(\beta)\big)$-module $\mathbf{k}[z] \otimes M$ with the action of $R(\beta)$ twisted by ψ_z.

For a non-zero $R(\beta)$-module M and a non-zero $R(\gamma)$-module N, let s be the largest non-negative integer such that the image of $R_{M_z,N}$ is contained in $z^s(N \circ M_z)$. We denote by $\mathrm{R}^{\mathrm{ren}}_{M_z,N}$ the $R(\beta)$-module homomorphism $z^{-s}R_{M_z,N}$ and call it the *renormalized R-matrix*. We denote by $\mathbf{r}_{M,N} := \mathrm{R}^{\mathrm{ren}}_{M_z,N}\big|_{z=0} \colon M \circ N \to N \circ M$ and call $\mathbf{r}_{M,N}$ the *R-matrix*. By the definition, $\mathbf{r}_{M,N}$ never vanishes.

We denote by $\Lambda(M,N)$ the homogeneous degree of $\mathbf{r}_{M,N}$ and set

$$\widetilde{\Lambda}(M,N) := \frac{1}{2}\big(\Lambda(M,N) + (\mathrm{wt}(M), \mathrm{wt}(N))\big), \quad \mathfrak{d}(M,N) := \frac{1}{2}\big(\Lambda(M,N) + \Lambda(N,M)\big).$$

The invariants Λ and $\mathfrak{d}$ enjoy the similar properties of Λ and $\mathfrak{d}$ for $\mathscr{C}_\mathfrak{g}$. For more details, we refer to [33, 38, 39].

Let $\beta \in \mathsf{Q}^+$. For $\imath \in I_\mathfrak{g}$, $1 \leqslant a \leqslant |\beta|$ and $M \in R(\beta)$-gmod, we set

$$E_\imath M := e_1(\imath)M \quad \text{and} \quad E^*_\imath M := e_{|\beta|}(\imath)M \quad \text{where } e_a(\imath) := \sum_{\nu \in I^\beta_\mathfrak{g}, \nu_a = \imath} e(\nu) \in R(\beta).$$

Then, $E_\imath$ and $E^*_\imath$ are functors from $R(\beta)$-gmod to $R(\beta - \alpha_\imath)$-gmod. For a non-zero module $M \in R(\beta)$-gmod, we also set

$$\varepsilon_\imath(M) = \max\{n \in \mathbb{Z}_{\geqslant 0} \mid E^n_\imath M \neq 0\} \in \mathbb{Z}_{\geqslant 0} \quad \text{and}$$
$$\varepsilon^*_\imath(M) = \max\{n \in \mathbb{Z}_{\geqslant 0} \mid E^{*n}_\imath M \neq 0\} \in \mathbb{Z}_{\geqslant 0}.$$

3.2. **Quantum unipotent coordinate ring and T-systems.** Let $U_q(\mathsf{g})$ be the quantum group associated to a finite simply-laced Cartan datum $(\mathsf{C}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ of g, which is the associative algebra over $\mathbb{Q}(q)$ generated by $e_\imath, f_\imath$ ($\imath \in I_\mathsf{g}$) and q^h ($h \in \mathsf{P}^\vee$).

Note that $U_q(\mathsf{g})$ admits a weight-space decomposition as follows:

$$U_q(\mathsf{g}) = \bigoplus_{\beta \in \mathsf{Q}} U_q(\mathsf{g})_\beta, \text{ where } U_q(\mathsf{g})_\beta := \left\{ x \in U_q(\mathsf{g}) \mid q^h x q^{-h} = q^{\langle h, \beta \rangle} x \text{ for any } h \in \mathsf{P}^\vee \right\}.$$

We denote by $U_q^+(\mathsf{g})$ the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathsf{g})$ generated by $\{e_\imath \mid \imath \in I_\mathsf{g}\}$. Let $U_q^+(\mathsf{g})_{\mathbb{Z}[q^{\pm 1}]}$ be the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $U_q^+(\mathsf{g})$ generated by $e_\imath^{(n)} := e_\imath^n/[n]!$ ($\imath \in I_\mathsf{g}$, $n \in \mathbb{Z}_{>0}$).

Let Δ_n be the algebra homomorphism $U_q^+(\mathsf{g}) \to U_q^+(\mathsf{g}) \otimes U_q^+(\mathsf{g})$ given by $\Delta_\mathsf{n}(e_\imath) = e_\imath \otimes 1 + 1 \otimes e_\imath$, where the algebra structure on $U_q^+(\mathsf{g}) \otimes U_q^+(\mathsf{g})$ is defined by

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = q^{-(\mathrm{wt}(x_2), \mathrm{wt}(y_1))} (x_1 y_1 \otimes x_2 y_2).$$

Set

$$A_q(\mathsf{n}) = \bigoplus_{\beta \in \mathsf{Q}^-} A_q(\mathsf{n})_\beta \quad \text{ where } A_q(\mathsf{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathsf{g})_{-\beta}, \mathbb{Q}(q)).$$

Then $A_q(\mathsf{n})$ is an algebra with the multiplication given by $(\psi \cdot \theta)(x) = \theta(x_{(1)})\psi(x_{(2)})$, when $\Delta_\mathsf{n}(x) = x_{(1)} \otimes x_{(2)}$ in Sweedler's notation.

Let us denote by $A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}$ the $\mathbb{Z}[q^{\pm 1}]$-submodule of $A_q(\mathsf{n})$ consisting of $\psi \in A_q(\mathsf{n})$ such that $\psi\big(U_q^+(\mathsf{g})_{\mathbb{Z}[q^{\pm 1}]}\big) \subset \mathbb{Z}[q^{\pm 1}]$. Then it is a $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $A_q(\mathsf{n})$. Moreover, the *upper global basis* $\mathbf{B}^{\mathrm{up}}$ is a $\mathbb{Z}[q^{\pm 1}]$-module basis of $A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}$. For the detail of the upper global basis, see [33, Section 1.3]. For $w \in \mathsf{W}_\mathsf{g}$, let $A_q(\mathsf{n}(w))_{\mathbb{Z}[q^{\pm 1}]}$ be the $\mathbb{Z}[q^{\pm 1}]$-submodule of $A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}$ consisting of the elements $\psi \in A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}$ such that $e_{\imath_1} \cdots e_{\imath_n} \psi = 0$ for any $\beta \in \big(\mathsf{Q}^+ \cap w\mathsf{Q}^+\big) \setminus \{0\}$ and any sequence $(\imath_1, \ldots, \imath_n) \in I_\mathsf{g}^\beta$. Then $A_q(\mathsf{n}(w))_{\mathbb{Z}[q^{\pm 1}]}$ is a $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}$ ([37, Theorem 2.20]). We call this algebra the *quantum unipotent coordinate ring associated with w*. The set $\mathbf{B}^{\mathrm{up}}(w) := \mathbf{B}^{\mathrm{up}} \cap A_q(\mathsf{n}(w))$ forms a $\mathbb{Z}[q^{\pm 1}]$-basis of $A_q(\mathsf{n}(w))_{\mathbb{Z}[q^{\pm 1}]}$ ([43]).

For a dominant weight $\lambda \in \mathsf{P}^+$, let $V(\lambda)$ be the irreducible highest weight $U_q(\mathsf{g})$-module with highest weight vector u_λ of weight λ. Let $(\ ,\ )_\lambda$ be the non-degenerate symmetric bilinear form on $V(\lambda)$ such that $(u_\lambda, u_\lambda)_\lambda = 1$ and $(xu, v)_\lambda = (u, \varphi(x)v)_\lambda$ for $u, v \in V(\lambda)$ and $x \in U_q(\mathsf{g})$, where φ is the algebra antiautomorphism on $U_q(\mathsf{g})$ defined by $\varphi(e_i) = f_i$, $\varphi(f_i) = e_i$ and $\varphi(q^h) = q^h$. For each $\mu, \zeta \in \mathsf{W}_\mathsf{g}\lambda$, the *unipotent quantum minor* $D(\mu, \zeta)$ is an element in $A_q(\mathsf{n})$ given by $D(\mu, \zeta)(x) = (xu_\mu, u_\zeta)_\lambda$ for $x \in U_q^+(\mathsf{g})$, where u_μ and u_ζ are the extremal weight vectors in $V(\lambda)$ of weight μ and ζ, respectively. Then we have $D(\mu, \zeta) \in \mathbf{B}^{\mathrm{up}} \sqcup \{0\}$.

For a reduced expression $\underline{w} = s_{\imath_1} s_{\imath_2} \cdots s_{\imath_l}$ of $w \in \mathsf{W}_{\mathfrak{g}}$, define $\underline{w}_{\leqslant k} := s_{\imath_1} \cdots s_{\imath_k}$ and $\lambda_k := \underline{w}_{\leqslant k} \Lambda_{\imath_k}$ for $1 \leqslant k \leqslant l$. Note that $\lambda_{k^-} = \underline{w}_{\leqslant k-1} \Lambda_{\imath_k}$ if $k^- > 0$. Here

$$k^- := \max\Big(\{ s \in [1, k-1] \mid \imath_s = \imath_k \} \sqcup \{0\} \Big).$$

For $0 \leqslant t \leqslant s \leqslant l$, we set

$$D_{\underline{w}}[t,s] := \begin{cases} D(\underline{w}_{\leqslant s} \Lambda_{\imath_s},\ \underline{w}_{\leqslant t-1} \Lambda_{\imath_t}) & \text{if } 1 \leqslant t \leqslant s \leqslant l \text{ and } \imath_s = \imath_t, \\ D(\underline{w}_{\leqslant s} \Lambda_{\imath_s},\ \Lambda_{i_s}) & \text{if } 0 = t < s \leqslant l, \\ \mathbf{1} & \text{if } 0 = t = s. \end{cases}$$

Then $D_{\underline{w}}[t,s]$ belongs to $\mathbf{B}^{\rm up}(w)$. The set $\big\{ D_{\underline{w}}[k] := D_{\underline{w}}[k,k] \mid 1 \leqslant k \leqslant l \big\}$ generates $A_q(\mathsf{n}(w))_{\mathbb{Z}[q^{\pm 1}]}$ as a $\mathbb{Z}[q^{\pm 1}]$-algebra.

We omit the subscript $_{\underline{w}}$ if there is no confusion about the choice of $\underline{w}$.

Remark 3.1. In [33], we use $D(s, t^+)$ instead of $D[t,s]$, where $k^+ := \min\Big(\{ t \in [k+1, l] \mid \imath_t = \imath_k \} \sqcup \{ l+1 \} \Big)$. We hope that there is no confusion by the differences of the notations.

Theorem 3.2 ([45, 59, 60, 63])**.** *There exists a $\mathbb{Z}[q^{\pm 1}]$-algebra isomorphism:*

$$\mathrm{ch}_q \colon K(R\text{-gmod}) \xrightarrow{\sim} A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}.$$

Furthermore, under the isomorphism ch_q, the upper global basis $\mathbf{B}^{\rm up}$ of $A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}$ corresponds to the set of the isomorphism classes of self-dual simple R-modules.

Since $D(\mu, \zeta)$ is a member of the upper global basis, there exists a unique real simple module $\mathsf{D}(\mu,\zeta)$ in $R(\zeta - \mu)$-gmod such that $\mathrm{ch}_q(\mathsf{D}(\mu,\zeta)) = D(\mu,\zeta)$. We call it *the determinantial module*. For a reduced expression $\underline{w} = s_{\imath_1} s_{\imath_2} \cdots s_{\imath_l}$ of $w \in \mathsf{W}_{\mathfrak{g}}$ and $0 \leqslant t \leqslant s \leqslant l$ such that $t = 0$ or $\imath_s = \imath_t$, let $\mathsf{D}_{\underline{w}}[t,s]$ be a simple module such that $\mathrm{ch}_q(\mathsf{D}_{\underline{w}}[t,s]) = D_{\underline{w}}[t,s]$. In particular, we call $\mathsf{D}_{\underline{w}}[k] := \mathsf{D}_{\underline{w}}[k,k]$ *the cuspidal module* for $1 \leqslant k \leqslant l$.

Proposition 3.3 ([33, Proposition 10.2.5], [37, Proposition 4.6])**.** *For a reduced expression $\underline{w} = s_{\imath_1} s_{\imath_2} \cdots s_{\imath_l}$ of $w \in \mathsf{W}_{\mathfrak{g}}$, we have the followings*:

(i) *For $1 \leqslant a < b \leqslant l$ with $\imath_a = \imath_b$, there exists an exact sequence in R-gmod :*

$$0 \to \mathop{\circ}_{\jmath \in I_{\mathfrak{g}};\, d(\imath_a, \jmath) = 1} \mathsf{D}[a(\jmath)^+, b(\jmath)^-] \to \mathsf{D}[a^+, b] \circ \mathsf{D}[a, b^-] \to \mathsf{D}[a,b] \circ \mathsf{D}[a^+, b^-] \to 0. \tag{3.1}$$

Here we understand $\mathsf{D}[t,s] \simeq \mathbf{1}$ for $t > s$. For the notations $k(\jmath)^{\pm}$, see (4.2) below.

(ii) *For* $1 \leqslant a \leqslant b < c \leqslant l$ *with* $\imath_a = \imath_b = \imath_c$, *we have*

$$\mathsf{D}[b^+, c] \nabla \mathsf{D}[a, b] \simeq \mathsf{D}[a, c].$$

We call an exact sequence of the form in (3.1) a *T-system.*

For an $R(\beta)$-module M, define

$$\mathrm{W}(M) := \left\{\gamma \in \mathsf{Q}^+ \cap (\beta - \mathsf{Q}^+) \mid e(\gamma, \beta - \gamma)M \neq 0\right\},$$
$$\mathrm{W}^*(M) := \left\{\gamma \in \mathsf{Q}^+ \cap (\beta - \mathsf{Q}^+) \mid e(\beta - \gamma, \gamma)M \neq 0\right\}.$$

For $w \in \mathsf{W}_{\mathsf{g}}$, we denote by R_w-gmod the full subcategory of R-gmod whose objects M satisfy $\mathrm{W}(M) \subset \mathsf{Q}^+ \cap w\mathsf{Q}^-$. Here R denote the quiver Hecke algebra associated to g.

Theorem 3.4 ([37])**.**

(a) R_w-gmod *is the smallest subcategory of* R-gmod *which is stable under taking subquotients, extensions, convolutions, grading shifts, and contains the determinantial modules* $\{\mathsf{D}_{\underline{w}}[k] \mid 1 \leqslant k \leqslant l\}$, *where* $\underline{w} = s_{\imath_1} s_{\imath_2} \cdots s_{\imath_l}$ *is a reduced expression of* w.
(b) *There exists a* $\mathbb{Z}[q^{\pm 1}]$*-algebra isomorphism between* $K(R_w\text{-gmod})$ *and* $A_{\mathbb{Z}[q^{\pm 1}]}(\mathsf{n}(w))$. *Here,* $A_{\mathbb{Z}[q^{\pm 1}]}(\mathsf{n}(w))$ *is the* $\mathbb{Z}[q^{\pm 1}]$*-subalgebra of* $A_q(\mathsf{n})_{\mathbb{Z}[q^{\pm 1}]}$ *generated by* $D[t, s]$ *with* $1 \leqslant t \leqslant s \leqslant l$.
(c) *The set of self-dual simple modules in* R_w-gmod *coincides with the upper global basis of* $A_{\mathbb{Z}[q^{\pm 1}]}(\mathsf{n}(w))$ *under the isomorphism* ch_q.

In particular, for any $\Lambda \in \mathsf{P}^+$, *the determinantial module* $\mathsf{D}(w\Lambda, \Lambda)$ *strongly commutes with all the simple modules in* R_w-gmod.

4. Quantum affine Schur-Weyl duality and T-systems

In this section, we briefly recall the quantum affine Schur-Weyl duality functor from the category R-gmod of finite-dimensional modules to $\mathscr{C}_{\mathfrak{g}}$, constructed by Kang-Kashiwara-Kim in [28] (cf. [40]), which preserves several invariants of those categories. We also review the affine cuspidal modules, PBW-pairs and reflections, defined in [40]. Then we will construct affine determinantial modules, which are real simple modules associated to i-boxes and PBW-pairs. In the last part, we will investigate the commuting condition and T-systems among affine determinantial modules, which can be understood as a vast generalization of the T-systems in R-gmod and among KR-modules.

4.1. **Quantum affine Schur-Weyl duality functor.** Let g be a simple Lie algebra with a simply-laced Cartan matrix $\mathsf{C} = (c_{\imath,\jmath})_{\imath,\jmath \in I_{\mathsf{g}}}$. Let $\mathcal{D} := \{\mathsf{L}_\imath\}_{\imath \in I_{\mathsf{g}}} \subset \mathscr{C}_{\mathfrak{g}}$ be a family of real simple modules of $\mathscr{C}_{\mathfrak{g}}$. The family $\mathcal{D}$ is called a *strong duality datum* associated with g if it satisfies the following conditions ([40, Definition 4.7]):

(1) $\mathsf{L}_\imath$ is a root module for each $\imath \in I_{\mathsf{g}}$,
(2) for any $\imath, \jmath \in I_{\mathsf{g}}$ with $\imath \neq \jmath$, $\mathfrak{d}(\mathsf{L}_\imath, \mathscr{D}^k \mathsf{L}_\jmath) = -\delta(k=0) c_{\imath,\jmath}$.

Let us denote by R_{C} the symmetric quiver Hecke algebra associated with C.

Theorem 4.1 ([28, 40])**.** *For a given strong duality datum $\mathcal{D}$ associated with g, there exists a functor*

$$\widehat{\mathcal{F}}_{\mathcal{D}} \colon R_{\mathsf{C}}\text{-gMod} \to U_q'(\mathfrak{g})\text{-Mod},$$

where $U_q'(\mathfrak{g})$-Mod denotes the category of integrable $U_q'(\mathfrak{g})$-modules. Moreover, $\widehat{\mathcal{F}}_{\mathcal{D}}$ satisfies the following properties:

(a) $\widehat{\mathcal{F}}_{\mathcal{D}}(L(\imath)) \simeq \mathsf{L}_\imath$.
(b) *The functor $\widehat{\mathcal{F}}_{\mathcal{D}}$ induces an exact monoidal functor on R-gmod which preserves simplicity*:

$$\mathcal{F}_{\mathcal{D}} \colon R_{\mathsf{C}}\text{-gmod} \to \mathscr{C}_{\mathfrak{g}}. \tag{4.1}$$

Namely, $\mathcal{F}_{\mathcal{D}}$ sends finite-dimensional graded R_{C}-modules to modules in $\mathscr{C}_{\mathfrak{g}}$ and, for any $M_1, M_2 \in R_{\mathsf{C}}$-gmod, we have isomorphisms

$$\mathcal{F}_{\mathcal{D}}(R_{\mathsf{C}}(0)) \simeq \mathbf{k}, \quad \mathcal{F}_{\mathcal{D}}(M_1 \circ M_2) \simeq \mathcal{F}_{\mathcal{D}}(M_1) \otimes \mathcal{F}_{\mathcal{D}}(M_2),$$

and $\mathcal{F}_{\mathcal{D}}$ sends simple modules to simple modules.

(c) *For $M \in R_{\mathsf{C}}$-gmod, we have*

$$\widehat{\mathcal{F}}_{\mathcal{D}}(M_z) \simeq \mathcal{F}_{\mathcal{D}}(M)_{\mathrm{e}^z}$$

(*see* [40] *for more details*).

We call the functor $\mathcal{F}_{\mathcal{D}}$ the *quantum affine Schur-Weyl duality functor.*

Definition 4.2 ([40, §5.2])**.** The category $\mathscr{C}_{\mathcal{D}}$ is defined to be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ such that

(1) it contains $\mathsf{L}_\imath$ for any $\imath \in I_{\mathsf{g}}$ and the trivial module $\mathbf{1}$,
(2) it is stable by taking subquotients, extensions, and tensor products.

Hence $\mathscr{C}_{\mathcal{D}}$ contains $\mathcal{F}_{\mathcal{D}}(M)$ for any module $M \in R_{\mathsf{C}}$-gmod.

The Grothendieck ring $K(R_{\mathsf{C}}\text{-gmod})$ has a $\mathbb{Z}[q^{\pm 1}]$-algebra structure. We set

$$K(R_{\mathsf{C}}\text{-gmod})|_{q=1} := K(R_{\mathsf{C}}\text{-gmod})/(q-1)K(R_{\mathsf{C}}\text{-gmod}).$$

Theorem 4.3 ([40])**.** *Let $\mathcal{D}$ be a strong duality datum. Then there exist a $\mathbb{Z}$-algebra isomorphism*

$$\mathrm{ch}\colon K(R_{\mathsf{C}}\text{-gmod})|_{q=1} \xrightarrow{\sim} K(\mathscr{C}_{\mathcal{D}}),$$

and a $\mathbb{C}$-algebra isomorphism

$$\mathbb{C}[N] \simeq \mathbb{C} \otimes \big(K(R_{\mathsf{C}}\text{-gmod})|_{q=1}\big),$$

where $K(\mathscr{C}_{\mathcal{D}})$ denotes the Grothendieck ring of $\mathscr{C}_{\mathscr{D}}$ and $\mathbb{C}[N]$ denotes the coordinate ring of the unipotent group N associated with the nilpotent Lie subalgebra n of g.

4.2. **Λ, $\widetilde{\Lambda}$ and $\mathfrak{d}$ under the duality functors.** Keep the notations g and $\mathsf{C} = (c_{\imath,\jmath})_{\imath,\jmath\in I_{\mathsf{g}}}$. Let $\mathcal{D}$ be a strong duality datum associated with g. For each $k \in \mathbb{Z}$, we define a subcategory $\mathscr{D}^k\mathscr{C}_{\mathcal{D}}$ as follows: $\mathscr{D}^k\mathscr{C}_{\mathcal{D}}$ is the smallest full subcategory of $\mathscr{C}^0_{\mathfrak{g}}$ such that (a) it contains $\mathscr{D}^k(\mathcal{F}_{\mathcal{D}}(L))$ for all simple modules L in R_{C}-gmod, and (b) it is stable by taking subquotients, extensions and tensor products. By [40, Proposition 5.4], there is no non-trivial module which is contained in distinct $\mathscr{D}^k\mathscr{C}_{\mathcal{D}}$ and $\mathscr{D}^l\mathscr{C}_{\mathcal{D}}$ simultaneously. Note that we have

$$\mathfrak{d}(M,N) = 0$$

for a simple module M in $\mathscr{D}^k\mathscr{C}_{\mathcal{D}}$ and a simple module N in $\mathscr{D}^l\mathscr{C}_{\mathcal{D}}$ if $|k-l| > 1$.

We recall one of the main results of [40], which tells that the invariants $\Lambda, \mathfrak{d}$ are preserved under the functor $\mathcal{F}_{\mathcal{D}}$:

Proposition 4.4 ([40, Theorem 4.12])**.**

(i) *For simple modules $M, N \in R_{\mathsf{C}}$-gmod, we have*

$$\Lambda(M,N) = \Lambda\big(\mathcal{F}_{\mathcal{D}}(M), \mathcal{F}_{\mathcal{D}}(N)\big),$$
$$\widetilde{\Lambda}(M,N) = \mathfrak{d}\big(\mathscr{D}\mathcal{F}_{\mathcal{D}}(M), \mathcal{F}_{\mathcal{D}}(N)\big).$$

(ii) *For a simple module M in R_{C}-gmod and $\imath \in I_{\mathsf{g}}$, we have*

$$\mathfrak{d}\big(\mathscr{D}\mathcal{F}_{\mathcal{D}}(L(\imath)), \mathcal{F}_{\mathcal{D}}(M)\big) = \varepsilon_\imath(M) \quad \textit{and} \quad \mathfrak{d}\big(\mathscr{D}^{-1}\mathcal{F}_{\mathcal{D}}(L(\imath)), \mathcal{F}_{\mathcal{D}}(N)\big) = \varepsilon^*_\imath(N).$$

4.3. **Affine cuspidal modules, PBW-pair and reflections.** Let g and $\mathsf{C} = (c_{\imath,\jmath})_{\imath,\jmath\in I_{\mathsf{g}}}$ be as in the last subsection. Let $\mathcal{D}:=\{\mathsf{L}_\imath\}_{\imath\in I_{\mathsf{g}}}$ be a strong duality datum associated to g, and let ℓ be the length of the longest element w_0 of the Weyl group of g, and let $\underline{w}_0 = s_{\imath_1}\ldots s_{\imath_\ell}$ be a reduced expression of w_0. We extend the definition of $(\imath_k)_{1\leqslant k\leqslant \ell}$ to $(\imath_k)_{k\in\mathbb{Z}}$ by

$$\imath_{k+\ell} = (\imath_k)^* \quad \text{for any } k \in \mathbb{Z}.$$

We set $\widehat{\underline{w}}_0 = (\imath_k)_{k\in\mathbb{Z}}$, and we call $(\mathcal{D}, \widehat{\underline{w}}_0)$ a *PBW-pair.*

Note that $s_{\imath_k}s_{\imath_{k+1}}\cdots s_{\imath_{k+\ell-1}}$ is a reduced expression of w_0 for any $k \in \mathbb{Z}$.

Definition 4.5. We define a sequence of simple $U_q'(\mathfrak{g})$-modules $\big(\mathsf{S}_k^{\mathcal{D},\widehat{\underline{w}}_0}\big)_{k\in\mathbb{Z}}$ in $\mathscr{C}_\mathfrak{g}$ as follows:

(1) For $1 \leqslant k \leqslant \ell$, $\mathsf{S}_k^{\mathcal{D},\widehat{\underline{w}}_0} := \mathcal{F}_\mathcal{D}(\mathsf{D}_{\underline{w}_0}[k])$ and, for all $k \in \mathbb{Z}$, it is defined by

(2) $\mathsf{S}_{k+\ell}^{\mathcal{D},\widehat{\underline{w}}_0} = \mathscr{D}(\mathsf{S}_k^{\mathcal{D},\widehat{\underline{w}}_0})$ for any $k \in \mathbb{Z}$.

The modules $\mathsf{S}_k^{\mathcal{D},\widehat{\underline{w}}_0}$ are called the *affine cuspidal modules* corresponding to the PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$. For simplicity of notation, we drop the superscript ${}^{\mathcal{D},\widehat{\underline{w}}_0}$ if there is no risk of confusion.

Throughout this section, we fix a PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ and frequently use the notations:

$$\begin{aligned} &s^+ := \min\{t \mid s < t,\ \imath_t = \imath_s\}, \qquad &&s^- := \max\{t \mid t < s,\ \imath_t = \imath_s\}, \\ &s(\jmath)^+ := \min\{t \mid s \leqslant t,\ \imath_t = \jmath\}, \quad &&s(\jmath)^- := \max\{t \mid t \leqslant s,\ \imath_t = \jmath\} \end{aligned} \tag{4.2}$$

for $s \in \mathbb{Z}$ and $\jmath \in I_\mathfrak{g}$.

Proposition 4.6 ([40, Proposition 5.7])**.**

(i) *For every $k \in \mathbb{Z}$, S_k is a root module.*

(ii) *For any $a, b \in \mathbb{Z}$ such that $|a-b| \geqslant \ell$, we have $\mathfrak{d}(\mathsf{S}_a, \mathsf{S}_b) = \delta(|a-b| = \ell)$.*

(iii) *For any $a < b$, the ordered pair $(\mathsf{S}_b, \mathsf{S}_a)$ is strongly unmixed.*

For any $\jmath \in I_\mathfrak{g}$, we set

$$\mathscr{S}_\jmath(\mathcal{D}) := \{\mathscr{S}_\jmath(\mathsf{L}_\imath)\}_{\imath\in I_\mathfrak{g}} \quad \text{and} \quad \mathscr{S}_\jmath^{-1}(\mathcal{D}) := \{\mathscr{S}_\jmath^{-1}(\mathsf{L}_\imath)\}_{\imath\in I_\mathfrak{g}}, \tag{4.3}$$

where

$$\mathscr{S}_\jmath(\mathsf{L}_\imath) := \begin{cases} \mathscr{D}\mathsf{L}_\imath & \text{if } \imath = \jmath, \\ \mathsf{L}_\jmath \nabla \mathsf{L}_\imath & \text{if } c_{\imath,\jmath} = -1, \\ \mathsf{L}_\imath & \text{if } c_{\imath,\jmath} = 0, \end{cases} \quad \text{and} \quad \mathscr{S}_\jmath^{-1}(\mathsf{L}_\imath) := \begin{cases} \mathscr{D}^{-1}\mathsf{L}_\imath & \text{if } \imath = \jmath, \\ \mathsf{L}_\imath \nabla \mathsf{L}_\jmath & \text{if } c_{\imath,\jmath} = -1, \\ \mathsf{L}_\imath & \text{if } c_{\imath,\jmath} = 0. \end{cases}$$

It is easy to see that $\mathscr{S}_\jmath \circ \mathscr{S}_\jmath^{-1}(\mathcal{D}) = \mathcal{D}$ and $\mathscr{S}_\jmath^{-1} \circ \mathscr{S}_\jmath(\mathcal{D}) = \mathcal{D}$ for any $\jmath \in I_\mathfrak{g}$.

Proposition 4.7 ([40, Proposition 5.8, Proposition 5.9])**.** *Let $\big(\mathcal{D}, \widehat{\underline{w}}_0 = (\imath_k)_{k\in\mathbb{Z}}\big)$ be a PBW-pair and $\jmath \in I_\mathfrak{g}$.*

(i) *The data $\mathscr{S}_\jmath(\mathcal{D})$ and $\mathscr{S}_\jmath^{-1}(\mathcal{D})$ are also strong duality data associated with C.*

(ii) *Set $\mathscr{S}^{\pm1}(\widehat{\underline{w}}_0) = (\imath'_k)_{k\in\mathbb{Z}}$ with $\imath'_k = \imath_{k\pm1}$. Then $\mathscr{S}(\mathcal{D}, \widehat{\underline{w}}_0) := \big(\mathscr{S}_{\imath_1}(\mathcal{D}), \mathscr{S}(\widehat{\underline{w}}_0)\big)$ and $\mathscr{S}^{-1}(\mathcal{D}, \widehat{\underline{w}}_0) := \big(\mathscr{S}_{\imath_1}^{-1}(\mathcal{D}), \mathscr{S}^{-1}(\widehat{\underline{w}}_0)\big)$ are PBW-pairs.*

(iii) *We have*

$$\mathsf{S}_k^{\mathscr{S}^{\pm1}(\mathcal{D},\widehat{\underline{w}}_0)} = \mathsf{S}_{k\pm1}^{\mathcal{D},\widehat{\underline{w}}_0} \quad \textit{for any } k \in \mathbb{Z}.$$

For each interval $[a,b]$, we denote by $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ satisfying the following conditions (see [40, §6.3]):

(1) it is stable under taking subquotients, extensions, tensor products and
(2) it contains S_k for all $a \leqslant k \leqslant b$ and the trivial module $\mathbf{1}$.

Hence we have

$$\mathscr{C}_{\mathfrak{g}}^{[a,b],\,\mathscr{S}(\mathcal{D},\widehat{\underline{w}}_0)} = \mathscr{C}_{\mathfrak{g}}^{[a+1,b+1],\,\mathcal{D},\widehat{\underline{w}}_0}.$$

Definition 4.8 ([40, Definition 6.1])**.** A strong duality datum $\mathcal{D}$ in $\mathscr{C}_{\mathfrak{g}}^0$ is called *complete* if, for any simple module $M \in \mathscr{C}_{\mathfrak{g}}^0$, there exist simple modules $M_k \in \mathscr{C}_{\mathcal{D}}$ ($k \in \mathbb{Z}$) such that

(1) $M_k \simeq \mathbf{1}$ for all but finitely many k,
(2) $M \simeq \mathrm{hd}(\cdots \otimes \mathscr{D}^2 M_2 \otimes \mathscr{D} M_1 \otimes \ M_0 \otimes \mathscr{D}^{-1} M_{-1} \otimes \cdots)$.

We say that a PBW-pair $(\mathcal{D},\widehat{\underline{w}}_0)$ is complete if $\mathcal{D}$ is a complete duality datum.

Note that if $\mathcal{D}$ is complete, then C is the Cartan matrix of the simply-laced root system associated with $\mathfrak{g}$ in §2.4 ([40, Proposition 6.2]).

Proposition 4.9 ([40, Theorem 6.9])**.** *For any complete PBW-pair $(\mathcal{D},\widehat{\underline{w}}_0)$, we have*

$$\mathscr{C}_{\mathfrak{g}}^{[-\infty,\infty],\mathcal{D},\widehat{\underline{w}}_0} = \mathscr{C}_{\mathfrak{g}}^0.$$

Lemma 4.10 ([40, Lemma 6.8])**.** *Let $(\mathcal{D},\widehat{\underline{w}}_0)$ be a complete PBW-pair, $[a,b]$ an interval, and M a simple module in $\mathscr{C}_{\mathfrak{g}}^0$. Then, M belongs to $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ if and only if*

$$\mathfrak{d}(\mathscr{D}\mathsf{S}_k, M) = 0 \text{ for } k > b \text{ and } \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, M) = 0 \text{ for } k < a.$$

Corollary 4.11. Let $(\mathcal{D},\widehat{\underline{w}}_0)$ be a complete PBW-pair, $[a,b]$ an interval, M a real simple module in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ and X a simple module in $\mathscr{C}_{\mathfrak{g}}^0$. If $M \nabla X$ and $X \nabla M$ belong to $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$, then so does X.

Proof. For $k > b$, we have $\mathfrak{d}(\mathscr{D}\mathsf{S}_k, M) = \mathfrak{d}(\mathscr{D}^2\mathsf{S}_k, M) = 0$. Hence Lemma 4.12 below implies

$$0 = \mathfrak{d}(\mathscr{D}\mathsf{S}_k, X \nabla M) = \mathfrak{d}(\mathscr{D}\mathsf{S}_k, X).$$

Similarly, for $k < a$, we have $\mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, M) = \mathfrak{d}(\mathscr{D}^{-2}\mathsf{S}_k, M) = 0$. Hence we have

$$0 = \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, X \nabla M) = \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, X).$$

Therefore, X belongs to $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$. □

Lemma 4.12. *Let L, M be real simple modules and X a simple module.*

(i) *If* $\mathfrak{d}(L, M) = \mathfrak{d}(\mathscr{D}L, M) = 0$, *then we have*

$$\mathfrak{d}(L, X \nabla M) = \mathfrak{d}(L, X).$$

(ii) *If* $\mathfrak{d}(L, M) = \mathfrak{d}(\mathscr{D}^{-1}L, M) = 0$, *then we have*

$$\mathfrak{d}(L, M \nabla X) = \mathfrak{d}(L, X).$$

Proof. (i) By the assumption, (L, X, M) and (X, M, L) is normal sequences. Hence by [38, Lemma 4.16], we have

$$\mathfrak{d}(L, X \nabla M) = \mathfrak{d}(L, X) + \mathfrak{d}(L, M) = \mathfrak{d}(L, X).$$

(ii) can be similarly proved by using the normality of (L, M, X) and (M, X, L). $\square$

4.4. **Affine determinantial modules associated with i-boxes.** In this subsection, we introduce the notion of i-boxes, assign a simple module to each i-box, and then study its properties.

Lemma 4.13. *For any $a \in \mathbb{Z}$, we have $\mathfrak{d}(\mathsf{S}_{a^+}, \mathsf{S}_a) = 1$.*

Proof. Set $\imath = \imath_a$. By Proposition 4.7, we may assume that $a = 1$. If $\ell = 1$, then the assertion easily follows from $\mathsf{S}_{a^+} = \mathscr{D}\mathsf{S}_a$ and Proposition 4.6 (i). Assume that $\ell > 1$. Then we have $a = 1 < a^+ \leqslant \ell$. Hence we have $\mathsf{S}_a \simeq \mathcal{F}_{\mathcal{D}}(L(\imath))$ and $\mathsf{S}_{a^+} \simeq \mathcal{F}_{\mathcal{D}}(M)$, where $M := \mathsf{D}(\underline{w}_{\leqslant a^+}\Lambda_\imath, s_\imath\Lambda_\imath)$. Since $s_\imath\underline{w}_{\leqslant a^+}\Lambda_\imath \succeq \underline{w}_{\leqslant a^+}\Lambda_\imath$ and $s_\imath\underline{w}_{\leqslant a^+}\Lambda_\imath = s_{\imath_2}\cdots s_{\imath_{a^+}}\Lambda_\imath \preceq s_\imath\Lambda_\imath$, [33, Proposition 10.2.4] implies that $\varepsilon_\imath(M) = -(\alpha_\imath, \underline{w}_{\leqslant a^+}\Lambda_\imath)$ and $\varepsilon_\imath^*(M) = 0$.

On the other hand, [37, Corollary 3.8] implies that

$$\begin{aligned}\mathfrak{d}(\mathsf{S}_{a^+}, \mathsf{S}_a) = \mathfrak{d}(L(\imath), M) &= \varepsilon_\imath(M) + \varepsilon_\imath^*(M) + \big(\alpha_\imath, \mathrm{wt}(M)\big) \\ &= -(\alpha_\imath, \underline{w}_{\leqslant a^+}\Lambda_\imath) + (\alpha_\imath, \underline{w}_{\leqslant a^+}\Lambda_\imath - s_\imath\Lambda_\imath) = 1. \quad \square\end{aligned}$$

For $a, b \in \mathbb{Z} \sqcup \{\pm\infty\}$, an *interval* $[a, b]$ is the set of integers between a and b:

$$[a, b] := \{k \in \mathbb{Z} \mid a \leqslant k \leqslant b\}.$$

If $a > b$, we understand $[a, b] = \emptyset$. We call $\max(b - a + 1, 0)$ the length of $[a, b]$ and denote it by $|\,[a, b]\,|$.

Definition 4.14. Let $(\mathcal{D}, \widehat{\underline{w}_0})$ be a PBW-pair.

(1) We say that an interval $\mathfrak{c} = [a, b]$ is an *i-box* if $-\infty < a \leqslant b < +\infty$ and $\imath_a = \imath_b$.
(2) For an i-box $[a, b]$, we set

$$|[a, b]|_\phi := \Big|\{s \mid s \in [a, b] \text{ and } \imath_a = \imath_s = \imath_b\}\Big|.$$

(3) For an i-box $[a,b]$, we define

$$M^{\mathcal{D},\widehat{\underline{w}}_0}[a,b] := \mathrm{hd}(\mathsf{S}_b \otimes \mathsf{S}_{b^-} \otimes \cdots \otimes \mathsf{S}_{a^+} \otimes \mathsf{S}_a).$$

We call $M^{\mathcal{D},\widehat{\underline{w}}_0}[a,b]$ *an affine determinantial module.*

Sometimes $M^{\mathcal{D},\widehat{\underline{w}}_0}[a,b]$ with $a > b$ appears, and we understand $M^{\mathcal{D},\widehat{\underline{w}}_0}[a,b] = \mathbf{1}$ in such a case.

For simplicity of notation, we sometimes drop the superscripts ${}^{\mathcal{D},\widehat{\underline{w}}_0}$ if there is no risk of confusion.

Proposition 4.15.

(i) *For any i-box $[a,b]$, the module $M[a,b]$ is simple.*
(ii) *For $a \leqslant b < a' \leqslant b'$, the pair $\big(M[a',b'], M[a,b]\big)$ is strongly unmixed.*

Proof. (i) follows from Proposition 2.18 and (ii) follows from (iii) in Proposition 4.6. □

Note that we shall see that $M[a,b]$ is real in Theorem 4.21 below.

Lemma 4.16. *For any i-box $[a,b] \subset [1,\ell]$, the module $M[a,b]$ is isomorphic to the image of a determinantial module under the functor $\mathcal{F}_{\mathcal{D}}$. More precisely, we have*

$$M^{\mathcal{D},\widehat{\underline{w}}_0}[a,b] \simeq \mathcal{F}_{\mathcal{D}}(\mathsf{D}_{\underline{w}_0}[a,b]).$$

Proof. This follows from [37, Proposition 4.1, Proposition 4.6] and the fact that $\mathcal{F}_{\mathcal{D}}$ is an exact monoidal functor sending simple modules to simple modules. □

Corollary 4.17. For an i-box $[a,b]$ with $b-a+1 \leqslant \ell$, we have an exact sequence in $\mathscr{C}_{\mathfrak{g}}$ as follows:

$$0 \to \bigotimes_{\jmath \in I_{\mathfrak{g}};\ d(\imath_a,\jmath)=1} M[a(\jmath)^+, b(\jmath)^-] \to M[a^+,b] \otimes M[a,b^-] \to M[a,b] \otimes M[a^+,b^-] \to 0.$$

Proof. The above exact sequence is the image of (3.1) under the exact functor $\mathcal{F}_{\mathcal{D}}$. □

4.5. **Commuting condition between affine determinantial modules.** In this subsection, we will give a sufficient condition that simple modules $M[a_1,b_1]$ and $M[a_2,b_2]$ commute. We start with the following definition on the pair of i-boxes $[a_1,b_1]$ and $[a_2,b_2]$.

Definition 4.18. We say that i-boxes $[a_1,b_1]$ and $[a_2,b_2]$ *commute* if we have either

$$a_1^- < a_2 \leqslant b_2 < b_1^+ \quad \text{or} \quad a_2^- < a_1 \leqslant b_1 < b_2^+.$$

The main goal of this subsection is to prove the theorem which tells us that affine determinantial modules $M[a_1, b_1]$ and $M[a_2, b_2]$ commute if $[a_1, b_1]$ and $[a_2, b_2]$ commute (see Theorem 4.21 below).

Proposition 4.19. *For any i-box $[a, b]$ and $s \in \mathbb{Z}$ such that $a^- < s < b^+$, we have*

$$\mathfrak{d}(M[a, b], \mathsf{S}_s) = 0.$$

Proof. We shall argue by induction on $|[a, b]|_\phi$.

(i) Assume first that $a < s < b$ and $\imath_a = \imath_b = \imath_s$. Note that

$$M[a, b] = \mathrm{hd}(M[s^+, b] \otimes \mathsf{S}_s \otimes M[a, s^-])$$

and hence

$$\mathfrak{d}(\mathsf{S}_s, M[a, b]) = \mathfrak{d}\left(\mathsf{S}_s, \mathrm{hd}(M[s^+, b] \otimes \mathsf{S}_s \otimes M[a, s^-])\right).$$

Since the pairs $(M[s^+, b], \mathsf{S}_s)$ and $(\mathsf{S}_s, M[a, s^-])$ are unmixed by Proposition 4.15, we have

$$\begin{aligned}
&\mathfrak{d}\left(\mathsf{S}_s, \mathrm{hd}(M[s^+, b] \otimes \mathsf{S}_s \otimes M[a, s^-])\right) \\
&\qquad = \mathfrak{d}\left(\mathsf{S}_s, \mathrm{hd}(M[a, s^+] \otimes \mathsf{S}_s)\right) + \mathfrak{d}\left(\mathsf{S}_s, \mathrm{hd}(\mathsf{S}_s \otimes M[s^-, b])\right) \\
&\qquad = \mathfrak{d}\left(\mathsf{S}_s, M[a, s]\right) + \mathfrak{d}\left(\mathsf{S}_s, M[s, b]\right) = 0
\end{aligned}$$

by Proposition 2.25. Here, the last equality follows from the induction hypothesis on $|[a, b]|_\phi$.

(ii) Assume that $a < s < b$ and $\imath_a = \imath_b \neq \imath_s$. Set $c = \min\{k \in [a, b] \mid \imath_k = \imath_b, s < k\}$. Then, we have $a \leqslant c^- < s < c \leqslant b$, and

$$M[a, b] = \mathrm{hd}(M[c, b] \otimes M[a, c^-]).$$

Now our assertion follows from the induction hypothesis.

(iii) It remains to prove the cases

$$\text{(a)} \quad a^- < s \leqslant a \quad \text{and} \quad \text{(b)} \quad b \leqslant s < b^+.$$

We shall prove only the case (b) since the other case is similarly proved.

(iii-1) Assume first that $s > a + \ell$. Then we have

$$M[a, b] = M[a^+, b] \mathbin{\nabla} \mathsf{S}_a.$$

On the other hand we have $\mathfrak{d}(\mathsf{S}_s, \mathsf{S}_a) = 0$ and $\mathfrak{d}(\mathsf{S}_s, M[a^+, b]) = 0$ by Proposition 4.6 and the induction hypothesis, respectively. Thus our assertion follows from Proposition 2.19.

(iii-2) Assume that $s \leqslant a+\ell$. We may assume that $a=1$ by Proposition 4.7. Hence it is enough to show that

$$M[1,b] \text{ commutes with } \mathsf{S}_s \text{ if } \imath_1 = \imath_b,\ b \leqslant s < b^+ \text{ and } s \leqslant 1+\ell.$$

Assume first that $s \leqslant \ell$. Then we have $\mathsf{S}_s = \mathcal{F}_{\mathcal{D}}(\mathsf{D}[s])$ and $\mathcal{F}_{\mathcal{D}}\big(\mathsf{D}(\underline{w}_{\leqslant s}\Lambda_{\imath_1}, \Lambda_{\imath_1})\big) \simeq M[1,b]$ by Lemma 4.16. Then the assertion follows from the fact that $\mathsf{D}[s] \in R_{\underline{w}_{\leqslant s}}$-gmod and $\mathsf{D}(\underline{w}_{\leqslant s}\Lambda_\imath, \Lambda_\imath)$ commutes with every simple module in $R_{\underline{w}_{\leqslant s}}$-gmod by Theorem 3.4.

Now we assume that $s = \ell+1$. In this case, we have $\mathsf{S}_s = \mathscr{D}\mathsf{S}_1$.

We divide this case into two sub-cases:

(1) Assume that $b = \ell+1 = s$. Then we have

$$M[a,b] = \mathsf{S}_b \nabla M[1,b^-] = \mathsf{S}_s \nabla \mathcal{F}_{\mathcal{D}}(\mathsf{D}(w_0\Lambda_{\imath_1}, \Lambda_{\imath_1})).$$

By Proposition 4.4 and Lemma 4.16, we have

$$\mathfrak{d}(\mathsf{S}_s, M[1,b^-]) = \mathfrak{d}(\mathscr{D}\mathsf{S}_1, M[1,b^-]) = \varepsilon_{\imath_1}(\mathsf{D}(w_0\Lambda_{\imath_1}, \Lambda_{\imath_1})).$$

Then [33, Lemma 9.15] tells that

$$\varepsilon_{\imath_1}(\mathsf{D}(w_0\Lambda_{\imath_1}, \Lambda_{\imath_1})) = \max\big(0, -(\alpha_{\imath_1}, w_0\Lambda_{\imath_1})\big) \leqslant 1.$$

Now [40, Proposition 2.17] implies that

$$\mathfrak{d}(\mathsf{S}_s, M[a,b]) = 0,$$

yielding our assertion in this case.

(2) Assume that $b \leqslant \ell$. Then we have $b < s = \ell+1 < b^+$ and hence $\imath_1^* = \imath_{\ell+1} \neq \imath_b = \imath_1$. Since $w_0\Lambda_{\imath_1} = s_{\imath_1}\cdots s_{\imath_\ell}\Lambda_{\imath_1} = s_{\imath_1}\cdots s_{\imath_b}\Lambda_{\imath_1}$, we have $M[1,b] = \mathsf{D}(w_0\Lambda_{\imath_1}, \Lambda_{\imath_1})$. Hence we have

$$\begin{aligned}\mathfrak{d}(\mathsf{S}_s, M[1,b]) = \mathfrak{d}(\mathscr{D}\mathsf{S}_1, M[1,b]) &= \varepsilon_{\imath_1}\big(\mathsf{D}(w_0\Lambda_{\imath_1}, \Lambda_{\imath_1})\big) \\ &= \max\big(0, -(\alpha_{\imath_1}, w_0\Lambda_{\imath_1})\big) = \max\big(0, (\alpha_{\imath_1^*}, \Lambda_{\imath_1})\big) = 0,\end{aligned}$$

where the last equality follows from $\imath_1^* \neq \imath_1$. Thus we obtain the desired result. □

Proposition 4.20. *For any i-box $[a,b]$, we have*

$$\mathfrak{d}(\mathsf{S}_{b^+}, M[a,b]) = \mathfrak{d}(\mathsf{S}_{a^-}, M[a,b]) = 1.$$

Proof. We shall only prove that $\mathfrak{d}(\mathsf{S}_{b^+}, M[a,b]) = 1$ since the other assertion can be proved similarly. Since $\mathsf{S}_{b^+} \nabla M[a,b] \simeq M[a,b^+]$, we have $M[a,b] \simeq M[a,b^+] \nabla \mathscr{D}\mathsf{S}_{b^+}$. Noting that S_{b^+} is a root module (see Definition 2.27), we have

$$0 \leqslant \mathfrak{d}(\mathsf{S}_{b^+}, M[a,b]) \leqslant \mathfrak{d}(\mathsf{S}_{b^+}, M[a,b^+]) + \mathfrak{d}(\mathsf{S}_{b^+}, \mathscr{D}\mathsf{S}_{b^+}) = 1.$$

Assume that $\mathfrak{d}(\mathsf{S}_{b^+}, M[a,b]) = 0$. Then we have $M[a,b^+] \simeq \mathsf{S}_{b^+} \otimes M[a,b]$, and

$$0 = \mathfrak{d}\big(\mathsf{S}_b, M[a,b^+]\big) = \mathfrak{d}\big(\mathsf{S}_b, \mathsf{S}_{b^+}\big) + \mathfrak{d}\big(\mathsf{S}_b, M[a,b]\big) = 1$$

by Lemma 4.13, which is a contradiction. Thus we conclude that $\mathfrak{d}(\mathsf{S}_{b^+}, M[a,b]) = 1$. $\square$

Theorem 4.21.

(a) *For any i-box $[a,b]$, $M[a,b]$ is a real simple module in $\mathscr{C}_{\mathfrak{g}}$.*

(b) *If two i-boxes $[a_1,b_1]$ and $[a_2,b_2]$ commute, then $M[a_1,b_1]$ and $M[a_2,b_2]$ commute.*

Proof. By induction on $|\,[a,b]\,|_\phi$, we may assume that $a \leqslant b^-$ and $M[a,b^-]$ is real simple. Hence (a) follows from $M[a,b] = \mathsf{S}_b \,\nabla\, M[a,b^-]$, Lemma 2.22 and Proposition 4.20.

(b) is an immediate consequence of Proposition 4.19. $\square$

Lemma 4.22. *For any i-box $[a,b]$, we have*

$$\mathfrak{d}(M[a,b], M[a^-,b^-]) \leqslant 1.$$

Proof. Note that $M[a^-,b^-] = M[a,b^-] \,\nabla\, \mathsf{S}_{a^-}$. Thus Theorem 4.21 and Proposition 4.20 tell that

$$\mathfrak{d}(M[a,b], M[a^-,b^-]) \leqslant \mathfrak{d}(M[a,b], M[a,b^-]) + \mathfrak{d}(M[a,b], \mathsf{S}_{a^-}) = 1. \quad \square$$

Later, we see that $\mathfrak{d}(M[a,b], M[a^-,b^-]) = 1$ in the course of the proof of Theorem 4.25.

Lemma 4.23. *For any i-box $[a,b]$, we have*

$$\mathfrak{d}(\mathscr{D}\mathsf{S}_b, M[a,b]) = 1 \quad \textit{and} \quad \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_a, M[a,b]) = 1.$$

Proof. Since $M[a,b] \simeq \mathsf{S}_b \,\nabla\, M[a,b^-]$ and $\mathfrak{d}(\mathscr{D}^k\mathsf{S}_b, M[a,b^-]) = 0$ for $k \geqslant 1$, the first equality $\mathfrak{d}(\mathscr{D}\mathsf{S}_b, M[a,b]) = 1$ follows from [40, Lemma 6.7]. The second equality can be proved similarly. $\square$

For an interval $[a,b]$ with $a \leqslant b$, we define i-boxes

$$[a,b\} := [a, b(\imath_a)^-] \quad \text{and} \quad \{a,b] := [a(\imath_b)^+, b]. \tag{4.4}$$

For notations, see (4.2).

Lemma 4.24. *If $a < b$, then $M[a,b\}$ and S_{b^-} commute.*

Proof. Set $c = b(\imath_a)^-$ so that $a \leqslant c \leqslant b < c^+$ and $M[a,b\} = M[a,c]$.

(i) If $b^- > a^-$, then $M[a,c]$ and S_{b^-} commute since $a^- < b^- < b < c^+$.

(ii) If $b^- \leqslant a^-$, then $\imath_a \neq \imath_b$ and hence $c < b$. Then $(b^-)^- < b^- \leqslant a^- < a \leqslant c < b = (b^-)^+$, which implies that $M[a,c]$ and $\mathsf{S}_{b^-} = M[b^-,b^-]$ commute. $\square$

4.6. **T-systems among affine determinantial modules.** A T-system was introduced in [46] as a system of differential equations associated with solvable lattice models. It was conjectured in [10] that the *q-characters* of Kirillov-Reshetikhin modules solve the T-system. The T-system can be written as a short exact sequence in terms of KR-modules. In [56, Appendix], T-systems are presented in terms of notations $W^{(i)}_{k,x}$ and also $V(i^k)_x$ for $i \in I_0$, $k \in \mathbb{Z}_{\geqslant 1}$ and $x \in \mathbf{k}^\times$. For instance, for a simply-laced untwisted $\mathfrak{g}$ and $i \in I_0$, T-system is given as follows:

$$0 \to \bigotimes_{j\in I_0;\, d(i,j)=1} W^{(j)}_{k,x(-q)} \to W^{(i)}_{k,x(-q)^2} \otimes W^{(i)}_{k,x} \to W^{(i)}_{k-1,x(-q)^2} \otimes W^{(i)}_{k+1,x} \to 0,$$

for any $i \in I_0$ and $x \in \mathbf{k}^\times$.

In our context, it is paraphrased as follows (see Theorem 6.14 below).

Theorem 4.25. *For an arbitrary quantum affine algebra $U_q'(\mathfrak{g})$ and an arbitrary i-box $[a,b]$ such that $a<b$, we have an exact sequence*

$$0 \to \bigotimes_{\jmath\in I_{\mathfrak{g}};\, d(\imath_a,\jmath)=1} M[a(\jmath)^+, b(\jmath)^-] \to M[a^+,b] \otimes M[a,b^-] \to M[a,b] \otimes M[a^+,b^-] \to 0. \tag{4.5}$$

We call it also a T-system.

Note that the left term and the right term in (4.5) are simple.

By this theorem, we have also an exact sequence

$$0 \to M[a,b] \otimes M[a^+,b^-] \to M[a,b^-] \otimes M[a^+,b] \longrightarrow \bigotimes_{\jmath\in I_{\mathfrak{g}};\, d(\imath_a,\jmath)=1} M[a(\jmath)^+, b(\jmath)^-] \to 0. \tag{4.6}$$

The rest of the subsection is devoted to the proof of this theorem. We begin with the following lemma.

Lemma 4.26. *Let X, Y, Z be real simple modules in $\mathscr{C}_{\mathfrak{g}}$, and we assume the following conditions:*

(i) *(X,Y,Z) is a normal sequence,*
(ii) $\Lambda(Y,X) + \Lambda(Y,Z) - \Lambda(Y, X \nabla Z) = 2\mathfrak{d}(X,Y)$.

Then, the composition $X \otimes Y \otimes Z \xrightarrow{\mathbf{r}_{X,Y}} Y \otimes X \otimes Z \twoheadrightarrow Y \nabla (X \nabla Z)$ is an epimorphism and it induces an isomorphism $\operatorname{hd}(X\otimes Y\otimes Z) \simeq Y \nabla (X \nabla Z)$.

Proof. Set $\lambda = \mathfrak{d}(X, Y)$. Then we have a commutative diagram:

$$\begin{tikzcd}
X \otimes Y_{e^z} \otimes Z \arrow[d, "\mathrm{R}^{\mathrm{ren}}_{X,Y_{e^z}}"] \arrow[dr, "z^{\lambda}\,\mathrm{id}_{X\otimes Y_{e^z}\otimes Z}"] \arrow[drr, bend left, "z^{\lambda}\mathrm{R}^{\mathrm{ren}}_{Y_{e^z},Z}"] \arrow[d, phantom, shift left=6ex, "\text{Ⓐ}"] & & \\
Y_{e^z} \otimes X \otimes Z \arrow[r, "\mathrm{R}^{\mathrm{ren}}_{Y_{e^z},X}"'] \arrow[d, two heads] & X \otimes Y_{e^z} \otimes Z \arrow[r, "\mathrm{R}^{\mathrm{ren}}_{Y_{e^z},Z}"'] \arrow[d, phantom, "\text{Ⓑ}"] & X \otimes Z \otimes Y_{e^z} \arrow[d, two heads] \\
Y_{e^z} \otimes (X \nabla Z) \arrow[rr, "z^{\lambda}\mathrm{R}^{\mathrm{ren}}_{Y_{e^z},X\nabla Z}"'] & & (X \nabla Z) \otimes Y_{e^z}
\end{tikzcd}$$

up to constant in $\mathbf{k}[[z]]^{\times}$. Here the commutativity of Ⓐ follows from Proposition 2.5 and the one of Ⓑ from Lemma 2.26 and the hypothesis (ii). Hence, the following diagram is commutative:

$$\begin{tikzcd}
X \otimes Y_{e^z} \otimes Z \arrow[d, "\mathrm{R}^{\mathrm{ren}}_{X,Y_{e^z}}"] \arrow[dr, "\mathrm{R}^{\mathrm{ren}}_{Y_{e^z},Z}"] & \\
Y_{e^z} \otimes X \otimes Z \arrow[d, two heads] & X \otimes Z \otimes Y_{e^z} \arrow[d, two heads] \\
Y_{e^z} \otimes (X \nabla Z) \arrow[r, "\mathrm{R}^{\mathrm{ren}}_{Y_{e^z},X}"'] & (X \nabla Z) \otimes Y_{e^z}
\end{tikzcd}$$

Setting $z = 0$, we obtain a commutative diagram:

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rrr, two heads] \arrow[d, "\mathbf{r}_{X,Y}"] \arrow[dr, "\mathbf{r}_{Y,Z}"] & & & \operatorname{hd}(X \otimes Y \otimes Z) \arrow[dd, tail] \\
Y \otimes X \otimes Z \arrow[d, two heads] & X \otimes Z \otimes Y \arrow[d, two heads] \arrow[dr, "\mathbf{r}_{X,Z}"] & & \\
Y \otimes (X \nabla Z) \arrow[r, "\mathbf{r}_{Y,X\nabla Z}"'] & (X \nabla Z) \otimes Y \arrow[r, tail] & Z \otimes X \otimes Y \arrow[r, "\mathbf{r}_{X,Y}"'] & Z \otimes Y \otimes X
\end{tikzcd}$$

Let S be the image of the composition

$$Y \otimes X \otimes Z \twoheadrightarrow Y \otimes (X \nabla Z) \xrightarrow{\mathbf{r}_{Y,X\nabla Z}} (X \nabla Z) \otimes Y \rightarrowtail Z \otimes X \otimes Y.$$

Then $S \simeq Y \nabla (X \nabla Z)$ is a simple module. By the above commutative diagram, the image K of the composition of $X \otimes Y \otimes Z \xrightarrow{\mathbf{r}_{Y,Z}} X \otimes Z \otimes Y \xrightarrow{\mathbf{r}_{X,Z}} Z \otimes X \otimes Y$ is contained in S. On the other hand, the image of $K \to Z \otimes Y \otimes X$ is isomorphic to the simple module $\mathrm{hd}(X \otimes Y \otimes Z)$. Therefore K is non-zero and hence $K = S$, and it is isomorphic to $\mathrm{hd}(X \otimes Y \otimes Z)$. □

Proof of Theorem 4.25. We argue by induction on $|\,[a,b]\,|_\phi \geqslant 2$.

Assume first that $|\,[a,b]\,|_\phi = 2$, i.e., $b = a^+$. If $\ell > 1$ then we have $b - a + 1 \leqslant \ell$ and hence we have the desired result by Corollary 4.17. If $\ell = 1$, then we have $\mathsf{S}_b = \mathscr{D}\mathsf{S}_a$ and we have an exact sequence

$$0 \longrightarrow \mathbf{1} \longrightarrow \mathscr{D}\mathsf{S}_a \otimes \mathsf{S}_a \longrightarrow \mathscr{D}\mathsf{S}_a \nabla \mathsf{S}_a \longrightarrow 0$$

since S_a is a root module. Hence, for any ℓ, we have obtained an exact sequence:

$$0 \to \bigotimes_{\jmath \in I_{\mathrm{g}};\ d(\imath_a, \jmath)=1} M[a(\jmath)^+, (a^+)(\jmath)^-] \to \mathsf{S}_{a^+} \otimes \mathsf{S}_a \to \mathsf{S}_{a^+} \nabla \mathsf{S}_a \to 0. \tag{4.7}$$

Now we can assume $|\,[a,b]\,|_\phi > 2$.

In order to prove the theorem, it is enough to show

(I) $\mathfrak{d}(M[a,b^-], M[a^+,b]) \leqslant 1$,
(II) $M[a^+,b] \nabla M[a,b^-] \simeq M[a,b] \otimes M[a^+,b^-]$,
(III) $M[a,b^-] \nabla M[a^+,b] \simeq \bigotimes_{d(\imath_a,\jmath)=1} M[a(\jmath)^+, b(\jmath)^-]$,
(IV) $M[a^+,b] \nabla M[a,b^-] \not\simeq M[a,b^-] \nabla M[a^+,b]$.

Indeed, (I) and (IV) imply $\mathfrak{d}(M[a,b^-], M[a^+,b]) = 1$, and [38, Proposition 4.7] implies the existence of a short exact sequence

$$0 \longrightarrow M[a,b^-] \nabla M[a^+,b] \longrightarrow M[a^+,b] \otimes M[a,b^-] \longrightarrow M[a^+,b] \nabla M[a,b^-] \longrightarrow 0.$$

Now let us show (I)–(IV).

(I) follows from Lemma 4.22.

Let us show (II). The composition

$$M[a^+,b] \otimes M[a,b^-] \rightarrowtail M[a^+,b] \otimes \mathsf{S}_a \otimes M[a^+,b^-] \twoheadrightarrow M[a,b] \otimes M[a^+,b^-]$$

does not vanish by Proposition 2.9. Since $M[a,b]$ and $M[a^+,b^-]$ commute by Theorem 4.21, $M[a,b] \otimes M[a^+,b^-]$ is a simple module. Hence we obtain an epimorphism $M[a^+,b] \otimes M[a,b^-] \twoheadrightarrow M[a,b] \otimes M[a^+,b^-]$. Thus we obtain (II).

Let us show (III).

We shall first prove

$$\mathrm{hd}\big(M[a^{++},b] \otimes \mathsf{S}_a \otimes \mathsf{S}_{a^+}\big) \simeq \mathsf{S}_a \nabla M[a^+,b]. \tag{4.8}$$

In order to prove (4.8), we shall apply Lemma 4.26 with $X := M[a^{++}, b]$, $Y := \mathsf{S}_a$, and $Z := \mathsf{S}_{a^+}$. Since (X, Z) is unmixed, (X, Y, Z) is a normal sequence.

We shall verify (ii) in Lemma 4.26. Note that (ii) is equivalent to

$$\Lambda(Y, Z) - \Lambda(Y, X \nabla Z) = \Lambda(X, Y).$$

We have $\Lambda(M[a^{++}, b] \nabla \mathsf{S}_{a^+}, \mathsf{S}_a) = \Lambda(M[a^{++}, b], \mathsf{S}_a) + \Lambda(\mathsf{S}_{a^+}, \mathsf{S}_a)$ because $(M[a^{++}, b], \mathsf{S}_a)$ is unmixed. On the other hand, we have

$$2 = 2\,\mathfrak{d}(M[a^+, b], \mathsf{S}_a) = \Lambda(M[a^{++}, b] \nabla \mathsf{S}_{a^+}, \mathsf{S}_a) + \Lambda(\mathsf{S}_a, M[a^{++}, b] \nabla \mathsf{S}_{a^+})$$

and

$$2 = 2\,\mathfrak{d}(\mathsf{S}_{a^+}, \mathsf{S}_a) = \Lambda(\mathsf{S}_{a^+}, \mathsf{S}_a) + \Lambda(\mathsf{S}_a, \mathsf{S}_{a^+}).$$

Hence we obtain

$$2 - \Lambda(\mathsf{S}_a, M[a^{++}, b] \nabla \mathsf{S}_{a^+}) = \Lambda(M[a^{++}, b], \mathsf{S}_a) + \big(2 - \Lambda(\mathsf{S}_a, \mathsf{S}_{a^+})\big),$$

which implies (ii). Hence, Lemma 4.26 implies (4.8).

Thus, we have a chain of morphisms

$$\begin{aligned} M[a^+, b^-] \otimes \mathsf{S}_a \otimes M[a^+, b] &\twoheadrightarrow M[a^+, b^-] \otimes \big(\mathsf{S}_a \nabla M[a^+, b]\big) \\ &\simeq M[a^+, b^-] \otimes \mathrm{hd}\big(M[a^{++}, b] \otimes \mathsf{S}_a \otimes \mathsf{S}_{a^+}\big) \\ &\simeq M[a^+, b^-] \otimes \big(M[a^{++}, b] \nabla (\mathsf{S}_a \nabla \mathsf{S}_{a^+})\big). \end{aligned}$$

On the other hand, the pair $(M[a^+, b^-], \mathsf{S}_a \nabla \mathsf{S}_{a^+})$ is unmixed since

$$\mathsf{S}_a \nabla \mathsf{S}_{a^+} \simeq \bigotimes_{\jmath \in I_{\mathrm{g}};\ d(\imath_a, \jmath) = 1} M[a(\jmath)^+, (a^+)(\jmath)^-] \quad \text{by (4.7)} \quad \text{and} \quad (a^+)(\jmath)^- < a^+.$$

Therefore, $(M[a^+, b^-], M[a^{++}, b],\ \mathsf{S}_a \nabla \mathsf{S}_{a^+})$ is a normal sequence and we have an epimorphism

$$M[a^+, b^-] \otimes \mathsf{S}_a \otimes M[a^+, b] \twoheadrightarrow \big(M[a^+, b^-] \nabla M[a^{++}, b]\big) \nabla (\mathsf{S}_a \nabla \mathsf{S}_{a^+}).$$

By the induction hypothesis, we have

$$M[a^+, b^-] \nabla M[a^{++}, b] \simeq \bigotimes_{d(\jmath, \imath_a) = 1} M[(a^+)(\jmath)^+, b(\jmath)^-].$$

Hence [33, Lemma 3.2.22] implies that

$$\begin{aligned} &\big(M[a^+, b^-] \nabla M[a^{++}, b]\big) \nabla (\mathsf{S}_a \nabla \mathsf{S}_{a^+}) \\ &\qquad \simeq \bigotimes_{\jmath \in I_{\mathrm{g}};\ d(\imath_a, \jmath) = 1} \big(M[(a^+)(\jmath)^+, b(\jmath)^-] \nabla M[a(\jmath)^+, (a^+)(\jmath)^-]\big) \\ &\qquad \simeq A, \end{aligned}$$

where $A := \bigotimes_{\jmath \in I_{\mathfrak{g}};\, d(\imath_a,\jmath)=1} M[a(\jmath)^+, b(\jmath)^-]$.

Thus we obtain an epimorphism

$$M[a^+, b^-] \otimes \mathsf{S}_a \otimes M[a^+, b] \twoheadrightarrow A.$$

Since $\mathfrak{d}(M[a^+, b^-], \mathsf{S}_a) = 1$, we have an exact sequence

$$0 \longrightarrow \mathsf{S}_a \nabla M[a^+, b^-] \longrightarrow M[a^+, b^-] \otimes \mathsf{S}_a \longrightarrow M[a, b^-] \longrightarrow 0.$$

Now, we shall show

$$(\mathsf{S}_a \nabla M[a^+, b^-]) \text{ and } M[a^+, b] \text{ commute.} \tag{4.9}$$

We have

$$\mathfrak{d}\big(\mathsf{S}_a \nabla M[a^+, b^-],\ M[a^+, b]\big) \leqslant \mathfrak{d}\big(\mathsf{S}_a, M[a^+, b]\big) + \mathfrak{d}\big(M[a^+, b^-], M[a^+, b]\big) = 1 + 0 = 1.$$

Hence it is enough to show

$$\mathfrak{d}\big(\mathsf{S}_a \nabla M[a^+, b^-],\ M[a^+, b]\big) \neq \mathfrak{d}\big(\mathsf{S}_a, M[a^+, b]\big) + \mathfrak{d}\big(M[a^+, b^-], M[a^+, b]\big).$$

If the equality holds, then [40, Lemma 2.23] implies that $\big(M[a^+, b], \mathsf{S}_a, M[a^+, b^-]\big)$ is a normal sequence. Hence we have

$$\Lambda\big(M[a^+, b], M[a^+, b^-]\big) + \Lambda\big(\mathsf{S}_a, M[a^+, b^-]\big) - \Lambda\big(M[a^+, b] \nabla \mathsf{S}_a, M[a^+, b^-]\big) = 0.$$

On the other hand, we have

$$\Lambda\big(M[a^+, b], M[a^+, b^-]\big) = -\Lambda\big(M[a^+, b^-], M[a^+, b]\big),$$

and

$$\begin{aligned}\Lambda\big(M[a^+, b] \nabla \mathsf{S}_a, M[a^+, b^-]\big) &= -\Lambda\big(M[a^+, b^-], M[a^+, b] \nabla \mathsf{S}_a\big) \\ &= -\Lambda\big(M[a^+, b^-], M[a^+, b]\big) - \Lambda\big(M[a^+, b^-], \mathsf{S}_a\big),\end{aligned}$$

where the last equality follows from the normality of $\big(M[a^+, b^-], M[a^+, b], \mathsf{S}_a\big)$.

Thus, we obtain

$$\begin{aligned} 0 &= \Lambda\big(M[a^+, b], M[a^+, b^-]\big) + \Lambda\big(\mathsf{S}_a, M[a^+, b^-]\big) - \Lambda\big(M[a^+, b] \nabla \mathsf{S}_a, M[a^+, b^-]\big) \\ &= -\Lambda(M[a^+, b^-], M[a^+, b]) + \Lambda\big(\mathsf{S}_a, M[a^+, b^-]\big) \\ &\qquad\qquad - \big(-\Lambda(M[a^+, b^-], M[a^+, b]) - \Lambda(M[a^+, b^-], \mathsf{S}_a)\big) \\ &= 2\,\mathfrak{d}\big(\mathsf{S}_a, M[a^+, b^-]\big), \end{aligned}$$

which contradicts $\mathfrak{d}\big(\mathsf{S}_a, M[a^+,b^-]\big) = 1$. Thus we obtain (4.9). In particular, $\big(\mathsf{S}_a \nabla M[a^+,b^-]\big) \otimes M[a^+,b]$ is a simple module.

- The simple modules $\big(\mathsf{S}_a \nabla M[a^+,b^-]\big) \otimes M[a^+,b]$ and A are not isomorphic.

Indeed, we have $\mathfrak{d}\big(\mathscr{D}\mathsf{S}_b,\ (\mathsf{S}_a \nabla M[a^+,b^-]) \otimes M[a^+,b]\big) \geqslant \mathfrak{d}\big(\mathscr{D}\mathsf{S}_b, M[a^+,b]\big) = 1$ by Lemma 4.23 and $\mathfrak{d}\big(\mathscr{D}\mathsf{S}_b, A\big) = 0$.

Thus the composition

$$\big(\mathsf{S}_a \nabla M[a^+,b^-]\big) \otimes M[a^+,b] \rightarrowtail M[a^+,b^-] \otimes \mathsf{S}_a \otimes M[a^+,b] \twoheadrightarrow A$$

vanishes, and hence $M[a^+,b^-] \otimes \mathsf{S}_a \otimes M[a^+,b] \twoheadrightarrow A$ factors through $M[a,b^-] \otimes M[a^+,b]$. Thus we obtain an epimorphism

$$M[a,b^-] \otimes M[a^+,b] \twoheadrightarrow A,$$

which completes the proof of (III).

Finally, (IV) follows from $\mathfrak{d}\big(\mathscr{D}\mathsf{S}_b, M[a,b^-] \nabla M[a^+,b]\big) = \mathfrak{d}\big(\mathscr{D}\mathsf{S}_b, M[a,b] \otimes M[a^+,b^-]\big) = 1$ and $\mathfrak{d}\big(\mathscr{D}\mathsf{S}_b, M[a^+,b] \nabla M[a,b^-]\big) = \mathfrak{d}\big(\mathscr{D}\mathsf{S}_b, \bigotimes_{\jmath \in I_{\mathfrak{g}};\ d(\imath_a,\jmath)=1} M[a(\jmath)^+, b(\jmath)^-]\big) = 0$. $\square$

5. Admissible chains of i-boxes

In this section, we construct commuting families of real simple modules consisting of affine determinantial modules and then investigate relations among the families, called box moves. Throughout this section, we fix a PBW-pair $(\mathcal{D}, \underline{\widehat{w}_0})$ and keep the data and notations in the previous section.

5.1. **Chains of i-boxes.** Recall $[a,b\}$ and $\{a,b]$ defined in (4.4).

Definition 5.1.

(1) A chain $\mathfrak{C}$ of i-boxes

$$(\mathfrak{c}_k = [a_k,b_k])_{1 \leqslant k \leqslant l} \quad \text{for } l \in \mathbb{Z}_{\geqslant 1} \sqcup \{\infty\}$$

is called *admissible* if

$$\widetilde{\mathfrak{c}}_k = [\widetilde{a}_k, \widetilde{b}_k] := \bigcup_{1 \leqslant j \leqslant k} [a_j, b_j] \text{ is an interval with } |\widetilde{\mathfrak{c}}_k| = k \text{ for } k = 1, \ldots, l$$

and

$$\text{either } [a_k, b_k] = [\widetilde{a}_k, \widetilde{b}_k\} \text{ or } [a_k, b_k] = \{\widetilde{a}_k, \widetilde{b}_k] \text{ for } k = 1, \dots, l.$$

(2) The interval $\widetilde{\mathfrak{c}}_k$ is called the *envelope* of $\mathfrak{c}_k$, and $\widetilde{\mathfrak{c}}_l$ is called the *range* of $\mathfrak{C}$.

Note that the chain is uniquely determined by its envelopes:

$$\mathfrak{c}_k = [a_k, b_k] := T_{k-1}[\widetilde{a}_k, \widetilde{b}_k] = \begin{cases} [\widetilde{a}_k, \widetilde{b}_k\} & \text{(i) if } \widetilde{a}_k = \widetilde{a}_{k-1} - 1, \\ \{\widetilde{a}_k, \widetilde{b}_k] & \text{(ii) if } \widetilde{b}_k = \widetilde{b}_{k-1} + 1, \end{cases} \tag{5.1}$$

for $1 < k \leqslant l$. In case (i) in (5.1), we write $T_{k-1} = \mathcal{L}$, while $T_{k-1} = \mathcal{R}$ in case (ii).

Thus, to an admissible chain of i-boxes of length l, we can associate a pair $(a, \mathfrak{T})$ consisting of an integer a and a sequence $\mathfrak{T} = (T_1, T_2, \dots, T_{l-1})$ such that $T_i \in \{\mathcal{L}, \mathcal{R}\}$ $(1 \leqslant i < l)$,

$$a = a_1 = b_1 = \widetilde{a}_1 = \widetilde{b}_1, \quad [\widetilde{a}_k, \widetilde{b}_k] = \begin{cases} [\widetilde{a}_{k-1} - 1, \widetilde{b}_{k-1}] & \text{if } T_{k-1} = \mathcal{L}, \\ [\widetilde{a}_{k-1}, \widetilde{b}_{k-1} + 1] & \text{if } T_{k-1} = \mathcal{R}. \end{cases}$$

Note that $[a_k, b_k] = [\widetilde{a}_k, \widetilde{b}_k\}$ or $\{\widetilde{a}_k, \widetilde{b}_k]$ according that $T_{k-1} = \mathcal{L}$ or $T_{k-1} = \mathcal{R}$.

Lemma 5.2. *Let $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leqslant k \leqslant l}$ be an admissible chain of i-boxes, and $(\widetilde{\mathfrak{c}}_k)_{1 \leqslant k \leqslant l}$ its envelopes. Then, we have*

(i) $(a_k)^- < \widetilde{a}_k \leqslant \widetilde{b}_k < (b_k)^+$,
(ii) $(a_k)^- < a_j \leqslant b_j < (b_k)^+$ *if* $1 \leqslant j \leqslant k \leqslant l$.
In particular, $\mathfrak{c}_j$ and $\mathfrak{c}_k$ commute.

Proof. (i) Set $\imath = \imath_{a_k} = \imath_{b_k}$. Then we have $a_k = \widetilde{a}_k(\imath)^+$. Hence we have $(a_k)^- < \widetilde{a}_k$. The other strict inequality on b_k can be proved similarly.

(ii) This assertion follows from

$$(a_k)^- < \widetilde{a}_k \leqslant a_j \leqslant b_j \leqslant \widetilde{b}_k < (b_k)^+. \qquad \square$$

For an admissible chain of i-boxes $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leqslant k \leqslant l}$ and $1 \leqslant t \leqslant l$, we set

$$\mathfrak{C}_{\leqslant t} := (\mathfrak{c}_k)_{1 \leqslant k \leqslant t}.$$

Proposition 5.3. *Let $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leqslant k \leqslant l}$ be an admissible chain of i-boxes and let $\mathfrak{c} = [a, b]$ be an i-box such that $\mathfrak{c} \subset \widetilde{\mathfrak{c}}_l$ and $\mathfrak{c}$ commutes with all $\mathfrak{c}_k$ $(1 \leqslant k \leqslant l)$. Then there exists $s \in [1, l]$ such that $\mathfrak{c} = \mathfrak{c}_s$.*

Proof. We can assume that $\mathfrak{c} \subset \widetilde{\mathfrak{c}}_l$ and $\mathfrak{c} \not\subset \widetilde{\mathfrak{c}}_{l-1}$ and $l \geqslant 2$, which implies

$$\text{(i)} \quad a = \widetilde{a}_l < \widetilde{a}_{l-1} \qquad \text{or} \qquad \text{(ii)} \quad b = \widetilde{b}_l > \widetilde{b}_{l-1}.$$

Since the case (ii) can be proved similarly, we only treat the case (i).

Then $\mathfrak{c}_l = [a_l, b_l]$ satisfies $a = a_l = \widetilde{a}_l$ and $b_l = \widetilde{b}_l(\imath_b)^-$. Hence, we have $b \leqslant b_l$. If $b = b_l$, we have $\mathfrak{c} = \mathfrak{c}_l$. Thus we may assume that $b < b_l$. Hence $b^+ \leqslant b_l \leqslant \widetilde{b}_l = \widetilde{b}_{l-1}$. Take the smallest $s \geqslant 1$ such that $b^+ \leqslant \widetilde{b}_s$. Then $1 \leqslant s \leqslant l-1$.

Assume first $s > 1$. Then we have $\widetilde{b}_{s-1} < b^+ \leqslant \widetilde{b}_s$, and hence $b^+ = \widetilde{b}_s = b_s$. Since $\mathfrak{c}$ and $\mathfrak{c}_s$ commute and $b^+ \leqslant b_s$, we have

$$a_s^- < a \leqslant b < b_s^+.$$

Hence we have $a_s^- < a < \widetilde{a}_{l-1} \leqslant a_s$, which contradicts $\imath_{a_s} = \imath_{b_s} = \imath_{b^+} = \imath_a$.

Hence we have $s = 1$. Therefore, $\widetilde{a}_l < b^+ \leqslant \widetilde{a}_1$. Now take the smallest $t \geqslant 1$ such that $\widetilde{a}_t \leqslant b^+$. Since $\widetilde{a}_l \leqslant a \leqslant b < b^+$, we have $\widetilde{a}_{l-1} = 1 + \widetilde{a}_l \leqslant b^+$. Hence we have $1 \leqslant t \leqslant l-1$.

If $t > 1$, then we have $\widetilde{a}_t \leqslant b^+ < \widetilde{a}_{t-1}$, and hence $b^+ = \widetilde{a}_t = a_t$. If $t = 1$, then $b^+ = \widetilde{a}_1 = a_1$. In any case, we have $b^+ = a_t$.

Since $\mathfrak{c}$ and $\mathfrak{c}_t$ commute and $b^+ \leqslant b_t$, we have

$$a_t^- < a \leqslant b < b_t^+.$$

Hence, we obtain $a_t^- < a = \widetilde{a}_l < \widetilde{a}_{l-1} \leqslant \widetilde{a}_t \leqslant a_t$, which contradicts $\imath_{a_t} = \imath_{b^+} = \imath_a$. $\square$

Corollary 5.4. Let $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leqslant k \leqslant l}$ be an admissible chain of i-boxes. Let $\mathfrak{c} = [a, b]$ be an i-box such that $a^- < \widetilde{a}_l \leqslant a$ and $b \leqslant \widetilde{b}_l < b^+$, i.e. $a = (\widetilde{a}_l)(\imath)^+$ and $b = (\widetilde{b}_l)(\imath)^-$ for some $\imath \in I_{\mathfrak{g}}$. Then $\mathfrak{c}$ is a member of $\mathfrak{C}$.

Proof. It follows from the fact that $\mathfrak{c} \subset \widetilde{\mathfrak{c}}_l$ commutes with all $\mathfrak{c}_k$ $(1 \leqslant k \leqslant l)$. $\square$

For an admissible chain $\mathfrak{C}$ of i-boxes, let us define a family of real simple modules $\mathsf{M}(\mathfrak{C})$ as follows:

$$\mathsf{M}(\mathfrak{C}) := \{M(\mathfrak{c}_k)\}_{1 \leqslant k \leqslant l}.$$

Theorem 5.5. *For any admissible chain $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leqslant k \leqslant l}$, $\mathsf{M}(\mathfrak{C})$ forms a commuting family of real simple modules.*

Proof. By Lemma 5.2, $\mathfrak{c}_{k_1}$ and $\mathfrak{c}_{k_2}$ commute for all $k_1, k_2 \in [1, l]$. Thus our assertion follows from Theorem 4.21. $\square$

5.2. **Box moves.** For an admissible chain $\mathfrak{C} = (\mathfrak{c}_k)_{1\leqslant k\leqslant l}$ with the associated pair $(a, \mathfrak{T})$ and for $1 \leqslant s < l$, we say that an i-box $\mathfrak{c}_s$ is *movable* if $s = 1$ or $T_{s-1} \neq T_s$ $(s \geqslant 2)$. Note that the last condition (for $s \geqslant 2$) is equivalent to the condition $\widetilde{a}_{s+1} = \widetilde{a}_{s-1} - 1$ and $\widetilde{b}_{s+1} = \widetilde{b}_{s-1} + 1$.

For a movable $\mathfrak{c}_s$ in $\mathfrak{C}$, we define a new admissible chain $B_s(\mathfrak{C})$ whose associated pair $(a', \mathfrak{T}')$ is given

$$\text{(i)} \begin{cases} a' = a \pm 1 & \text{if } s = 1 \text{ and } T_1 = \mathcal{R} \text{ (resp. } \mathcal{L}), \\ a' = a & \text{if } s > 1, \end{cases}$$

$$\text{(ii) } T'_k = T_k \quad \text{for } k \notin \{s-1, s\} \qquad \text{and} \qquad T'_k \neq T_k \quad \text{for } k \in \{s-1, s\}.$$

That is, $B_s(\mathfrak{C})$ is the admissible chain obtained from $\mathfrak{C}$ by moving $\widetilde{\mathfrak{c}}_s$ by 1 to the right or to the left inside $\widetilde{\mathfrak{c}}_{s+1}$. We call $B_s(\mathfrak{C})$ the *box move* of $\mathfrak{C}$ at s.

Proposition 5.6. *Let $\mathfrak{C} = (\mathfrak{c}_k)_{1\leqslant k\leqslant l}$ be an admissible chain of i-boxes and let k_0 be a movable i-box $(1 \leqslant k_0 < l)$. Set $B_{k_0}(\mathfrak{C}) = (\mathfrak{c}'_k)_{1\leqslant k\leqslant l}$. Assume that $\imath_{\widetilde{a}_{k_0+1}} \neq \imath_{\widetilde{b}_{k_0+1}}$, i.e., $\widetilde{\mathfrak{c}}_{k_0+1}$ is not an i-box. Then we have*

$$\mathfrak{c}'_k = \mathfrak{c}_{\mathfrak{s}_{k_0}(k)},$$

where $\mathfrak{s}_{k_0} \in \mathfrak{S}_l$ is the transposition of k_0 and $k_0 + 1$.

Proof. Set $p = k_0 + 1$.By the assumption, we have

$$[\widetilde{a}_p, \widetilde{b}_p\} = [\widetilde{a}_p, \widetilde{b}_p - 1\} \quad \text{and} \quad \{\widetilde{a}_p, \widetilde{b}_p] = \{\widetilde{a}_p + 1, \widetilde{b}_p],$$

which implies the desired result. □

Proposition 5.7. *Let $\mathfrak{C} = (\mathfrak{c}_k)_{1\leqslant k\leqslant l} = (c, \mathfrak{T})$ be an admissible chain of i-boxes and let $\mathfrak{c}_{k_0}$ be a movable i-box. Assume that $\imath_{\widetilde{a}_{k_0+1}} = \imath_{\widetilde{b}_{k_0+1}}$, i.e., $\widetilde{\mathfrak{c}}_{k_0+1}$ is an i-box. Set $\mathfrak{c}_{k_0+1} = [a, b]$ with $a = \widetilde{a}_{k_0+1}$ and $b = \widetilde{b}_{k_0+1}$, and set $B_{k_0}(\mathfrak{C}) = (\mathfrak{c}'_k)_{1\leqslant k\leqslant l}$. Then we have*

(i) *$\mathfrak{c}_{k_0} = [a^+, b]$ and $\mathfrak{c}'_{k_0} = [a, b^-]$ if $T_{k_0-1} = \mathcal{R}$,*
(ii) *$\mathfrak{c}_{k_0} = [a, b^-]$ and $\mathfrak{c}'_{k_0} = [a^+, b]$ if $T_{k_0-1} = \mathcal{L}$,*

In particular, we have an exact sequence

$$0 \to \bigotimes_{d(\imath_a, \jmath)=1} M[a(\jmath)^+, b(\jmath)^-] \to X \otimes Y \to M(\mathfrak{c}_{k_0+1}) \otimes M[a^+, b^-] \to 0. \tag{5.2}$$

where $(X, Y) = \big(M(\mathfrak{c}_{k_0}), M(\mathfrak{c}'_{k_0})\big)$ in case (i) and $(X, Y) = \big(M(\mathfrak{c}'_{k_0}), M(\mathfrak{c}_{k_0})\big)$ in case (ii).

Proof. By the assumption that $\imath_{\widetilde{a}_{k_0+1}} = \imath_{\widetilde{b}_{k_0+1}}$, $[a,b] := [a_{k_0+1}, b_{k_0+1}] = [\widetilde{a}_{k_0+1}, \widetilde{b}_{k_0+1}]$.
In case (i), we have

$$[a_{k_0}, b_{k_0}] = \{a_{k_0+1}+1, \widetilde{b}_{k_0+1}] = [a^+, b] \quad \text{and} \quad \mathfrak{c}'_{k_0} = [\widetilde{a}_{k_0+1}, \widetilde{b}_{k_0+1}-1\} = [a, b^-].$$

The proof in case (b) is similar.
The last statement follows from Theorem 4.25. □

Remark 5.8. When $\mathfrak{c}_{k_0}$ is movable and $\widetilde{\mathfrak{c}}_{k_0+1} = \mathfrak{c}_{k_0+1} = [a,b]$ is an i-box, Corollary 5.4 tells that the i-boxes $\{[a(\jmath)^+, b(\jmath)^-] \mid d(\imath_a, \jmath) = 1\}$ appearing in (5.2) are all contained in $\mathfrak{C}_{\leqslant k_0+1}$ and hence in $\mathfrak{C}$. Note that $[a^+, b^-]$ commutes with i-boxes in $\{[a^+,b], [a,b^-], [a,b]\} = \{\mathfrak{c}_{k_0-1}, \mathfrak{c}_{k_0}, \mathfrak{c}_{k_0+1}\}$, and it also commutes with all i-boxes in $\mathfrak{C}_{\leqslant k_0-2}$ since $(a^+)^- = \widetilde{a}_{k_0+1} < \widetilde{a}_k \leqslant a_k \leqslant b_k \leqslant \widetilde{b}_k < \widetilde{b}_{k_0+1} = (b^-)^+$ for $1 \leqslant k \leqslant k_0-2$. It follows that $[a^+, b^-]$ is contained in $\mathfrak{C}$. Thus, the operation B_{k_0} under this situation can be understood as a combinatorial analogue of T-system (see Lemma 7.17 below, also [14, Section 13] and [15, Section 12]).

Definition 5.9. For admissible chains $\mathfrak{C}^{(1)}$ and $\mathfrak{C}^{(2)}$ of the same length $l \in \mathbb{Z}_{\geqslant 1}$, we say that they are *$T$-equivalent*, denoted by $\mathfrak{C}^{(1)} \overset{T}{\sim} \mathfrak{C}^{(2)}$, if there exists a sequence $(p_1, p_2, \ldots, p_r) \in \{1, 2, \ldots, l-1\}^r$ $(r \in \mathbb{Z}_{\geqslant 1})$ such that

$$B_{p_r}(\cdots(B_{p_2}(B_{p_1}(\mathfrak{C}^{(1)}))\cdots) = \mathfrak{C}^{(2)}.$$

The following lemma is almost obvious.

Lemma 5.10.

(i) *The binary relation $\overset{T}{\sim}$ for admissible chains of finite length is an equivalence relation.*
(ii) *If $\mathfrak{C}^{(1)}$ and $\mathfrak{C}^{(2)}$ are T-equivalent, then they have the same range.*
(iii) *Two admissible chains $\mathfrak{C}^{(1)}$ and $\mathfrak{C}^{(2)}$ with the same range are T-equivalent.*

Example 5.11. Let us consider a PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$ with $\mathfrak{g} = A_3$ and

$$\begin{aligned} \widehat{\underline{w}}_0 &= \cdots\, s_{\imath_{-2}}\ s_{\imath_{-1}}\ s_{\imath_0}\ s_{\imath_1}\ s_{\imath_2}\ s_{\imath_3} \cdots \\ &= \cdots\ \ s_3\ \ \ s_2\ \ \ s_3\ \ s_1\ \ s_2\ \ s_3\ \cdots \end{aligned}$$

and an admissible chain $\mathfrak{C}^{(1)} = (\mathfrak{c}^{(1)}_k)_{1 \leqslant k \leqslant 3} = (0, \mathfrak{T} = (\mathcal{L}, \mathcal{L})) = ([0], [-1,0\}, [-2,0\}) = ([0], [-1], [-2,0])$ associated to $\widehat{\underline{w}}_0$.

Note there exists only one movable box $\mathfrak{c}^{(1)}_1$ in $\mathfrak{C}^{(1)}$. Thus we have

$$\begin{aligned} \mathfrak{C}^{(2)} = (\mathfrak{c}^{(2)}_k)_{1 \leqslant k \leqslant 3} := B_1(\mathfrak{C}^{(1)}) &= \big(-1, \mathfrak{T} = (\mathcal{R}, \mathcal{L})\big) \\ &= \big([-1], \{-1,0], [-2,0\}\big) = \big([-1], [0], [-2,0]\big), \end{aligned}$$

since $[-1,0]$ is not an i-box (see Proposition 5.6).

For $\mathfrak{C}^{(2)}$, the second i box $\mathfrak{c}_2^{(2)}$ is movable and hence we have

$$\begin{aligned}\mathfrak{C}^{(3)} = (\mathfrak{c}_k^{(3)})_{1\leqslant k\leqslant 3} := B_1(\mathfrak{C}^{(2)}) &= \big(-1, \mathfrak{T} = (\mathcal{L},\mathcal{R})\big)\\ &= \big([-1],[-2,-1\},\{-2,0]\big) = \big([-1],[-2],[-2,0]\big),\end{aligned}$$

since $[-2,0]$ is an i-box (see Proposition 5.7).

Finally, $\mathfrak{c}_1^{(3)}$ is movable hence we have

$$\mathfrak{C}^{(4)} = B_1(\mathfrak{C}^{(3)}) = \big(-2, \mathfrak{T} = (\mathcal{R},\mathcal{R})\big) = \big([-2],\{-2,-1],\{-2,0]\big) = \big([-2],[-1],[-2,0]\big).$$

6. Q-data and associated PBW-pairs

In this section, we recall the notion of Q-data introduced in [13]. A Q-datum is a generalization of a Dynkin quiver with a height function. Then we attach a complete PBW-pair to each Q-datum.

6.1. Q-data. For each untwisted quantum affine algebra $U_q'(\mathfrak{g})$, we associate a finite simple Lie algebra $\mathfrak{g}_{\rm fin}$ of simply-laced type as follows (see § 2.4 and [39]):

(6.1)

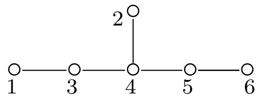
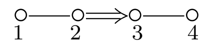
$\mathfrak{g}$	$A_n^{(1)}$ $(n \geqslant 1)$	$B_n^{(1)}$ $(n \geqslant 2)$	$C_n^{(1)}$ $(n \geqslant 3)$	$D_n^{(1)}$ $(n \geqslant 4)$	$E_{6,7,8}^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$
$\mathfrak{g}_{\rm fin}$	A_n	A_{2n-1}	D_{n+1}	D_n	$E_{6,7,8}$	E_6	D_4
$\mathsf{h}^\vee$	$n+1$	$2n-1$	$n+1$	$2n-2$	12, 18, 30	9	4

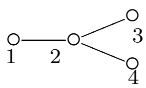
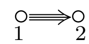
Here $\mathsf{h}^\vee := \langle c,\rho\rangle = \sum_{i\in I}\mathsf{c}_i$ is the dual Coxeter number of $\mathfrak{g}$.

Let $\triangle_{\mathfrak{g}_{\rm fin}}$ be the Dynkin diagram for $\mathfrak{g}_{\rm fin}$. Then there exists a Dynkin diagram automorphism σ of $\triangle_{\mathfrak{g}_{\rm fin}}$ whose orbit set $\triangle_{\mathfrak{g}_{\rm fin}}^\sigma$ yields the Dynkin diagram $\triangle_{\mathfrak{g}_0}$: for $\mathfrak{g} = (ADE)_n^{(1)}$-case, we have $\sigma = \mathrm{id}$. In the remaining cases, σ is given by $\vee$ or $\widetilde{\vee}$ in the following diagrams:

$$\Big(\triangle_{A_{2n-1}}: \underset{1}{\circ}-\underset{2}{\circ}-\cdots\cdots\underset{}{\circ}-\underset{2n-2}{\circ}-\underset{2n-1}{\circ}\ ,\ k^\vee = 2n-k\Big) \Longrightarrow \triangle_{B_n}: \underset{1}{\circ}-\underset{2}{\circ}-\cdots\cdots\underset{n-1}{\circ}\Rightarrow\underset{n}{\circ}\ ,$$

$$\left(\triangle_{D_{n+1}}: \underset{1}{\circ}-\underset{2}{\circ}\cdots\cdots\underset{n-1}{\circ}\genfrac{}{}{0pt}{}{<\ \underset{n}{\circ}}{<\ \underset{n+1}{\circ}},\ k^\vee = \begin{cases} k & \text{if } k\leqslant n-1,\\ n+1 & \text{if } k=n,\\ n & \text{if } k=n+1\end{cases}\right) \Longrightarrow \triangle_{C_n}: \underset{1}{\circ}-\underset{2}{\circ}-\cdots\cdots\underset{n-1}{\circ}\Leftarrow\underset{n}{\circ}\ ,$$

$\left(\triangle_{E_6}:$  $, \begin{cases} 1^\vee = 6,\ 6^\vee = 1, \\ 3^\vee = 5,\ 5^\vee = 3, \\ 4^\vee = 4,\ 2^\vee = 2 \end{cases} \right) \Longrightarrow \triangle_{F_4}:$  ,

$\left(\triangle_{D_4}:$  $, \begin{cases} 1^{\widetilde{\vee}} = 3,\ 3^{\widetilde{\vee}} = 4,\ 4^{\widetilde{\vee}} = 1, \\ 2^{\widetilde{\vee}} = 2 \end{cases} \right) \Longrightarrow \triangle_{G_2}:$  .

We denote by $I_{\rm fin} = \{1,2,\ldots,|I_{\rm fin}|\}$ the index set of $\triangle_{\mathfrak{g}_{\rm fin}}$. Let $(\mathsf{C}, \mathsf{P}_{\rm fin}, \Pi_{\rm fin}, \mathsf{P}^\vee_{\rm fin}, \Pi^\vee_{\rm fin})$ be the finite Cartan datum associated to $\mathfrak{g}_{\rm fin}$. Let us denote by $\mathsf{W}_{\rm fin}$ the Weyl group, by $\{\Lambda_\imath \mid \imath \in I_{\rm fin}\}$ the set of fundamental weights, by Φ the set of roots, by $\mathsf{P}^+_{\rm fin}$ the set of dominant integral weights, by $\alpha_\imath$ the $\imath$-th simple root, by Q the root lattice of $\mathfrak{g}_{\rm fin}$, and by $(\ ,\ )$ the symmetric bilinear form on Q, all of which are associated to $\mathfrak{g}_{\rm fin}$. We also write Φ^+ (resp. Φ^-) for the set of positive (resp. negative) roots of $\mathfrak{g}_{\rm fin}$ and Q^+ (resp. Q^-) for the positive (resp. negative) root lattice of $\mathfrak{g}_{\rm fin}$. For $\beta = \sum_{\imath \in I_{\rm fin}} k_\imath \alpha_\imath \in \mathsf{Q}$, we set $|\beta| := \sum_{\imath \in I_{\rm fin}} |k_\imath|$.

Let $I_{\mathfrak{g}_0} = \{1,2,\ldots,n\}$ be the index set of $\triangle_{\mathfrak{g}_0}$. Note that $I_{\rm fin} = I_{\mathfrak{g}_0}$ when $\mathfrak{g} = (ADE)^{(1)}_n$. To distinguish the index sets $I_{\rm fin}$ and $I_{\mathfrak{g}_0}$, we use $\imath$ for indices in $I_{\rm fin}$ and i for indices in $I_{\mathfrak{g}_0}$. We write $\pi\colon I_{\rm fin} \to I_{\mathfrak{g}_0}$ for the projection. Then the orbit of $\imath$ is given as

$$\pi^{-1}\pi(\imath) = \{\sigma^m(\imath) \mid m \in \mathbb{Z}\}.$$

We denote by $\mathsf{A} = (a_{i,j})_{i,j \in I_{\mathfrak{g}_0}}$ the Cartan matrix associated to $\mathfrak{g}_0$. Then we have

$$a_{i,j} = -|\{\jmath \in \pi^{-1}(j) \mid d(\imath,\jmath) = 1\}|.$$

for $i \neq j \in I_{\mathfrak{g}_0}$ and $\imath \in \pi^{-1}(i)$. Set $\mathsf{d}_i = |\pi^{-1}(i)|$ and $\mathsf{D} = \mathrm{diag}(\mathsf{d}_i \mid i \in I_{\mathfrak{g}_0})$ the diagonal matrix. Then, DA is symmetric.

For $\imath \in I_{\rm fin}$, we write also $\mathsf{d}_\imath$ for $\mathsf{d}_{\pi(\imath)}$, i.e.,

$$\mathsf{d}_\imath = |\pi^{-1}\pi(\imath)|.$$

Note that $\mathsf{d}_\imath \in \{1, \mathrm{ord}(\sigma)\}$ and

$$(\boldsymbol{\alpha}_{\pi(\imath)}, \boldsymbol{\alpha}_{\pi(\imath)}) = 2\,\mathsf{d}_\imath/\mathrm{ord}(\sigma).$$

In this subsection, we fix $\mathfrak{g}$ and a pair $(\triangle_{\mathfrak{g}_{\rm fin}}, \sigma)$ of the Dynkin diagram $\triangle_{\mathfrak{g}_{\rm fin}}$ of finite type $\mathfrak{g}_{\rm fin}$ and an automorphism $\sigma = \mathrm{id}, \vee$ or $\widetilde{\vee}$. If there is no afraid of confusion, we simply write $\triangle$ for $\triangle_{\mathfrak{g}_{\rm fin}}$.

Definition 6.1 ([13, Definition 2.5])**.** A function $\xi\colon I_{\rm fin} \to \mathbb{Z}$ is called a *height function on* $(\triangle, \sigma)$ if the following two conditions are satisfied.

(1) For any $\imath, \jmath \in I_{\rm fin}$ such that $d(\imath,\jmath) = 1$ and $\mathsf{d}_\imath = \mathsf{d}_\jmath$, we have $|\xi_\imath - \xi_\jmath| = \mathsf{d}_\imath$.

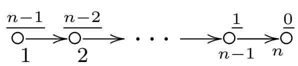
(2) For any $\imath, \jmath \in I_{\mathrm{fin}}$ such that $d(\imath,\jmath) = 1$ and $1 = \mathsf{d}_\imath < \mathsf{d}_\jmath = \mathrm{ord}(\sigma)$, there exists a unique element $j^\circ \in \pi^{-1}\pi(\jmath)$ such that $|\xi_\imath - \xi_{j^\circ}| = 1$ and $\xi_{\sigma^k(j^\circ)} = \xi_{j^\circ} + 2k$ for any $0 \leqslant k < \mathrm{ord}(\sigma)$.

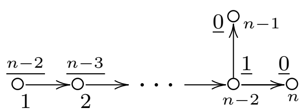
We call the triple $\mathscr{Q} = (\triangle, \sigma, \xi)$ a Q-*datum* for $\mathfrak{g}$.

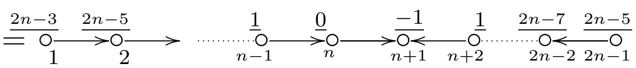
Note that, when $\sigma = \mathrm{id}$, a Q-datum coincides with a usual Dynkin quiver with a height function ([44]).

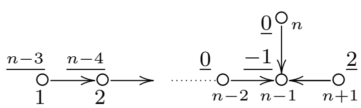
Remark 6.2. The convention for a Q-datum in this paper is associated to the *sink-adapted orientation* which is different from the *source-adapted orientation* in [13, 21]. We take this convention in order to match $M[a,b]$ as the image of a certain unipotent quantum minor under the quantum affine Schur-Weyl duality functor (see § 4.1 and § 6.3 below).

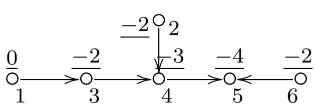
Example 6.3. For an untwisted affine type $\mathfrak{g}$, we give examples of Q-data $\mathscr{Q}$:

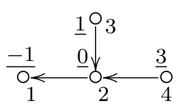
(1) $\mathscr{Q} =$  for $\mathfrak{g} = A_n^{(1)}$, $\mathrm{ord}(\sigma) = 1$,

(2) $\mathscr{Q} =$  for $\mathfrak{g} = D_n^{(1)}$, $\mathrm{ord}(\sigma) = 1$,

(3) $\mathscr{Q} =$  for $\mathfrak{g} = B_n^{(1)}$, $\mathrm{ord}(\sigma) = 2$,

(4) $\mathscr{Q} =$  for $\mathfrak{g} = C_n^{(1)}$, $\mathrm{ord}(\sigma) = 2$,

(5) $\mathscr{Q} =$  for $\mathfrak{g} = F_4^{(1)}$, $\mathrm{ord}(\sigma) = 2$,

(6) $\mathscr{Q} =$  for $\mathfrak{g} = G_2^{(1)}$, $\mathrm{ord}(\sigma) = 3$.

Here,

(1) an underline integer $\underline{*}$ is the value of $\xi_\imath$ at each vertex $\imath \in \triangle$,
(2) an arrow $\imath \to \jmath$ means that $\xi_\imath > \xi_\jmath$ and $d(\imath,\jmath) = 1$.

For a Q-datum $\mathscr{Q}$, we call a vertex $\imath \in I_{\mathrm{fin}}$ a *sink* of $\mathscr{Q}$ if $\xi_\imath < \xi_\jmath$ for all $\jmath \in I_{\mathrm{fin}}$ such that $d(\imath,\jmath) = 1$. We also call a vertex a *source* of $\mathscr{Q}$ if $\xi_\imath - 2\mathsf{d}_\imath > \xi_\jmath - 2\mathsf{d}_\jmath$ for all $\jmath \in I_{\mathrm{fin}}$ such that $d(\imath,\jmath) = 1$.

For a Q-datum $\mathscr{Q} = (\triangle, \sigma, \xi)$ and its sink $\imath$, we denote by $s_\imath \mathscr{Q}$ the Q-datum $(\triangle, \sigma, s_\imath \xi)$ where $s_\imath \xi$ is the height function defined as follows ([13, Lemma 2.11]):

$$(s_\imath \xi)_\jmath = \xi_\jmath + \delta_{\imath\jmath} \times 2\,\mathsf{d}_\imath,$$

Then $s_\imath \mathscr{Q}$ becomes a Q-datum associated to the same $\mathfrak{g}$ of $\mathscr{Q}$. Similarly, we can define the Q-datum $s_\imath^{-1}\mathscr{Q}$ for a source $\imath \in I_{\rm fin}$ of $\mathscr{Q}$. We call these operations as (combinatorial) *reflection functors* on Q-data associated to $\mathfrak{g}$.

For a reduced expression $\underline{w} = s_{\imath_1} s_{\imath_2} \cdots s_{\imath_l}$ of $w \in \mathsf{W}_{\rm fin}$ and $1 \leqslant k \leqslant l$, we set

$$\underline{w}_{\leqslant k} := s_{\imath_1} s_{\imath_2} \cdots s_{\imath_k}, \quad \underline{w}_{< k} := s_{\imath_1} s_{\imath_2} \cdots s_{\imath_{k-1}} \quad \text{and} \quad \beta^{\underline{w}}_k := s_{\imath_1} \cdots s_{\imath_{k-1}} \alpha_{\imath_k}. \tag{6.2}$$

Let $\mathscr{Q}$ be a Q-datum associated to $\mathfrak{g}$ and let $\mathsf{W}_{\rm fin}$ be the Weyl group of type $\triangle_{\mathfrak{g}_{\rm fin}}$. For $w \in \mathsf{W}_{\rm fin}$ and its reduced expression $\underline{w} = s_{\imath_1} s_{\imath_2} \cdots s_{\imath_l}$, we say that $\underline{w}$ is *adapted to* $\mathscr{Q}$ (or simply $\mathscr{Q}$*-adapted*) if

$$\imath_k \text{ is a sink of } s_{\imath_{k-1}} s_{\imath_{k-2}} \cdots s_{\imath_1} \mathscr{Q} \text{ for all } 1 \leqslant k \leqslant l.$$

Proposition 6.4 ([13, Corollary 2.21])**.** *Let $w_0 = s_{\imath_1} \cdots s_{\imath_\ell}$ be a reduced expression of w_0. Then the following conditions* (i) *and* (ii) *are equivalent*:

(i) *The following two conditions hold*:
  (a) *For any $\imath$, $\jmath \in I_{\rm fin}$ such that $d(\imath, \jmath) = 1$ and any s such that $1 \leqslant s < s^+ \leqslant \ell$ and $\imath = \imath_s$, we have*

$$-a_{\pi(\jmath),\pi(\imath)} = \begin{cases} |\{t \mid s < t < s^+, \pi(\jmath) = \pi(\imath_t)\}| & \text{if } \mathsf{d}_\imath < \mathsf{d}_\jmath, \\ |\{t \mid s < t < s^+, \jmath = \imath_t\}| & \text{otherwise.} \end{cases}$$

  Here $s^+ := \min\big(\{p \mid s < p \leqslant \ell,\ \imath_p = \imath_s\} \cup \{+\infty\}\big)$ and $s^- := \max\big(\{p \mid s > p \geqslant 1,\ \imath_p = \imath_s\} \cup \{-\infty\}\big)$.
  (b) *If $s^- < t < s < s^+ \leqslant \ell$ and $d(\imath_s, \imath_t) = 1$, $\mathsf{d}_{\imath_s} < \mathsf{d}_{\imath_t}$, then there exists t' such that $s < t' < s^+$ and $\imath_{t'} = \sigma(\imath_t)$.*

(ii) *The expression $s_{\imath_1} \cdots s_{\imath_\ell}$ is adapted to some* Q*-datum $\mathscr{Q}$ associated to $\mathfrak{g}$.*

In [13], for each Q-datum $\mathscr{Q}$, a unique element $\tau_{\mathscr{Q}} = s_{\imath_1} \cdots s_{\imath_r} \sigma \in \mathsf{W}_{\rm fin}\sigma$ is defined, which can be understood as a generalization of a Coxeter element associated to a Dynkin quiver in the $\sigma = \mathrm{id}$ case. Here, we regard $\mathsf{W}_{\rm fin}\sigma$ as the subset of the automorphism group of the root lattice of $\triangle$. We call $\tau_{\mathscr{Q}}$ the $\mathscr{Q}$*-adapted Coxeter element*. We refer the reader to [13] for its definition, but give some of its properties instead.

Remark 6.5 ([13, §2])**.**

(a) The element $(\tau_{\mathscr{Q}})^{\mathrm{ord}(\sigma)}$ is contained in $\mathsf{W}_{\mathrm{fin}}$, of length $\mathsf{t} := \mathrm{ord}(\sigma) \times |I_{\mathfrak{g}_0}|$ and has a $\mathscr{Q}$-adapted reduced expression $s_{\imath_1} s_{\imath_2} \cdots s_{\imath_{\mathsf{t}}}$.
(b) Any $\mathscr{Q}$-adapted reduced expression $s_{\imath_1} s_{\imath_2} \cdots s_{\imath_{\mathsf{t}}}$ of $(\tau_{\mathscr{Q}})^{\mathrm{ord}(\sigma)}$ satisfies the following property. Set $\mathscr{Q}' := s_{\imath_{\mathsf{t}}} \cdots s_{\imath_1} \mathscr{Q}$, and let ξ and ξ' be the height function of $\mathscr{Q}$ and $\mathscr{Q}'$. Then we have
$$\xi'_\imath = \xi_\imath + 2 \times \mathrm{ord}(\sigma) \text{ for any } \imath \in I_{\mathrm{fin}}.$$
(c) Let $\mathsf{h}^\vee$ be the dual Coxeter number of $\mathfrak{g}_0$. Then one can check (see [13, Table 1]) that
$$\mathsf{h}^\vee = \frac{2\,|\Phi^+|}{\mathrm{ord}(\sigma)\,|I_{\mathfrak{g}_0}|}.$$
(d) Let $\underline{w}_0 = s_{\imath_1} \cdots s_{\imath_\ell}$ be a $\mathscr{Q}$-adapted reduced expression of the longest element w_0. Set $\mathscr{Q}' = \underline{w}_0^{-1} \mathscr{Q} := s_{\imath_\ell} \cdots s_{\imath_1} \mathscr{Q}$ and let ξ and ξ' be the height function of $\mathscr{Q}$ and $\mathscr{Q}'$. Then we have
$$\xi'_\imath = \xi_{\imath^*} + \mathrm{ord}(\sigma)\mathsf{h}^\vee \text{ for any } \imath \in I_{\mathrm{fin}}.$$
In particular, setting $\imath_{\ell+1} = (\imath_1)^*$, $s_{\imath_2} \cdots s_{\imath_\ell} s_{\imath_{\ell+1}}$ is an $(s_{\imath_1}\mathscr{Q})$-adapted reduced expression of w_0.

For a Q-datum $\mathscr{Q} = (\triangle, \sigma, \xi)$ associated to $\mathfrak{g}$, we define

$$\widehat{I}_{\mathscr{Q}} := \{(\imath, p) \in I_{\mathrm{fin}} \times \mathbb{Z} \mid p - \xi_\imath \in 2\mathsf{d}_\imath \mathbb{Z}\}. \tag{6.3}$$

We define the quiver $\Psi_{\mathscr{Q}}$ whose set of vertices is $\widehat{I}_{\mathscr{Q}}$ and arrows are assigned in the following way: for $(\imath, p), (\jmath, q) \in \widehat{I}_{\mathscr{Q}}$, we have

$$(\imath, p) \to (\jmath, q) \quad \text{if} \quad d(\imath, \jmath) = 1 \quad \text{and} \quad q - p = \min\{\mathsf{d}_\imath, \mathsf{d}_\jmath\}. \tag{6.4}$$

Let $\widehat{\Phi} := \Phi^+ \times \mathbb{Z}$. For each $\imath \in I_{\mathrm{fin}}$, we define
$$\gamma_\imath^{\mathscr{Q}} := (1 - \tau_{\mathscr{Q}}^{\mathsf{d}_\imath})\Lambda_\imath \in \Phi^+.$$
In [13, 21], it is shown that there exists a unique bijection $\phi_{\mathscr{Q}} \colon \widehat{I}_{\mathscr{Q}} \to \widehat{\Phi}$ defined inductively as follows:

$$\begin{aligned}
&(1)\ \phi_{\mathscr{Q}}(\imath, \xi_\imath) = (\gamma_\imath^{\mathscr{Q}}, 0) \\
&(2)\ \text{if } \phi_{\mathscr{Q}}(\imath, p) = (\beta, m), \text{ then we define} \\
&\quad \text{(a) } \phi_{\mathscr{Q}}(\imath, p + 2\mathsf{d}_\imath) = (\tau_{\mathscr{Q}}^{\mathsf{d}_\imath}(\beta), m) && \text{if } \tau_{\mathscr{Q}}^{\mathsf{d}_\imath}(\beta) \in \Phi^+, \\
&\quad \text{(b) } \phi_{\mathscr{Q}}(\imath, p + 2\mathsf{d}_\imath) = (-\tau_{\mathscr{Q}}^{\mathsf{d}_\imath}(\beta), m+1) && \text{if } \tau_{\mathscr{Q}}^{\mathsf{d}_\imath}(\beta) \in \Phi^-, \\
&\quad \text{(c) } \phi_{\mathscr{Q}}(\imath, p - 2\mathsf{d}_\imath) = (\tau_{\mathscr{Q}}^{-\mathsf{d}_\imath}(\beta), m) && \text{if } \tau_{\mathscr{Q}}^{-\mathsf{d}_\imath}(\beta) \in \Phi^+, \\
&\quad \text{(d) } \phi_{\mathscr{Q}}(\imath, p - 2\mathsf{d}_\imath) = (-\tau_{\mathscr{Q}}^{-\mathsf{d}_\imath}(\beta), m-1) && \text{if } \tau_{\mathscr{Q}}^{-\mathsf{d}_\imath}(\beta) \in \Phi^-.
\end{aligned} \tag{6.5}$$

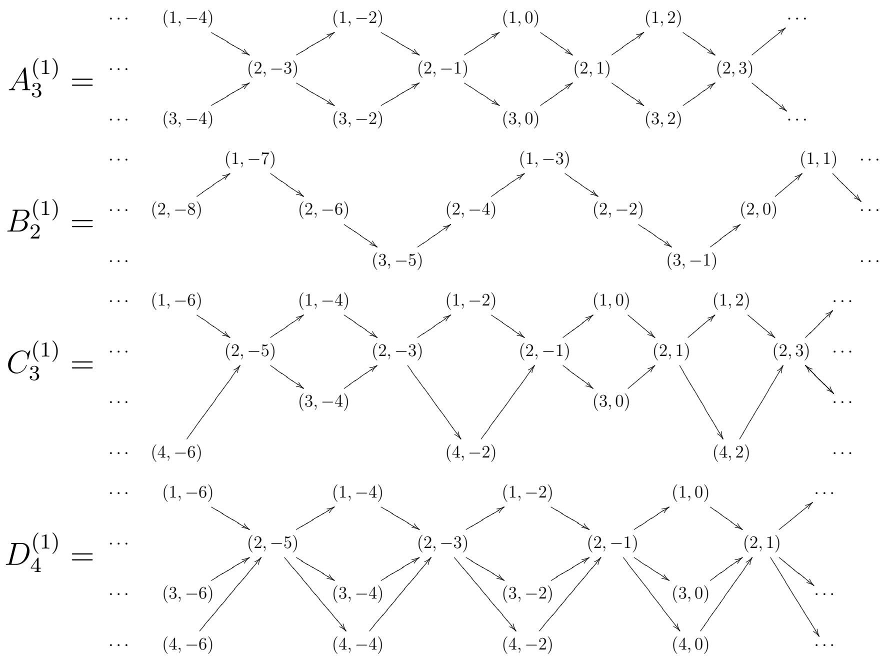


FIGURE 1. Some examples of the quivers $\Psi_{\mathfrak{g}}$.

Note that the full subquiver $\Gamma_{\mathscr{Q}}$ of $\widehat{I}_{\mathscr{Q}}$ with

$$I_{\mathscr{Q}} := \phi_{\mathscr{Q}}^{-1}(\Phi^+ \times \{0\}) \subset I_{\rm fin} \times \mathbb{Z}$$

as the set of vertices is isomorphic to the Auslander-Reiten (AR) quiver if $\mathscr{Q}$ is associated to $\mathfrak{g} = (ADE)_n^{(1)}$, to the twisted AR-quiver if $\mathscr{Q}$ is associated to $\mathfrak{g} = (BCF)_n^{(1)}$, and to the triply-twisted AR-quiver if $\mathscr{Q}$ is associated to $\mathfrak{g} = G_2^{(1)}$ ([57]). Throughout this paper, we call $\Gamma_{\mathscr{Q}}$ the AR-quiver in a uniform way.

Proposition 6.6 ([13, Corollary 2.39]). *Let $\mathsf{h}^\vee$ be the dual Coxeter number of $\mathfrak{g}_0$. Then we have*

$$\phi_{\mathscr{Q}}(\imath^*, \xi_\imath + \mathrm{ord}(\sigma)\mathsf{h}^\vee) = (\gamma_\imath^{\mathscr{Q}}, 1).$$

Hence, for $k \in \mathbb{Z}$ and $\phi_{\mathscr{Q}}(\imath, p) = (\beta, 0)$ for $\beta \in \Phi^+$, we have

$$\phi_{\mathscr{Q}}(\imath^{k*}, p + k\,\mathrm{ord}(\sigma)\mathsf{h}^\vee) = \big((-w_0)^k\beta, k\big)$$

Here $\imath^{k*} := \imath^{\overbrace{*\cdots*}^{|k|\text{-times}}}$. *In particular, we have*

$$I_{\mathscr{Q}} = \{(\imath, p) \in \widehat{I}_{\mathscr{Q}} \mid \xi_\imath \leqslant p < \xi_{\imath^*} + \mathrm{ord}(\sigma)\mathsf{h}^\vee\}.$$

Proposition 6.7 ([2, 13, 57])**.** *For a* Q*-datum* $\mathscr{Q}$*, we have the followings*:

(i) *If a sequence* $\big((i_k, p_k)\big)_{1\leqslant k\leqslant \ell}$ *of elements of* $I_{\mathscr{Q}}$ *satisfies*
  (a) $I_{\mathscr{Q}} = \{(\imath_k, p_k) \mid 1 \leqslant k \leqslant \ell\}$*, and*
  (b) $(\imath_k, p_k) \to (\imath_{k'}, p_{k'})$ *in* $\Gamma_{\mathscr{Q}}$ *implies* $k < k'$*,*
  then $\underline{w}_0 = s_{\imath_1} \cdots s_{\imath_\ell}$ *is a* $\mathscr{Q}$*-adapted reduced expression of* w_0*, and*
  $$p_k := \xi_{\imath_k}(s_{\imath_{k-1}} \cdots s_{\imath_1}\mathscr{Q}).$$
  Here, we set $\xi_\jmath(\mathscr{Q}) := \xi_\jmath$ *for a* Q*-datum* $\mathscr{Q} = (\triangle, \sigma, \xi)$*.*

(ii) *Conversely, if* $\underline{w}_0 = s_{\imath_1} \cdots s_{\imath_\ell}$ *is a* $\mathscr{Q}$*-adapted reduced expression of* w_0*, then* $\big((i_k, p_k)\big)_{1\leqslant k\leqslant \ell}$ *with* $p_k := \xi_{\imath_k}(s_{\imath_{k-1}} \cdots s_{\imath_1}\mathscr{Q})$ *satisfies* (a) *and* (b) *in* (i)*.*

(iii) *For any sequence* $\big((i_k, p_k)\big)_{1\leqslant k\leqslant \ell}$ *satisfying* (a) *and* (b) *in* (i)*, we have* $\phi_{\mathscr{Q}}(\imath_k, p_k) = (\beta_k, 0)$ $(1 \leqslant k \leqslant \ell)$*, where* $\beta_k = s_{\imath_1} \cdots s_{\imath_{k-1}}(\alpha_{\imath_k})$*.*

Remark 6.8. The statements below follow from [21] and [13].

(a) For Q-data $\mathscr{Q}$ and $\mathscr{Q}'$ associated to the same $\mathfrak{g}$, $\widehat{I}_{\mathscr{Q}}$ and $\widehat{I}_{\mathscr{Q}'}$ differ by constant integer (see [21] and [13]); that is, there exists a unique $k \in \{0, 1, \ldots, 2\mathrm{ord}(\sigma) - 1\}$ such that

$$\widehat{I}_{\mathscr{Q}'} = \{(\imath, p + k) \mid (\imath, p) \in \widehat{I}_{\mathscr{Q}}\}.$$

Note also that there exists a unique pair $(\epsilon, m) \in \{0, 1\} \times [0, \mathrm{ord}(\sigma) - 1]$ such that

$$\widehat{I}_{\mathscr{Q}'} = \{(\imath, p + \epsilon) \mid (\sigma^m(\imath), p) \in \widehat{I}_{\mathscr{Q}}\}.$$

Thus we use the notation $\widehat{I}_{\mathfrak{g}}$ and $\Psi_{\mathfrak{g}}$ instead of $\widehat{I}_{\mathscr{Q}}$ and $\Psi_{\mathscr{Q}}$ when we can neglect integer shift.

(b) A subset K of $I_{\mathrm{fin}} \times \mathbb{Z}$ is equal to $\widehat{I}_{\mathscr{Q}}$ for some Q-datum $\mathscr{Q}$ if and only if K satisfies the following condition:

$$\begin{cases} \text{(i) For any } \imath \in I_{\mathrm{fin}}, \text{ we have } \{m \in \mathbb{Z} \mid (\imath, m) \in \mathsf{K}\} = 2\mathsf{d}_\imath\mathbb{Z} + a \text{ for some } a \in \mathbb{Z}, \\ \text{(ii) if } d(\imath, \jmath) = 1,\ \mathsf{d}_\imath \leqslant \mathsf{d}_\jmath \text{ and } (\jmath, p) \in \mathsf{K}, \text{ then we have } (\imath, p + \mathsf{d}_\imath) \in \mathsf{K}, \\ \text{(iii) for any } (\imath, m) \in \mathsf{K}, \text{ we have } \big(\sigma(\imath), m + 2\big) \in \mathsf{K}. \end{cases} \tag{6.6}$$

Definition 6.9. We say that an infinite sequence $\mathfrak{s} := \big((\imath_k, p_k)\big)_{k\in\mathbb{Z}}$ in $I_{\mathrm{fin}} \times \mathbb{Z}$ is *admissible* if the following conditions are satisfied:

(1) $p_{s^+} = p_s + 2\mathsf{d}_{\imath_s}$,

(2) $p_t = p_s + \min(\mathsf{d}_{\imath_s}, \mathsf{d}_{\imath_t})$ if $d(\imath_s, \imath_t) = 1$ and $t^- < s < t < s^+$,
(3) $p_t - (p_s + 2) \in 2\mathsf{d}_{\imath_s}\mathbb{Z}$ if $s, t \in \mathbb{Z}$ satisfy $\imath_t = \sigma(\imath_s)$,
(4) $s_{\imath_k} \cdots s_{\imath_{k+\ell-1}} = w_0$ for every $k \in \mathbb{Z}$, where ℓ denotes the length of the longest element $w_0 \in \mathsf{W}_{\mathrm{fin}}$.

Note that (4) implies that $\imath_{k+\ell} = (\imath_k)^*$ for any $k \in \mathbb{Z}$. Note also that, when $s < t$ and $\imath_s \neq \imath_t$, the condition $t^- < s < t < s^+$ is equivalent to $\imath_s, \imath_t \notin \{\imath_k \mid s < k < t\}$.

Lemma 6.10. *Let $\mathfrak{s} = \big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$ be an admissible sequence, and assume that $s, t \in \mathbb{Z}$ satisfy $d(\imath_s, \imath_t) = 1$. Then we have the followings*:

(i) *$p_s < p_t$ if and only if $s < t$.*
(ii) *$p_s - p_t - a \in 2a\mathbb{Z}$ where $a = \min(\mathsf{d}_{\imath_s}.\mathsf{d}_{\imath_t})$.*

Proof. It is enough to show that $p_s < p_t$ and $p_s - p_t - a \in 2a\mathbb{Z}$ under the condition $s < t$.

Take $t' \in \mathbb{Z}$ such that $\imath_{t'} = \imath_t$ and $(t')^- < s < t'$ and then take $s' \in \mathbb{Z}$ such that $\imath_{s'} = \imath_s$ and $s' < t' < (s')^+$. Then we have $(t')^- < s' < t' < (s')^+$, and hence

$$p_{t'} = p_{s'} + a$$

by (2). Since $s < s'$ and $t' < t$, (1) implies $p_s \leqslant p_{s'}$ and $p_{t'} \leqslant p_t$. Thus we obtain

$$p_s \leqslant p_{s'} < p_{t'} \leqslant p_t$$

and

$$p_s + a \equiv p_{s'} + a \equiv p_{t'} \equiv p_t \mod 2a.$$

Note that $2\mathsf{d}_{\imath_s}, 2\mathsf{d}_{\imath_t} \in 2a\mathbb{Z}$. □

Proposition 6.11. *Let A be the set of pairs $(\mathscr{Q}, \underline{w}_0)$ of a Q-datum $\mathscr{Q}$ and a $\mathscr{Q}$-adapted reduced expression $\underline{w}_0$ of w_0, and let B be the set of admissible sequences $\mathfrak{s}$. Then there exists a bijective map $\varrho\colon A \longrightarrow B$ defined as follows.*

(i) *For a pair $(\mathscr{Q}, \underline{w}_0)$ of a Q-datum $\mathscr{Q} = (\triangle, \sigma, \xi)$ and a $\mathscr{Q}$-adapted reduced expression $\underline{w}_0 = s_{\jmath_1} \cdots s_{\jmath_\ell}$ of w_0, we define $\big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$ by:*

$$\imath_{s+m\ell} = \begin{cases} \jmath_s & \text{if } m \text{ is even,} \\ (\jmath_s)^* & \text{if } m \text{ is odd,} \end{cases} \quad \text{for } s, m \in \mathbb{Z} \text{ such that } 1 \leqslant s \leqslant \ell,$$

$$p_k = \begin{cases} \xi_{\imath_k}\big(s_{\imath_{k-1}} \cdots s_{\imath_1} \mathscr{Q}\big) & \text{if } k \geqslant 1, \\ \xi_{\imath_k}\big((s_{\imath_k})^{-1}(s_{\imath_{k+1}})^{-1} \cdots (s_{\imath_0})^{-1} \mathscr{Q}\big) & \text{if } k \leqslant 0. \end{cases}$$

Here, we set $\xi_\jmath(\mathscr{Q}) := \xi_\jmath$ for a Q-datum $\mathscr{Q} = (\triangle, \sigma, \xi)$.
Then $\big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$ is an admissible sequence and we set $\varrho(\mathscr{Q}, \underline{w}_0) = \big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$.

(ii) *Conversely, for an admissible sequence* $\mathfrak{s} = \big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$, $(\mathscr{Q}, \underline{w}_0) = \varrho^{-1}(\mathfrak{s})$ *is given by*

$$\underline{w}_0 = s_{\imath_1} \cdots s_{\imath_\ell} \quad \text{and} \quad \xi_\imath = p_{\vartheta_\imath},$$

where $\vartheta_\imath := \min\{k \in \mathbb{Z}_{\geqslant 1} \mid \imath_k = \imath\}$.

Moreover, if $(\mathscr{Q}, \underline{w}_0)$ *and* $\mathfrak{s}$ *correspond by* ϱ*, the following properties hold.*

(a) *The map* $s \mapsto (\imath_s, p_s)$ *gives a bijection* $\mathbb{Z} \xrightarrow{\sim} \widehat{I}_{\mathscr{Q}}$.

(b) *For* $1 \leqslant k \leqslant \ell$, $\phi_{\mathscr{Q}}(\imath_k, p_k) = (\beta_k, 0)$, *where* $\beta_k = s_{\imath_1} \cdots s_{\imath_{k-1}} \alpha_{\imath_k}$.

(c) $\imath_{s+\ell} = (\imath_s)^*$ *and* $p_{s+\ell} = p_s + \mathrm{ord}(\sigma)\, \mathsf{h}^\vee$*, where* $\mathsf{h}^\vee$ *is the dual Coxeter number of* $\mathfrak{g}_0$.

(d) *Set* ${\underline{w}_0}' = s_{\imath_2} s_{\imath_3} \cdots s_{\imath_\ell} s_{\imath_{\ell+1}}$ *and* $\mathfrak{s}' := \big((\imath'_k, p'_k)\big)_{k\in\mathbb{Z}}$ *where* $(\imath'_k, p'_k) = (\imath_{k+1}, p_{k+1})$. *Then* $\imath_1$ *is a sink of* $\mathscr{Q}$*, and* $\mathfrak{s}' = \varrho\big(s_{\imath_1}\mathscr{Q}, {\underline{w}_0}'\big)$.

(e) *If* $s^- < t < s$, $d(\imath_s, \imath_t) = 1$ *and* $\mathsf{d}_{\imath_s} < \mathsf{d}_{\imath_t}$*, then there exists* t' *such that* $s < t' < s^+$ *and* $\imath_{t'} = \sigma(\imath_t)$.

Proof. (i) By the construction of $(\imath_k)_{k\in\mathbb{Z}}$, it is evident that we have $w_0 = s_{\imath_{k+1}} \cdots s_{\imath_{k+\ell}}$ for any k. By Remark 6.5, $\imath_k$ is a sink of $s_{\imath_{k-1}} \cdots s_{\imath_1}\mathscr{Q}$ for any $k \geqslant 1$. Similarly $\imath_k$ is a source of $(s_{\imath_k})^{-1}(s_{\imath_{k+1}})^{-1} \cdots (s_{\imath_0})^{-1}\mathscr{Q}$. Hence, there exists a sequence of Q-data $(\mathscr{Q}_k)_{k\in\mathbb{Z}}$ such that

$$\begin{aligned}
&\mathscr{Q}_0 = \mathscr{Q}, \\
&\imath_{k+1} \text{ is a sink of } \mathscr{Q}_k \text{ and } \mathscr{Q}_{k+1} = s_{\imath_{k+1}}\mathscr{Q}_k, \\
&\imath_k \text{ is a source of } \mathscr{Q}_k \text{ and } \mathscr{Q}_{k-1} = (s_{\imath_k})^{-1}\mathscr{Q}_k,
\end{aligned}$$

Hence, $p_k = \xi_{\imath_k}(\mathscr{Q}_{k-1})$ and $(\imath_k, p_k) \in \widehat{I}_{\mathscr{Q}}$ for any k.

Now, the conditions (1), (3) and (4) in Definition 6.9 are almost obvious by the construction (see also (6.6)).

Let us show (2). The following properties are almost obvious by the construction:

$$\text{(6.7)} \qquad \begin{aligned}
&(\alpha)\ p_s < p_t \text{ if } d(\imath_s, \imath_t) = 1 \text{ and } s < t. \\
&(\beta)\ p_t - p_s - \min(\mathsf{d}_{\imath_s}, \mathsf{d}_{\imath_t}) \in 2\min(\mathsf{d}_{\imath_s}, \mathsf{d}_{\imath_t})\mathbb{Z} \text{ if } d(\imath_s, \imath_t) = 1.
\end{aligned}$$

Now assume $d(\imath_s, \imath_t) = 1$ and $t^- < s < t < s^+$. Then (1) and (α) implies $p_{t^-} = p_t - 2\mathsf{d}_{\imath_t} < p_s < p_t < p_{s^+} = p_s + 2\mathsf{d}_{\imath_s}$. Hence we obtain

$$p_s < p_t < p_s + 2\min(\mathsf{d}_{\imath_s}, \mathsf{d}_{\imath_t}).$$

Then (β) implies that $p_t = p_s + \min(\mathsf{d}_{\imath_s}, \mathsf{d}_{\imath_t})$. Thus $\mathfrak{s}$ is an admissible sequence.

(ii) Conversely, let $\mathfrak{s} = \big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$ be an admissible sequence. Let us show that $(\xi_\imath)_{\imath\in I_{\mathrm{fin}}}$ is a height function.

(ii-a) Let us show (1) in Definition 6.1. Assume that $\imath, \jmath \in I_{\rm fin}$ satisfy $d(\imath, \jmath) = 1$ and $\mathsf{d}_\imath = \mathsf{d}_\jmath$. In order to see $|\xi_\imath - \xi_\jmath| = \mathsf{d}_\imath$, we may assume that $\vartheta_\imath < \vartheta_\jmath$. Then we have $(\vartheta_\jmath)^- \leqslant 0 < \vartheta_\imath < \vartheta_\jmath$. Hence $p_{(\vartheta_\jmath)^-} = \xi_\jmath - 2\mathsf{d}_\jmath < p_{\vartheta_\imath} = \xi_\imath < p_{\vartheta_\jmath} = \xi_\jmath$ by Lemma 6.10 (i). Since $\xi_\imath - (\xi_\jmath - \mathsf{d}_\imath) \in 2\mathsf{d}_\imath\mathbb{Z}$ by Lemma 6.10 (ii). we obtain $\xi_\imath = \xi_\jmath - \mathsf{d}_\imath$.

(ii-b) Let us show (ii) in Definition 6.1. Assume that $\imath, \jmath \in I_{\rm fin}$ satisfy $d(\imath, \jmath) = 1$ and $\mathsf{d}_\imath = 1 < \mathsf{d}_\jmath$. We may assume that $\vartheta_{\jmath'} \geqslant \vartheta_\jmath$ for any $\jmath' \in \pi^{-1}\pi(\jmath)$. Set $s_0 = \vartheta_\jmath$. Then by (1) and (3) in Definition 6.9, for any $k \in \mathbb{Z}$, there exists $s_k \in \mathbb{Z}$ such that $\imath_{s_k} = \sigma^k(\jmath)$ and $p_{s_k} = p_{s_0} + 2k$. Since $p_{\vartheta_i} \equiv p_{s_0} + 1 \bmod 2$, there exists t_k such that $\imath_{t_k} = \imath$ and $p_{t_k} = p_{s_0} + 2k + 1$. Then we have $p_{s_k} < p_{t_k} < p_{s_{k+1}}$, which implies $s_k < t_k < s_{k+1}$ by Lemma 6.10 (i). Hence $(s_k)_{k\in\mathbb{Z}}$ is a strictly increasing sequence. Since $\imath_{s_{-1}} = \jmath$ and $s_{-1} < s_0 = \vartheta_\jmath$, we have $s_{-1} < 0$. Hence $s_{-1} \leqslant 0 < \vartheta_\imath$ and $(\vartheta_\imath)^- \leqslant 0 < s_0$. Then Lemma 6.10 (i) implies $\xi_\jmath - 2 < \xi_\imath$ and $\xi_\imath - 2 < \xi_\jmath$. Hence we have $|\xi_\imath - \xi_\jmath| \leqslant 1$. Since $\xi_\jmath \neq \xi_\imath$ by Lemma 6.10 (ii), we obtain $|\xi_\imath - \xi_\jmath| = 1$. On the other hand, if $1 \leqslant k \leqslant \operatorname{ord}(\sigma) - 1$, then we have $(s_k)^- = s_{k-\operatorname{ord}(\sigma)} \leqslant s_{-1} < 0$, which implies $s_k = \vartheta_{\sigma^k(\jmath)}$, and we obtain $\xi_{\sigma^k(\jmath)} = p_{s_k} = p_{s_0} + 2k = \xi_\jmath + 2k$.

(a) is obvious.

(b) follows from Lemma 6.7.

(c) follows from the definition and Remark 6.5 (d).

(d) follows from (i).

(e) By (1) and (3) in Definition 6.9, there exists t' such that $\imath_{t'} = \sigma(\imath_t)$ and $p_{t'} = p_t + 2$. Then we have $p_s \leqslant p_t + 1 < p_{t'} = p_t + 2 < p_s + 2 = p_{s^+}$, where the first inequality follows from $p_s - 2 = p_{s^-} < p_t$. Hence $s < t' < s^+$ by Lemma 6.10 (i). □

6.2. **Associated fundamental modules, and twisted case.** For each untwisted affine $\mathfrak{g}$ and $(\imath, p) \in I_{\rm fin} \times \mathbb{Z}$, we define the fundamental module $V_\mathfrak{g}(\imath, p) \in \mathscr{C}_\mathfrak{g}$ in the following way: set $q_{\rm sh} := q^{1/\operatorname{ord}(\sigma)}$ and

$$V_\mathfrak{g}(\imath, p) := \begin{cases} V(\varpi_{\pi(\imath)})_{(-q_{\rm sh})^p} & \text{if } \mathfrak{g} = A_n^{(1)},\ C_n^{(1)},\ D_n^{(1)},\ E_{6,7,8}^{(1)},\ G_2^{(1)}, \\ V(\varpi_{\pi(\imath)})_{(-1)^{d(\imath,n)}(q_{\rm sh})^p} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ V(\varpi_{\pi(\imath)})_{(-1)^{d(\imath,2)}(q_{\rm sh})^p} & \text{if } \mathfrak{g} = F_4^{(1)}. \end{cases} \tag{6.8}$$

Note that $q_{\pi(\imath)} = (q_{\rm sh})^{\mathsf{d}_\imath}$. By [21, 57], $V_\mathfrak{g}(\imath, p)$'s are distinct. The fundamental modules $(V_\mathfrak{g}(\imath, p))_{(\imath,p)\in I_{\rm fin}\times\mathbb{Z}}$ do not depend on the choice of Q-data.

Now let us consider the twisted affine types $\mathsf{g}^{(2)} = A^{(2)}_{2n}$ $(n \geqslant 1)$, $A^{(2)}_{2n-1}$ $(n \geqslant 2)$, $D^{(2)}_{n+1}$ $(n \geqslant 3)$, $E^{(2)}_6$ and $\mathsf{g}^{(3)} = D^{(3)}_4$. By [19, Theorem 4.15], there exists a ring isomorphism

$$\chi_t\colon K(\mathscr{C}^0_{\mathsf{g}^{(1)}}) \xrightarrow{\sim} K(\mathscr{C}^0_{\mathsf{g}^{(t)}}) \qquad (t = 2, 3) \tag{6.9}$$

sending fundamental modules to fundamental modules, and KR-modules to KR-modules (see also [31]). Here, $K(\mathscr{C}^0_{\mathsf{g}^{(1)}})$ and $K(\mathscr{C}^0_{\mathsf{g}^{(t)}})$ denote the Grothendieck rings of $\mathscr{C}^0_{\mathsf{g}^{(1)}}$ and $\mathscr{C}^0_{\mathsf{g}^{(t)}}$, which are called the Hernandez-Leclerc subcategory of $U'_q(\mathsf{g}^{(1)})$ and $U'_q(\mathsf{g}^{(t)})$, respectively (see Subsection 6.3 below for definitions). Note that χ_t commutes with $\mathscr{D}$.

The image of fundamental modules by χ_t can be described as follows.

For $(\imath, p) \in I_{\mathrm{fin}} \times \mathbb{Z}$, we set

$$V_{\mathsf{g}^{(t)}}(\imath, p) := V(\varpi_{\pi(\imath)})_{A(\imath,p)},$$

where

$$\pi(\imath) := \begin{cases} \imath & \text{if (i) } \mathsf{g}^{(2)} = A^{(2)}_N \text{ and } \imath \leqslant \lceil N/2 \rceil \\ & \quad \text{or (ii) } \mathsf{g}^{(2)} = D^{(2)}_{n+1} \text{ and } \imath < n, \\ N+1-\imath & \text{if } \mathsf{g}^{(2)} = A^{(2)}_N \text{ and } \imath > \lceil N/2 \rceil, \\ n & \text{if } \mathsf{g}^{(2)} = D^{(2)}_{n+1} \text{ and } \imath = n, n+1, \end{cases}$$

$$\begin{cases} \pi(1) = \pi(6) = 1, \quad \pi(3) = \pi(5) = 2, \quad \pi(4) = 3, \quad \pi(2) = 4 & \text{if } \mathsf{g}^{(2)} = E^{(2)}_6, \\ \pi(1) = \pi(3) = \pi(4) = 1, \quad \pi(2) = 2 & \text{if } \mathsf{g}^{(3)} = D^{(3)}_4, \end{cases}$$

and

$$A(\imath, p) := \begin{cases} (-q)^p & \text{if (i) } \mathsf{g}^{(2)} = A^{(2)}_N \text{ and } \imath \leqslant \lceil N/2 \rceil \\ & \quad \text{or (ii) } \mathsf{g}^{(2)} = E^{(2)}_6 \text{ and } \imath = 1, 3, \\ (-1)^N (-q)^p & \text{if } \mathsf{g}^{(2)} = A^{(2)}_N \text{ and } \imath > \lceil N/2 \rceil, \\ (\sqrt{-1})^{n+1-\imath}(-q)^p & \text{if } \mathsf{g}^{(2)} = D^{(2)}_{n+1} \text{ and } \imath < n, \\ (-1)^\imath (-q)^p & \text{if } \mathsf{g}^{(2)} = D^{(2)}_{n+1} \text{ and } \imath = n, n+1, \\ -(-q)^p & \text{if } \mathsf{g}^{(2)} = E^{(2)}_6 \text{ and } \imath = 5, 6, \\ (\sqrt{-1})(-q)^p & \text{if } \mathsf{g}^{(2)} = E^{(2)}_6 \text{ and } \imath = 2, 4, \\ \big(\delta_{\imath,1} - \delta_{\imath,2} + \delta_{\imath,3}\omega + \delta_{\imath,4}\omega^2\big)(-q)^p & \text{if } \mathsf{g}^{(3)} = D^{(3)}_4. \end{cases}$$

Here ω is the third root of unity. Recall that we follow the enumeration of vertices of the Dynkin diagram of $\mathfrak{g}$ as [27] except the $A^{(2)}_{2n}$ case given in (2.1).

Then we have

$$\chi_t\big([V_{\mathsf{g}^{(1)}}(\imath,p)]\big) = [V_{\mathsf{g}^{(t)}}(\imath,p)] \quad \text{for any } (\imath,p) \in \widehat{I}_{\mathsf{g}^{(1)}}. \tag{6.10}$$

For each twisted quantum affine algebra $U_q'(\mathfrak{g})$, we associate a finite simple Lie algebra $\mathfrak{g}_{\mathrm{fin}}$ and $\Psi_{\mathfrak{g}}$ as follows:

(6.11)

$\mathfrak{g}$	$A_n^{(2)}$ $(n \geqslant 3)$	$D_{n+1}^{(2)}$ $(n \geqslant 3)$	$D_4^{(3)}$	$E_6^{(2)}$
$(\mathfrak{g}_{\mathrm{fin}}, \Psi_{\mathfrak{g}})$	$\big(A_n, \Psi_{A_n^{(1)}}\big)$	$\big(D_{n+1}, \Psi_{D_{n+1}^{(1)}}\big)$	$\big(D_4, \Psi_{D_4^{(1)}}\big)$	$\big(E_6, \Psi_{E_6^{(1)}}\big)$

Also, the set of Q-data associated to $U_q'(\mathsf{g}^{(t)})$ $(t \geqslant 2)$ are the same as the one for $U_q'(\mathsf{g}^{(1)})$. In particular, $\sigma = \mathrm{id}$ in the twisted case. Note also that the dual Coxeter number of $\mathsf{g}^{(1)}$ and the one of $\mathsf{g}^{(t)}$ coincide.

In the sequel, we write simply $V(\imath,p)$ for $V_{\mathfrak{g}}(\imath,p)$ if there is no afraid of confusion.

6.3. **PBW-pair associated with a Q-datum.** For each Q-datum $\mathscr{Q}$, the subcategory $\mathscr{C}_{\mathscr{Q}}$ of $\mathscr{C}_{\mathfrak{g}}$, introduced in [21], is defined as the smallest subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing $V(\imath,p)$ for all $(\imath,p) \in I_{\mathscr{Q}}$ and the trivial module $\mathbf{1}$ and is stable under taking subquotients, extensions and tensor products.

Theorem 6.12 and Proposition 6.13 below are proved for untwisted (resp. twisted) affine A and D types in [29, Theorem 4.3.1, Theorem 4.3.4] (resp. [32, Theorem 5.1]), for untwisted affine B and C types in [42, Theorem 6.3, Theorem 6.5], and for the remaining exceptional affine types [55, Theorem 6.3, Theorem 6.7, Theorem 6.13, Theorem 6.15] (see also [40, Proposition 6.5]):

Theorem 6.12 ([29, 32, 42, 55]). *Let $U_q'(\mathfrak{g})$ be a quantum affine algebra and let $\mathscr{Q}$ be a* Q*-datum associated to $\mathfrak{g}$. Set*

$$V_{\mathscr{Q}}(\alpha_\jmath) := V(\imath,p) \quad \textit{for } \jmath \in I_{\mathrm{fin}}, \textit{ where } \phi_{\mathscr{Q}}^{-1}(\alpha_\jmath,0) = (\imath,p). \tag{6.12}$$

Then, we have

(a) *The family $\mathcal{D}_{\mathscr{Q}} := \{V_{\mathscr{Q}}(\alpha_\jmath)\}_{\jmath \in I_{\mathrm{fin}}}$ is a* complete *duality datum associated with the Cartan matrix C of type $\mathfrak{g}_{\mathrm{fin}}$. Hence the functor*

$$\mathcal{F}_{\mathscr{Q}} := \mathcal{F}_{\mathcal{D}_{\mathscr{Q}}} \colon R_{\mathsf{C}}\text{-gmod} \to \mathscr{C}_{\mathscr{Q}} \textit{ in } (4.1) \textit{ is exact.}$$

(b) *The functor $\mathcal{F}_{\mathscr{Q}}$ sends simple modules to simple modules.*

For an admissible sequence $\mathfrak{s} = \big((\imath_k,p_k)\big)_{k\in\mathbb{Z}}$ in $I_{\mathrm{fin}} \times \mathbb{Z}$, we say that $(\mathcal{D}_{\mathscr{Q}}, \widehat{\underline{w}}_0)$ is the associated PBW-pair, where $\mathscr{Q}$ is the corresponding Q-datum and $\widehat{\underline{w}}_0 = (\imath_k)_{k\in\mathbb{Z}}$.

Proposition 6.13 ([29, 32, 42, 55]). *Let $\mathfrak{s} = ((\imath_k, p_k))_{k\in\mathbb{Z}}$ be an admissible sequence in $I_{\rm fin} \times \mathbb{Z}$, and let $(\mathcal{D}_{\mathscr{Q}}, \widehat{\underline{w}}_0)$ be the associated PBW-pair. Then we have*

$$\mathsf{S}_k^{\mathfrak{s}} := \mathsf{S}_k^{\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0} \simeq V(\imath_k, p_k).$$

We can say more. By (2.2), we have the following theorem.

Theorem 6.14. *For an admissible sequence $\mathfrak{s}$ in $\widehat{I}_{\mathfrak{g}}$ and any i-box $[a, b]$,*

$$M^{\mathfrak{s}}[a, b] := M^{\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}[a, b] \text{ is a KR-module over } U_q'(\mathfrak{g}).$$

Hence we can understand KR-modules as a special case of affine determinantial modules.

Remark 6.15. Let $\mathfrak{s}$ be an admissible sequence. Then, for any i-box $[a, b]$ with $\imath_a = \imath_b = \imath$ with $|[a, b]|_\phi = k$, we have

$$(\imath_b, p_b) = (\imath, p_a + 2\,\mathsf{d}_\imath \times (k-1)) \quad \text{and} \quad M^{\mathfrak{s}}[a, b] = W^{(\pi(\imath))}_{k,\epsilon\,(q_{\rm sh})^{p_a}},$$

where $q_{\rm sh} = q^{1/{\rm ord}(\sigma)}$ and $\epsilon \in \mathbb{C}^\times$ is determined by (6.8) and (6.9).

Definition 6.16. For each interval $[a, b]$ and an admissible sequence $\mathfrak{s}$, we set

$$\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}} := \mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}.$$

Namely, $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$ is the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ satisfying the following conditions:

(1) it is stable under taking subquotients, extensions, tensor products and
(2) it contains $\mathsf{S}_s^{\mathfrak{s}} \simeq V_{\mathscr{Q}}(\imath_s, p_s)$ for all $a \leqslant s \leqslant b$ and the trivial module $\mathbf{1}$.

Now let us compare $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{s}}$ with the subcategories introduced in [20, 21, 22], which are stable under taking subquotients, extensions, tensor products. Those categories depends on the choice of $\widehat{I}_{\mathfrak{g}}$, and we choose one.

(a) The subcategory $\mathscr{C}_{\mathfrak{g}}^0$ of $\mathscr{C}_{\mathfrak{g}}$, introduced in [20, 36], is defined as the smallest subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing $V(\imath, p)$ for all $(\imath, p) \in \widehat{I}_{\mathfrak{g}}$. Thus $\mathscr{C}_{\mathfrak{g}}^0$ can be identified with $\mathscr{C}_{\mathfrak{g}}^{[-\infty,\infty],\mathfrak{s}}$ for any admissible sequence $\mathfrak{s}$.
(b) Let $\widehat{I}_{\mathfrak{g}}^- := \widehat{I}_{\mathfrak{g}} \cap (I_{\rm fin} \times \mathbb{Z}_{\leqslant 0})$. The subcategory $\mathscr{C}_{\mathfrak{g}}^-$ of $\mathscr{C}_{\mathfrak{g}}$, introduced in [22], is defined as the smallest subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing $V(\imath, p)$ for all $(\imath, p) \in \widehat{I}_{\mathfrak{g}}^-$ and stable under taking subquotients, extensions, tensor products. Take a unique Q-datum $\mathscr{Q}$ with $\xi_\imath \in [1, 2\,\mathsf{d}_\imath]$ and $\widehat{I}_{\mathscr{Q}} = \widehat{I}_{\mathfrak{g}}$, and let $\underline{w}_0$ be a $\mathscr{Q}$-adapted reduced expression of w_0. Now let $\mathfrak{s}$ be the admissible sequence corresponding to $(\mathscr{Q}, \underline{w}_0)$ (see Proposition 6.11). Then, $\mathscr{C}_{\mathfrak{g}}^-$ coincides with the category $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathfrak{s}}$. In a similar way, we can define $\mathscr{C}_{\mathfrak{g}}^{<t}$ for $t \in \mathbb{Z}$

generated by $V(\imath,p)$ with $(\imath,p) \in \widehat{I}_{\mathfrak{g}}^{<t} := \widehat{I}_{\mathfrak{g}} \cap (I_{\mathrm{fin}} \times \mathbb{Z}_{<t})$. Then it is equal to $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathfrak{s}}$ taking $\mathfrak{s}$ such that $\widehat{I}_{\mathfrak{g}}^{<t} = \{(\imath_k,p_k)\}_{k\leqslant 0}$.

(c) For each untwisted affine $\mathfrak{g}$ of simply-laced type and $N \in \mathbb{Z}_{\geqslant 1}$, the subcategory $\mathscr{C}_{\mathfrak{g}}^{N}$ of $\mathscr{C}_{\mathfrak{g}}$, introduced in [20], is defined as the smallest subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing $V(\imath,p)$ for all $(\imath,p) \in \widehat{I}_{\mathfrak{g}} \cap (I_{\mathrm{fin}} \times [-2N+1,0])$. By taking a Q-datum $\mathscr{Q}$ with $[1,2] \ni \xi_\imath$ and $(\imath,\xi_\imath) \in \widehat{I}_{\mathfrak{g}}$, and a $\mathscr{Q}$-adapted reduced expression $\underline{w}_0$, $\mathscr{C}_{\mathfrak{g}}^{N}$ coincides with $\mathscr{C}_{\mathfrak{g}}^{[a,0],\mathfrak{s}}$, where $\mathfrak{s}$ is the admissible sequence corresponding to $(\mathscr{Q},\underline{w}_0)$ and $a = 1-(N\times|I_{\mathrm{fin}}|)$.

(d) For each Q-datum $\mathscr{Q}$, the subcategory $\mathscr{C}_{\mathscr{Q}}$ of $\mathscr{C}_{\mathfrak{g}}$ coincides with $\mathscr{C}_{\mathfrak{g}}^{[1,\ell],\mathfrak{s}}$ for any corresponding admissible sequence $\mathfrak{s}$.

Sometimes, we write $\mathscr{C}_{\mathfrak{g}}^{<\xi}$ for the monoidal category $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathfrak{s}}$, since it depends only on the choice of a Q-datum $\mathscr{Q} = (\triangle,\sigma,\xi)$. Indeed, by setting

$$\widehat{I}_{\mathfrak{g}}^{<\xi} := \{(\imath,p) \mid (\imath,p) \in \widehat{I}_{\mathscr{Q}}, p < \xi_\imath\}, \tag{6.13}$$

we have

$$\widehat{I}_{\mathfrak{g}}^{<\xi} = \{(\imath_k,p_k) \mid k \leqslant 0\}.$$

7. Cluster algebra structure and monoidal categorification

In this section, we briefly recall the definition of a cluster algebra with small modifications as in [38]. We also briefly review the main result of [38] on Λ-monoidal categorification of cluster algebras, which is an application of the invariants Λ, Λ^∞ and $\mathfrak{d}$ on $\mathscr{C}_{\mathfrak{g}}$, and can be understood as a quantum affine analogue of the result for quiver Hecke algebras in [33]. After reviewing several properties of monoidal seeds of various kinds developed in [35] and [39], we will construct Λ-admissible monoidal seeds associated to PBW-pairs $(\mathcal{D},\widehat{\underline{w}}_0)$ and the quivers $\mathrm{Q}_{\mathrm{GLS}}(\widehat{\underline{w}}_0)$ introduced in [14]. In the last part, we will review the result in [22] which gives a cluster algebra structure on $K(\mathscr{C}_{\mathfrak{g}}^{-})$ associated to the initial quiver Q_{HL}, and prove that the initial seed of $K(\mathscr{C}_{\mathfrak{g}}^{-})$ in [22] lifts to a Λ-admissible monoidal seed. For more details on cluster algebras and monoidal categorification, we refer the reader to [3, 9] and [38].

From now on, $\mathcal{C}$ is a full subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing the trivial module $\mathbf{1}$ and stable under taking tensor products, subquotients and extensions. Note that its Grothendieck group $K(\mathcal{C})$ has a ring structure with the $\mathbb{Z}$-basis consisting of the isomorphism classes of simple modules.

7.1. **Cluster algebras.** Fix a countable index set $\mathsf{K} = \mathsf{K}^{\mathrm{ex}} \sqcup \mathsf{K}^{\mathrm{fr}}$ which decomposes into a subset K^{ex} of *exchangeable* indices and a subset K^{fr} of *frozen* indices.

Let $\widetilde{B} = (b_{ij})_{(i,j)\in \mathsf{K}\times\mathsf{K}^{\mathrm{ex}}}$ be an integer-valued matrix such that

$$
\begin{array}{ll}
(1) & \text{for each } j \in \mathsf{K}^{\mathrm{ex}}, \text{ there exist finitely many } i \in \mathsf{K} \text{ such that } b_{ij} \neq 0, \\
(2) & \text{the } \textit{principal part } B := (b_{ij})_{i,j\in\mathsf{K}^{\mathrm{ex}}} \text{ is skew-symmetric.}
\end{array}
\tag{7.1}
$$

We call $\widetilde{B}$ an *exchange matrix.* We extend the definition of b_{ij} for $(i,j) \in \mathsf{K}\times\mathsf{K}$ by:

$$b_{ij} = -b_{ji} \text{ if } i \in \mathsf{K}^{\mathrm{ex}} \text{ and } j \in \mathsf{K}^{\mathrm{fr}}, \text{ and } b_{ij} = 0 \text{ for } i,j \in \mathsf{K}^{\mathrm{fr}},$$

so that $(b_{ij})_{i,j\in\mathsf{K}}$ is skew-symmetric.

To the matrix $\widetilde{B}$, we associate the quiver $\mathcal{Q}_{\widetilde{B}}$ such that the set of vertices is K and the number of arrows from $i \in \mathsf{K}$ to $j \in \mathsf{K}$ is $\max(0, b_{ij})$. Then, $\mathcal{Q}_{\widetilde{B}}$ satisfies the following conditions:

$$
\begin{cases}
\text{(a) the set of vertices of } \mathcal{Q}_{\widetilde{B}} \text{ is labeled by } \mathsf{K}, \\
\text{(b) } \mathcal{Q}_{\widetilde{B}} \text{ does not have any loop, any 2-cycle, nor arrow between frozen vertices,} \\
\text{(c) each exchangeable vertex } v \text{ of } \mathcal{Q}_{\widetilde{B}} \text{ has } \textit{finite degree}\text{; that is, the number of arrows incident with } v \text{ is finite.}
\end{cases}
\tag{7.2}
$$

Conversely, for a quiver satisfying (7.2), we can associate a matrix $\widetilde{B}$ satisfying (7.1) by taking

$$b_{ij} := (\text{the number of arrows from } i \text{ to } j) - (\text{the number of arrows from } j \text{ to } i).$$

We say that a $\mathbb{Z}$-valued skew-symmetric $\mathsf{K}\times\mathsf{K}$-matrix $L = (\lambda_{ij})_{i,j\in\mathsf{K}}$ is *compatible* with $\widetilde{B}$ (or $(L, \widetilde{B})$ is a compatible pair), if

$$\sum_{k\in\mathsf{K}} \lambda_{ik} b_{kj} = 2\delta_{i,j} \qquad \text{for each } i \in \mathsf{K} \text{ and } j \in \mathsf{K}^{\mathrm{ex}}.$$

Let $\{X_i\}_{i\in\mathsf{K}}$ be the set of mutually commuting indeterminates.

Definition 7.1. For a commutative ring $\mathscr{A}$, we say that a triple $\mathcal{S} = (\{x_i\}_{i\in\mathsf{K}}, L, \widetilde{B})$ is a Λ-*seed* in $\mathscr{A}$ if

(1) $\{x_i\}_{i\in\mathsf{K}}$ is a family of elements of $\mathscr{A}$ and there exists an injective algebra homomorphism $\mathbb{Z}[X_i; i \in \mathsf{K}]$ to $\mathscr{A}$ such that $X_i \mapsto x_i$,
(2) $(L, \widetilde{B})$ is a compatible pair.

For a Λ-seed $\mathcal{S} = (\{x_i\}_{i\in\mathsf{K}}, L, \widetilde{B})$, we call the set $\{x_i\}_{i\in\mathsf{K}}$ the *cluster* of $\mathcal{S}$ and its elements the *cluster variables*. An element of the form $x^{\mathbf{a}}$ $\big(\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus\mathsf{K}}\big)$ is called a *cluster monomial*, where

$$x^{\mathbf{c}} := \prod_{i\in\mathsf{K}} x_i^{c_i} \quad \text{for} \quad \mathbf{c} = (c_i)_{i\in\mathsf{K}} \in \mathbb{Z}^{\oplus\mathsf{K}}.$$

Let $\mathcal{S} = (\{x_i\}_{i\in\mathsf{K}}, L, \widetilde{B})$ be a Λ-seed in a field $\mathfrak{K}$ of characteristic 0. For $k \in \mathsf{K}^{\mathrm{ex}}$, we define

$$\text{(a)} \quad \mu_k(L)_{ij} = \begin{cases} -\lambda_{kj} + \sum\limits_{t\in\mathsf{K}} \max(0, -b_{tk})\lambda_{tj} & \text{if } i = k,\ j \neq k, \\ -\lambda_{ik} + \sum\limits_{t\in\mathsf{K}} \max(0, -b_{tk})\lambda_{it} & \text{if } i \neq k,\ j = k, \\ \lambda_{ij} & \text{otherwise}, \end{cases}$$

$$\text{(b)} \quad \mu_k(\widetilde{B})_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + (-1)^{\delta(b_{ik}<0)} \max(b_{ik}b_{kj}, 0) & \text{otherwise}, \end{cases}$$

$$\text{(c)} \quad \mu_k(x)_i = \begin{cases} x^{\mathbf{a}'} + x^{\mathbf{a}''}, & \text{if } i = k, \\ x_i & \text{if } i \neq k, \end{cases}$$

where $\mathbf{a}' := (a'_i)_{i\in\mathsf{K}} \in \mathbb{Z}^{\oplus\mathsf{K}}$ and $\mathbf{a}'' := (a''_i)_{i\in\mathsf{K}} \in \mathbb{Z}^{\oplus\mathsf{K}}$ are defined as follows:

$$a'_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{ik}) & \text{if } i \neq k, \end{cases} \qquad a''_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{ik}) & \text{if } i \neq k. \end{cases}$$

Then the triple

$$\mu_k(\mathcal{S}) := (\{\mu_k(x)_i\}_{i\in\mathsf{K}}, \mu_k(L), \mu_k(\widetilde{B}))$$

becomes a new Λ-seed in $\mathfrak{K}$ and we call it the *mutation* of $\mathcal{S}$ at k.

The *cluster algebra* $\mathcal{A}(\mathcal{S})$ *associated to the* Λ*-seed* $\mathcal{S}$ is the $\mathbb{Z}$-subalgebra of the field $\mathfrak{K}$ generated by all the cluster variables in the Λ-seeds obtained from $\mathcal{S}$ by all possible successive mutations.

A *cluster algebra structure associated to a* Λ*-seed* $\mathcal{S}$ on a $\mathbb{Z}$-algebra $\mathcal{A}$ is a family $\mathscr{F}$ of Λ-seeds in $\mathcal{A}$ such that

(1) for any Λ-seed $\mathcal{S}$ in $\mathscr{F}$, the cluster algebra $\mathcal{A}(\mathcal{S})$ is isomorphic to $\mathcal{A}$,
(2) any mutation of a Λ-seed in $\mathscr{F}$ is in $\mathscr{F}$,
(3) for any pair $\mathcal{S}$, $\mathcal{S}'$ of Λ-seeds in $\mathscr{F}$, $\mathcal{S}'$ can be obtained from $\mathcal{S}$ by a finite sequence of mutations.

7.2. **Monoidal seeds and their mutations.** Let $\mathcal{C}$ be a full subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing the trivial module $\mathbf{1}$ and stable under taking tensor products, subquotients and extensions.

Definition 7.2.

(1) A *monoidal seed in* $\mathcal{C}$ is a quadruple $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ consisting of a commuting family $\{M_i\}_{i\in\mathsf{K}}$ of real simple modules in $\mathcal{C}$, an integer-valued $\mathsf{K}\times\mathsf{K}^{\mathrm{ex}}$-matrix $\widetilde{B} = (b_{ij})_{(i,j)\in\mathsf{K}\times\mathsf{K}^{\mathrm{ex}}}$ satisfying the conditions in (7.1), an index set K and an index set $\mathsf{K}^{\mathrm{ex}} \subset \mathsf{K}$ of exchangeable vertices.

(2) For $i \in \mathsf{K}$, we call M_i the i-th *cluster variable module* of $\mathscr{S}$.

For a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$, let $\Lambda^{\mathscr{S}} = (\Lambda^{\mathscr{S}}_{ij})_{i,j\in\mathsf{K}}$ be the skew-symmetric matrix given by $\Lambda^{\mathscr{S}}_{ij} = \Lambda(M_i, M_j)$.

Definition 7.3. We say that a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ in $\mathcal{C}$ is *admissible* if it satisfies the following conditions:

(1) for each $k \in \mathsf{K}^{\mathrm{ex}}$, there exists a simple object M'_k of $\mathcal{C}$ such that there is an exact sequence in $\mathcal{C}$

$$0 \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to M_k \otimes M'_k \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to 0,$$

(2) M'_k commutes with M_i for each $k \in \mathsf{K}^{\mathrm{ex}}$ and any $i \in \mathsf{K}\setminus\{k\}$.

Note that we have also an exact sequence

$$0 \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to M'_k \otimes M_k \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to 0.$$

Note also that M'_k is unique up to an isomorphism if it exists, since $M_k \nabla M'_k \simeq \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})}$.

Lemma 7.4. *If a monoidal seed* $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ *is* admissible, M'_k *in Definition 7.3 is real for any* $k \in \mathsf{K}^{\mathrm{ex}}$.

Proof. It follows from Proposition 2.21 (ii). □

By this lemma, if $\mathscr{S}$ is admissible, then the quadruple

$$\mu_k(\mathscr{S}) := \big(\{M_i\}_{i\neq k} \cup \{M'_k\}, \mu_k(\widetilde{B}); \mathsf{K}, \mathsf{K}^{\mathrm{ex}}\big)$$

is a monoidal seed in $\mathcal{C}$. We call $\mu_k(\mathscr{S})$ the *mutation* of $\mathscr{S}$ in direction k.

Proposition 7.5 ([38, Proposition 6.4])**.** *Let $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ be an admissible monoidal seed in $\mathcal{C}$. Let $k \in \mathsf{K}^{\mathrm{ex}}$ and let M'_k be as in* Definition 7.3. *Then we have the following properties.*

(i) *For any $j \in \mathsf{K}$, we have $(\Lambda^{\mathscr{S}}\ \widetilde{B})_{jk} = -2\,\mathfrak{d}(M_j, M'_k)$.*
(ii) *For any $j \in \mathsf{K}$, we have*

$$\Lambda(M_j, M'_k) = -\Lambda(M_j, M_k) - \sum_{b_{ik}<0} \Lambda(M_j, M_i) b_{ik},$$
$$\Lambda(M'_k, M_j) = -\Lambda(M_k, M_j) + \sum_{b_{ik}>0} \Lambda(M_i, M_j) b_{ik}.$$

Definition 7.6 ([38, Definition 6.5])**.** Let $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ be an admissible monoidal seed.

(1) We say that a monoidal seed $\mathscr{S}$ is Λ-*admissible* if M'_k in Definition 7.3 satisfies $\mathfrak{d}(M_k, M'_k) = 1$.
(2) If $\mathscr{S}$ is Λ-admissible, we say that the mutation $\mu_k(\mathscr{S})$ of $\mathscr{S}$ at $k \in \mathsf{K}^{\mathrm{ex}}$ is a Λ-*mutation*,
(3) We say that a monoidal seed $\mathscr{S}$ is *completely* Λ-*admissible* if $\mathscr{S}$ admits successive Λ-mutations in all possible directions.

For a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B})$ in $\mathcal{C}$, we define the triple $[\mathscr{S}]$ in $K(\mathcal{C})$ by

$$[\mathscr{S}] := \big(\{[M_i]\}_{i\in\mathsf{K}}, -\Lambda^{\mathscr{S}}, \widetilde{B}\big).$$

If $\mathscr{S}$ is a Λ-admissible monoidal seed, then $[\mathscr{S}]$ is a Λ-seed.

Definition 7.7 ([38, Definition 6.7])**.** A category $\mathcal{C}$ is called a Λ-*monoidal categorification* of a cluster algebra $\mathcal{A}$ if

(1) the Grothendieck ring $K(\mathcal{C})$ is isomorphic to $\mathcal{A}$,
(2) there exists a completely Λ-admissible monoidal seed $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ in $\mathcal{C}$ such that
$$[\mathscr{S}] := (\{[M_i]\}_{i\in\mathsf{K}}, -\Lambda^{\mathscr{S}}, \widetilde{B})$$
is an initial Λ-seed of $\mathcal{A}$.

Now we present the main result of [38]:

Theorem 7.8 ([38, Theorem 6.10])**.** *Let $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ be a Λ-admissible monoidal seed in $\mathcal{C}$, and set*

$$[\mathscr{S}] := (\{[M_i]\}_{i\in\mathsf{K}}, -\Lambda^{\mathscr{S}}, \widetilde{B}).$$

We assume that the algebra $K(\mathcal{C})$ is isomorphic to the cluster algebra $\mathcal{A}([\mathscr{S}])$. Then, we have

- *$\mathscr{S}$ is completely Λ-admissible, and*
- *$\mathcal{C}$ gives a Λ-monoidal categorification of $\mathcal{A}([\mathscr{S}])$.*

Definition 7.9. A family of real simple modules $\{M_i\}_{i\in\mathsf{K}}$ in $\mathcal{C}$ is called a *real commuting family in* $\mathcal{C}$ if it satisfies:

(1) $\{M_i\}_{i\in\mathsf{K}}$ is mutually commuting.

It is called a *maximal real commuting family in* $\mathcal{C}$ if it satisfies further :

(2) if a simple module X commutes with all the M_i's, then X is isomorphic to $\bigotimes_{i\in\mathsf{K}} M_i^{\otimes a_i}$ for some $\mathbf{a}=\{a_i\}_{i\in\mathsf{K}}\in\mathbb{Z}_{\geqslant 0}^{\oplus\mathsf{K}}$.

Corollary 7.10 ([38, Corollary 6.11])**.** Let $\mathscr{S}=(\{M_i\}_{i\in\mathsf{K}},\widetilde{B};\mathsf{K},\mathsf{K}^{\mathrm{ex}})$ be a Λ-admissible monoidal seed in $\mathcal{C}$ and assume that the algebra $K(\mathcal{C})$ is isomorphic to $\mathcal{A}([\mathscr{S}])$. Then the following statements hold:

(i) Any cluster monomial in $K(\mathcal{C})$ is the isomorphism class of a real simple object in $\mathcal{C}$.
(ii) Any cluster monomial in $K(\mathcal{C})$ is a Laurent polynomial of the initial cluster variables with coefficient in $\mathbb{Z}_{\geqslant 0}$.
(iii) For $k\in\mathsf{K}^{\mathrm{ex}}$ and the k-th cluster variable module $\widetilde{M}_k$ of a monoidal seed $\widetilde{\mathscr{S}}$ obtained by successive Λ-mutations from the initial monoidal seed $\mathscr{S}$, we have
$$\mathfrak{d}(\widetilde{M}_k,\widetilde{M}'_k)=1.$$
Here $\widetilde{M}'_k$ is the k-th cluster variable module of $\mu_k(\widetilde{\mathscr{S}})$.
(iv) Any monoidal cluster $\{\widetilde{M}_i\}_{i\in\mathsf{K}}$ is a maximal real commuting family.

7.3. **Properties of Λ-admissible monoidal seeds.** Recall the definitions of $\mathcal{W}_0$, Δ_0, $\mathsf{E}(M)$ $(M\in\mathscr{C}_{\mathfrak{g}})$ given in § 2.4.

Lemma 7.11. *Let $\mathscr{S}=(\{M_i\}_{i\in\mathsf{K}},\widetilde{B};\mathsf{K},\mathsf{K}^{\mathrm{ex}})$ be an admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}$. Then we have*
$$\sum_{i\in\mathsf{K}}\mathsf{E}(M_i)b_{ik}=0\quad\textit{for any } k\in\mathsf{K}^{\mathrm{ex}},$$
where $\widetilde{B}=(b_{ij})_{(i,j)\in\mathsf{K}\times\mathsf{K}^{\mathrm{ex}}}$.

Proof. By the definition, there is a short exact sequence with a simple M'_k:
$$0\to\bigotimes_{b_{ik}>0}M_i^{\otimes b_{ik}}\to M_k\otimes M'_k\to\bigotimes_{b_{ik}<0}M_i^{\otimes(-b_{ik})}\to 0,$$

Then we have

$$\begin{aligned}\sum_{b_{ik}>0} \mathsf{E}(M_i)b_{ik} &= \mathsf{E}\big(\bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}}\big)\\ &= \mathsf{E}(M_k)+\mathsf{E}(M_k') = \mathsf{E}\big(\bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})}\big) = \sum_{b_{ik}<0} \mathsf{E}(M_i)(-b_{ik})\end{aligned}$$

by [39, Lemma 3.1]. Hence we have the desired result. □

Lemma 7.12 (cf. [35, Lemma 3.2])**.** *Let $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ be a Λ-admissible in $\mathscr{C}_{\mathfrak{g}}^0$, and $\widetilde{B} = (b_{ij})_{(i,j)\in\mathsf{K}\times\mathsf{K}^{\mathrm{ex}}}$. Assume that K is a finite set.*

(i) *Then we have $\dim\big(\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)\big) \leqslant |\mathsf{K}^{\mathrm{fr}}|$.*
(ii) *Assume further that $\dim\big(\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)\big) = |\mathsf{K}^{\mathrm{fr}}|$. Then, for any $k\in\mathsf{K}^{\mathrm{ex}}$, $(b_{ik})_{i\in\mathsf{K}}$ is a unique element $(v_i)_{i\in\mathsf{K}}$ of $\mathbb{Q}^{\mathsf{K}}$ such that*

$$\sum_{i\in\mathsf{K}} \mathsf{E}(M_i)v_i = 0 \quad \textit{and} \quad \sum_{i\in\mathsf{K}} (\Lambda^{\mathscr{S}})_{ji}v_i = -2\delta_{j,k} \quad \textit{for any } j\in\mathsf{K}^{\mathrm{ex}}.$$

Proof. Let $f\colon \mathbb{Q}^{\oplus\mathsf{K}} \longrightarrow \mathbb{Q}^{\oplus\mathsf{K}\mathrm{ex}} \bigoplus \mathcal{W}_0$ be the linear map given by $(\Lambda^{\mathscr{S}})_{ki}$ and $\mathsf{E}(M_i)$. Then, $\mathrm{Im}(f)$ contains $\mathbb{Q}^{\oplus\mathsf{K}\mathrm{ex}} \bigoplus 0$ by Proposition 7.5 and Lemma 7.11. Moreover, the image of the composition $f\colon \mathbb{Q}^{\oplus\mathsf{K}} \longrightarrow \mathbb{Q}^{\oplus\mathsf{K}\mathrm{ex}} \bigoplus \mathcal{W}_0 \to \mathcal{W}_0$ is $\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)$, which implies that $\mathrm{Im}(f) = \mathbb{Q}^{\oplus\mathsf{K}\mathrm{ex}} \bigoplus \big(\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)\big)$. Hence the dimension of $\mathrm{Im}(f)$ is equal to $|\mathsf{K}^{\mathrm{ex}}| + \dim\big(\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)\big)$. Then, $|\mathsf{K}| \geqslant \dim\big(\mathrm{Im}(f)\big)$ implies (i).

(ii) If $\dim\big(\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)\big) = |\mathsf{K}^{\mathrm{fr}}|$, then f is injective. □

Proposition 7.13. *Let $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ be a Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^0$ with $\widetilde{B} = (b_{ij})_{(i,j)\in\mathsf{K}\times\mathsf{K}^{\mathrm{ex}}}$, and let $k\in\mathsf{K}^{\mathrm{ex}}$. Assume that*

(i) *K is a finite set and $\dim\big(\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)\big) \geqslant |\mathsf{K}^{\mathrm{fr}}|$,*
(ii) *there exist a real simple module $X\in\mathcal{C}$ and an exact sequence*

$$0 \to A \to M_k \otimes X \to B \to 0,$$

such that
(a) *X commutes with M_j for all $j\in\mathsf{K}\setminus\{k\}$,*
(b) *$\mathfrak{d}(M_k, X) = 1$,*
(c) *$A = \bigotimes_{i\in\mathsf{K}} M_i^{\otimes m_i}$, $B = \bigotimes_{i\in\mathsf{K}} M_i^{\otimes n_i}$ for some $m_i, n_i \in \mathbb{Z}_{\geqslant 0}$.*

Then we have $b_{ik} = m_i - n_i$.

If we have furthermore $m_i n_i = 0$ for all $i \in \mathsf{K}$, then we have

$$X \simeq M'_k,$$

where M'_k is given in Definition 7.3.

Proof. We shall apply Lemma 7.12. Set $\lambda_{ij} = \Lambda(M_i, M_j)$. Then we have

$$\sum_{i\in\mathsf{K}} m_i \mathsf{E}(M_i) = \mathsf{E}(A) = \mathsf{E}(M_k) + \mathsf{E}(X) = \mathsf{E}(B) = \sum_{i\in\mathsf{K}} n_i \mathsf{E}(M_i).$$

For any $j \in \mathsf{K}$, we have

$$\sum_{i\in\mathsf{K}} \lambda_{ij} m_i = \Lambda(A, M_j) = \Lambda(X \nabla M_k, M_j) = \Lambda(X, M_j) + \Lambda(M_k, M_j)$$
$$\sum_{i\in\mathsf{K}} \lambda_{ji} n_i = \Lambda(M_j, B) = \Lambda(M_j, M_k \nabla X) = \Lambda(M_j, M_k) + \Lambda(M_j, X)$$

Hence we have

$$\begin{aligned}\sum_{i\in\mathsf{K}} \lambda_{ji}(n_i - m_i) &= \Lambda(X, M_j) + \Lambda(M_k, M_j) + \Lambda(M_j, M_k) + \Lambda(M_j, X)\\ &= 2\big(\mathfrak{d}(M_j, X) + \mathfrak{d}(M_j, M_k)\big) = 2\delta_{j,k}.\end{aligned}$$

Thus, Lemma 7.12 implies that $b_{ik} = m_i - n_i$.

Now assume that $m_i n_i = 0$ for all $i \in \mathsf{K}$. Then we have $n_i = \max\{0, -b_{ik}\}$ and

$$B \simeq \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})}.$$

Hence, we obtain

$$M_k \nabla X \simeq B \simeq M_k \nabla M'_k,$$

which implies that $X \simeq M'_k$. ∎

As an immediate application of Proposition 7.13, we can show that the exchange matrix is uniquely determined for a Λ-admissible monoidal seed.

Proposition 7.14. *Let $\mathscr{S}_0 = (\{M_i\}_{i\in\mathsf{K}_0}, \widetilde{B}_0; \mathsf{K}_0, \mathsf{K}_0^{\mathrm{ex}})$ and $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ be two Λ-admissible monoidal seeds in $\mathscr{C}_{\mathfrak{g}}^0$ such that $\mathsf{K}_0 \subset \mathsf{K}$ and $\mathsf{K}_0^{\mathrm{ex}} \subset \mathsf{K}^{\mathrm{ex}}$. Assume that K is a finite set and $\dim\big(\sum_{i\in\mathsf{K}} \mathbb{Q}\mathsf{E}(M_i)\big) \geqslant |\mathsf{K}^{\mathrm{fr}}|$. Then*

$$\widetilde{B}|_{\mathsf{K}_0\times\mathsf{K}_0^{\mathrm{ex}}} = \widetilde{B}_0 \quad \text{and} \quad \widetilde{B}|_{(\mathsf{K}\setminus\mathsf{K}_0)\times\mathsf{K}_0^{\mathrm{ex}}} = 0.$$

The following lemma is almost obvious by the definition.

Lemma 7.15. *Let $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ be a monoidal seed in $\mathcal{C}$. Let K_0 be a subset of K with a decomposition $\mathsf{K}_0 = \mathsf{K}_0^{\mathrm{ex}} \sqcup \mathsf{K}_0^{\mathrm{fr}}$ such that $\mathsf{K}_0^{\mathrm{ex}} \subset \mathsf{K}^{\mathrm{ex}}$. Set*

$$\mathscr{S}|_{(\mathsf{K}_0,\, \mathsf{K}_0^{\mathrm{ex}})} := \bigl(\{M_i\}_{i\in\mathsf{K}_0},\ \widetilde{B}|_{(\mathsf{K}_0)\times\mathsf{K}_0^{\mathrm{ex}}}; \mathsf{K}_0,\ \mathsf{K}_0^{\mathrm{ex}}\bigr).$$

Assume that

$$b_{ij} = 0 \text{ if } i \in \mathsf{K}\setminus\mathsf{K}_0 \text{ and } j \in \mathsf{K}_0^{\mathrm{ex}}. \tag{7.3}$$

Then, we have

(i) $\bigl(\mu_s(\widetilde{B})\bigr)_{ij} = 0$ *if* $s \in \mathsf{K}_0^{\mathrm{ex}}$, $i \in \mathsf{K}\setminus\mathsf{K}_0$ *and* $j \in \mathsf{K}_0^{\mathrm{ex}}$,

(ii) *if $\mathscr{S}$ is Λ-admissible, then we have*

$$\bigl(\mu_s\mathscr{S}\bigr)|_{(\mathsf{K}_0,\mathsf{K}_0^{\mathrm{ex}})} = \begin{cases} \mu_s(\mathscr{S}|_{(\mathsf{K}_0,\, \mathsf{K}_0^{\mathrm{ex}})}) & \text{if } s \in \mathsf{K}_0^{\mathrm{ex}}, \\ \mathscr{S}|_{(\mathsf{K}_0,\, \mathsf{K}_0^{\mathrm{ex}})} & \text{if } s \in \mathsf{K}\setminus\mathsf{K}_0. \end{cases}$$

In particular, if $\mathscr{S}$ is a completely Λ-admissible monoidal seed in $\mathcal{C}$, then so is $\mathscr{S}|_{(\mathsf{K}_0,\, \mathsf{K}_0^{\mathrm{ex}})}$.

7.4. **Monoidal seeds and admissible chains of i-boxes.** Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair. Throughout this subsection we consider admissible chains $\mathfrak{C}$ of i-boxes associated with $(\mathcal{D}, \widehat{\underline{w}}_0)$.

Let $\mathfrak{C} = (\mathfrak{c}_k)_{1\leqslant k\leqslant l}$ be an admissible chain of i-boxes with a range $[a,b]$. We define

$$\begin{aligned} \mathsf{K}(\mathfrak{C}) &:= [1,l], \\ \mathsf{K}^{\mathrm{fr}}(\mathfrak{C}) &:= \{s \in \mathsf{K}(\mathfrak{C}) \mid \mathfrak{c}_s = [a(\imath)^+, b(\imath)^-] \text{ for some } \imath \in I_{\mathsf{g}}\}, \\ \mathsf{K}^{\mathrm{ex}}(\mathfrak{C}) &:= \mathsf{K}(\mathfrak{C}) \setminus \mathsf{K}^{\mathrm{fr}}(\mathfrak{C}), \\ \mathsf{M}(\mathfrak{C}) &:= \{M(\mathfrak{c}_k)\}_{k\in\mathsf{K}(\mathfrak{C})}. \end{aligned} \tag{7.4}$$

Here, if $l = \infty$, we understand that $\mathsf{K}^{\mathrm{fr}}(\mathfrak{C})$ is the empty set. Recall that $\mathsf{M}(\mathfrak{C})$ is a commuting family of real simple modules (see Theorem 5.5).

We shall first prove the following lemma that assures that Proposition 7.13 is applicable to $\mathscr{S}(\mathfrak{C})$ for any admissible chain $\mathfrak{C}$ of i-boxes with a finite range.

Lemma 7.16. *Let $\mathfrak{C} = (\mathfrak{c}_k)_{1\leqslant k\leqslant l}$ be an admissible chain of i-boxes associated with $\widehat{\underline{w}}_0$ and assume that its range $[a,b]$ is finite. Then we have*

$$\dim\Bigl(\sum_{1\leqslant k\leqslant l} \mathbb{Q}\mathsf{E}\bigl(M(\mathfrak{c}_k)\bigr)\Bigr) = |\mathsf{K}^{\mathrm{fr}}(\mathfrak{C})| = \Bigl|\{\imath_s \mid s \in [a,b]\}\Bigr|.$$

Proof. By Proposition 5.6 and Proposition 5.7, $\sum_{1 \leqslant k \leqslant l} \mathbb{Q}\mathsf{E}\big(M(\mathfrak{c}_k)\big)$ does not change by box moves. Hence, we have $\mathsf{E}(\mathsf{S}_s) \in \sum_{1 \leqslant k \leqslant l} \mathbb{Q}\mathsf{E}\big(M(\mathfrak{c}_k)\big)$ for any $s \in [a, b]$. Therefore, we have

$$\sum_{1 \leqslant k \leqslant l} \mathbb{Q}\,\mathsf{E}\big(M(\mathfrak{c}_k)\big) = \sum_{s \in [a,b]} \mathbb{Q}\,\mathsf{E}(\mathsf{S}_s) \simeq \sum_{s \in [a,b]} \mathbb{Q}\,\alpha_{\imath_s},$$

which implies that $\dim\Big(\sum_{1 \leqslant k \leqslant l} \mathbb{Q}\mathsf{E}\big(M(\mathfrak{c}_k)\big)\Big) = \Big|\{\imath_s \mid s \in [a, b]\}\Big|$. □

The following lemma says that box moves correspond to mutations.

Lemma 7.17. *Let $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leqslant k \leqslant l}$ be an admissible chain of i-boxes associated with $\widehat{\underline{w}}_0$ and a finite range such that $\mathscr{S} := \big(\mathsf{M}(\mathfrak{C}), \widetilde{B}; \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C})\big)$ is a Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^0$ for some exchange matrix $\widetilde{B}$. If $k_0 \in \mathsf{K}^{\mathrm{ex}}(\mathfrak{C})$ and $\mathfrak{c}_{k_0}$ is a movable i-box such that $\widetilde{\mathfrak{c}}_{k_0+1} = \mathfrak{c}_{k_0+1} = [a, b]$, then we have*

$$\begin{aligned} \mu_{k_0}(\mathscr{S}) &= \big(\mathsf{M}(B_{k_0}(\mathfrak{C})), \mu_{k_0}(\widetilde{B}); \mathsf{K}(B_{k_0}(\mathfrak{C})), \mathsf{K}^{\mathrm{ex}}(B_{k_0}(\mathfrak{C}))\big) \\ &= \big(\{M_i\}_{i \in \mathsf{K} \setminus \{k_0\}} \sqcup \{M'_{k_0}\}, \mu_{k_0}(\widetilde{B}); \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C})\big), \end{aligned}$$

where

$$M'_{k_0} := \begin{cases} M[a, b^-] & \text{if } \mathfrak{c}_k = [a^+, b], \\ M[a^+, b] & \text{if } \mathfrak{c}_k = [a, b^-]. \end{cases}$$

Thus B_k in Remark 5.8 *corresponds to μ_k in this case and the mutation μ_k corresponds to T-system in* (4.5)*.*

Proof. By Theorem 4.21 and Remark 5.8, the modules $M[a(\jmath)^+, b(\jmath)^-]$ $(d(\imath_a, \jmath) = 1)$ and $M[a^+, b^-]$ commute with $M[a, b]$, and they are contained in $\mathsf{M}(\mathfrak{C})$. Thus our assertion follows from Proposition 5.7, Proposition 7.13 together with Lemma 7.16. □

Together with Proposition 5.6, we obtain the following corollary.

Corollary 7.18. For a finite interval $[a, b]$, let $\mathfrak{C}$ and $\mathfrak{C}'$ be admissible chains of i-boxes associated with the same $\widehat{\underline{w}}_0$ and the same range $[a, b]$. Assume that $(\mathsf{M}(\mathfrak{C}), \widetilde{B}; \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C}))$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^0$ for some exchange matrix $\widetilde{B}$. Then, $(\mathsf{M}(\mathfrak{C}'), \widetilde{B}'; \mathsf{K}(\mathfrak{C}'), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C}'))$ is also a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^0$ for some exchange matrix $\widetilde{B}'$.

Proposition 7.19. *Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a complete PBW-pair and let $\mathscr{S} = \big(\{M_i\}_{i \in \mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}}\big)$ be a monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b], \mathcal{D}, \widehat{\underline{w}}_0}$. If $\mathscr{S}$ is (completely) Λ-admissible in $\mathscr{C}_{\mathfrak{g}}^0$, then it is (completely) Λ-admissible in $\mathscr{C}_{\mathfrak{g}}^{[a,b], \mathcal{D}, \widehat{\underline{w}}_0}$.*

Proof. It is enough to show that the mutation M'_k of $\mathscr{S}$ at a vertex $k \in \mathsf{K}^{\mathrm{ex}}$ belongs to $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$. Since M_k, $M_k \nabla M'_k$ and $M'_k \nabla M_k$ belong to $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$, our assertion follows from Corollary 4.11. □

7.5. **An example of Λ-admissible monoidal seeds.** Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair. Let $\mathrm{Q}_{\mathrm{GLS}}(\widehat{\underline{w}}_0)$ be the quiver structure on $\mathbb{Z}$ with two types of arrows (with the notations in (4.2)):

$$
\begin{array}{llll}
\text{vertical arrow} & : & s \longrightarrow t & \text{if } s^- < t^- < s < t \text{ and } d(\imath_s, \imath_t) = 1, \\
\text{horizontal arrow} & : & s \longrightarrow s^- &
\end{array}
\tag{7.5}
$$

If there is no afraid of confusion, we write shortly $\mathrm{Q}_{\mathrm{GLS}}$ for $\mathrm{Q}_{\mathrm{GLS}}(\widehat{\underline{w}}_0)$.

For an interval $[a, b]$, we denote by $\mathrm{Q}_{\mathrm{GLS}}^{[a,b]}(\widehat{\underline{w}}_0)$ its induced quiver on $[a, b]$, and by $\widetilde{B}_{\mathrm{GLS}}^{[a,b],\widehat{\underline{w}}_0}$ the exchange matrix associated with the quiver $\mathrm{Q}_{\mathrm{GLS}}^{[a,b]}(\widehat{\underline{w}}_0)$.

Theorem 7.20. *Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair. For $-\infty \leqslant a \leqslant b < +\infty$, the monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$*

$$
\mathscr{S}_-^{[a,b],\mathcal{D},\widehat{\underline{w}}_0} := \big(\{M^{\mathcal{D},\widehat{\underline{w}}_0}[s,b]\}_{s\in[a,b]}, \widetilde{B}_{\mathrm{GLS}}^{[a,b],\widehat{\underline{w}}_0}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}}\big)
\tag{7.6}
$$

is Λ-admissible. Here $\mathsf{K} = [a, b]$, $\mathsf{K}^{\mathrm{ex}} = \{s \in [a, b] \mid a \leqslant s^-\}$.

Note that $\mathscr{S}_-^{[a,b],\mathcal{D},\widehat{\underline{w}}_0} = \mathscr{S}(\mathfrak{C}_-^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$ with the admissible chain of i-boxes

$$
\mathfrak{C}_-^{[a,b],\mathcal{D},\widehat{\underline{w}}_0} := \big([b+1-k, b\}\big)_{1\leqslant k\leqslant b-a+1}.
$$

We devote this subsection for the proof of Theorem 7.20. We employ the framework of the proof of [33, Theorem 11.2.2].

Set $\mathsf{M}_s = M^{\mathcal{D},\widehat{\underline{w}}_0}[s,b\}$, $\mathsf{K} = [a, b]$ and

$$
\begin{aligned}
\mathsf{K}^{\mathrm{fr}} &= \{s \in [a,b] \mid s^- < a \leqslant s\}, \\
\mathsf{K}^{\mathrm{ex}} &= \mathsf{K} \setminus \mathsf{K}^{\mathrm{fr}} = \{s \in \mathbb{Z} \mid a \leqslant s^- < s \leqslant b\}, \\
\widetilde{B}_{\mathrm{GLS}}^{[a,b],\widehat{\underline{w}}_0} &= (b_{ij})_{(i,j)\in \mathsf{K}\times\mathsf{K}^{\mathrm{ex}}}.
\end{aligned}
$$

In order to prove Theorem 7.20, we will show

$$(7.7)\quad \begin{cases} \text{(a) } \{\mathsf{M}_s \mid s \in \mathsf{K}\} \text{ is a real commuting family,} \\ \text{(b) for } s \in \mathsf{K}^{\mathrm{ex}}\text{, there exist a simple module } \mathsf{M}'_s \text{ and an exact sequence} \\ \qquad 0 \longrightarrow \bigotimes_{b_{ts}>0} \mathsf{M}_t^{\otimes b_{ts}} \longrightarrow \mathsf{M}_s \otimes \mathsf{M}'_s \longrightarrow \bigotimes_{b_{ts}<0} \mathsf{M}_t^{\otimes(-b_{st})} \longrightarrow 0, \\ \text{(c) } \mathfrak{d}(\mathsf{M}_s, \mathsf{M}'_s) = 1, \\ \text{(d) } \mathsf{M}'_s \text{ is real and } \mathsf{M}'_s \text{ commutes with } \mathsf{M}_t \text{ for any } a \leqslant t \neq s \leqslant b. \end{cases}$$

Note that, (a) is already proved in Theorem 5.5. Thus it suffices to prove (b), (c) and (d).

For a vertex $s \in \mathsf{K}$, the arrows incident to s can be classified into four types as follows (see (7.5)): (i) horizontal incoming arrows : $s^+ \longrightarrow s$ (ii) horizontal outgoing arrows $s \longrightarrow s^-$ (iii) vertically incoming arrow : $t \longrightarrow s$ with $d(\imath_s, \imath_t) = 1$ and $t^- < s^- < t < s$, and (iv) vertically outgoing arrow : $s \longrightarrow t$ with $d(\imath_s, \imath_t) = 1$ and $s^- < t^- < s < t$.

Let us fix a vertex $s \in \mathsf{K}^{\mathrm{ex}}$ and set $\imath = \imath_s$. We define the subsets of K as follows:

$$(7.8)\quad \begin{aligned} \mathrm{Vi}(s) :=& \{t \in \mathbb{Z} \mid d(\imath, \imath_t) = 1 \text{ and } t^- < s^- < t < s\} \\ =& \{s^-(\jmath)^+ \mid d(\imath, \jmath) = 1,\ s^- < s(\jmath)^-\}, \\ \mathrm{Vo}(s) :=& \{t \in \mathbb{Z} \mid d(\imath, \imath_t) = 1 \text{ and } s^- < t^- < s < t \leqslant b\} \\ =& \{s(\jmath)^+ \mid d(\imath, \jmath) = 1,\ s^- < s(\jmath)^-,\ s(\jmath)^+ \leqslant b\}. \end{aligned}$$

We set

$$\mathsf{M}_s^{\mathrm{Vi}} := \bigotimes_{t \in \mathrm{Vi}(s)} \mathsf{M}_t \quad \text{and} \quad \mathsf{M}_s^{\mathrm{Vo}} := \bigotimes_{t \in \mathrm{Vo}(s)} \mathsf{M}_t.$$

Then we have

$$\bigotimes_{b_{ts}>0} \mathsf{M}_t^{\otimes b_{ts}} \simeq \mathsf{M}_{s^+} \otimes \mathsf{M}_s^{\mathrm{Vi}} \quad \text{and} \quad \bigotimes_{b_{ts}<0} \mathsf{M}_t^{\otimes(-b_{ts})} \simeq \mathsf{M}_{s^-} \otimes \mathsf{M}_s^{\mathrm{Vo}}.$$

Now let us show the following lemma.

Lemma 7.21. *The module*

$$(7.9)\quad \mathsf{M}'_s := \mathsf{M}_s^{\mathrm{Vo}} \nabla \mathsf{S}_{s^-}$$

satisfies the following properties:

(i) M'_s *is a simple module.*

(ii) $\mathsf{M}_s \nabla \mathsf{M}'_s \simeq \bigotimes_{b_{ts}<0} \mathsf{M}_t^{\otimes(-b_{ts})} \simeq \mathsf{M}_{s^-} \otimes \mathsf{M}_s^{\mathrm{Vo}}$.

(iii) $\mathsf{M}'_s \mathbin{\nabla} \mathsf{M}_s \simeq \bigotimes_{b_{ts}>0} \mathsf{M}_t^{\otimes b_{ts}} \simeq \mathsf{M}_{s^+} \otimes \mathsf{M}_s^{\mathrm{Vi}}$.

Note that M'_s in (7.9) is simple by Theorem 2.10 because $\mathsf{M}_s^{\mathrm{Vo}}$ is simple.

Sublemma 1. *We have*

$$\mathsf{M}_s \mathbin{\nabla} \mathsf{M}'_s \simeq \mathsf{M}_{s^-} \otimes \mathsf{M}_s^{\mathrm{Vo}}.$$

Proof. We have

$$\mathsf{M}_s \mathbin{\nabla} \mathsf{S}_{s^-} \simeq \mathsf{M}_{s^-}.$$

On the other hand, we have

$$\mathsf{M}_s \otimes \big(\mathsf{M}_s^{\mathrm{Vo}} \mathbin{\nabla} \mathsf{S}_{s^-}\big) \rightarrowtail \mathsf{M}_s \otimes \mathsf{S}_{s^-} \otimes \mathsf{M}_s^{\mathrm{Vo}} \twoheadrightarrow \mathsf{M}_{s^-} \otimes \mathsf{M}_s^{\mathrm{Vo}},$$

whose composition does not vanish by Proposition 2.9 (i). Thus our assertion follows from Theorem 2.10. □

Note that we have

$$M[s,b\} \simeq M[s^+,b\} \mathbin{\nabla} \mathsf{S}_s \rightarrowtail \mathsf{S}_s \otimes M[s^+,b\}.$$

Thus we have the following monomorphism:

$$\big(\mathsf{M}_s^{\mathrm{Vo}} \mathbin{\nabla} \mathsf{S}_{s^-}\big) \otimes \mathsf{M}_s \rightarrowtail \big(\mathsf{M}_s^{\mathrm{Vo}} \mathbin{\nabla} \mathsf{S}_{s^-}\big) \otimes \mathsf{S}_s \otimes \mathsf{M}_{s^+}.$$

Hence, in order to prove (iii) in Lemma 7.21, it is suffices to show that there exists an epimorphism

$$\big(\mathsf{M}_s^{\mathrm{Vo}} \mathbin{\nabla} \mathsf{S}_{s^-}\big) \otimes \mathsf{S}_s \twoheadrightarrow \mathsf{M}_s^{\mathrm{Vi}}, \tag{7.10}$$

by Proposition 2.9 (i).

Note that we have a surjective homomorphism

$$\mathsf{M}_s^{\mathrm{Vo}} \otimes \mathsf{S}_{s^-} \otimes \mathsf{S}_s \twoheadrightarrow \big(\mathsf{M}_s^{\mathrm{Vo}} \mathbin{\nabla} \mathsf{S}_{s^-}\big) \otimes \mathsf{S}_s.$$

Sublemma 2. *The sequence* $\big(\mathsf{M}_s^{\mathrm{Vo}}, \mathsf{S}_{s^-}, \mathsf{S}_s\big)$ *is normal. In particular,* $\mathrm{hd}\big(\mathsf{M}_s^{\mathrm{Vo}} \otimes \mathsf{S}_{s^-} \otimes \mathsf{S}_s\big)$ *is simple.*

Proof. Note that if $t \in \mathrm{Vo}(s)$, then $t > s$. Hence we have

$$\mathfrak{d}(\mathsf{M}_s^{\mathrm{Vo}}, \mathscr{D}^{-1}\mathsf{S}_s) = 0,$$

by (iii) in Proposition 4.6, i.e. $(\mathsf{M}_s^{\mathrm{Vo}}, \mathsf{S}_s)$ is unmixed. Hence, our assertion follows from Proposition 2.14. □

Hence we have

$$\mathrm{hd}\left(\mathsf{M}_s^{\mathrm{Vo}} \otimes \mathsf{S}_{s^-} \otimes \mathsf{S}_s\right) \simeq \left(\mathsf{M}_s^{\mathrm{Vo}} \nabla \mathsf{S}_{s^-}\right) \nabla \mathsf{S}_s.$$

Sublemma 3. *For $s \in \mathsf{K}^{\mathrm{ex}}$, we have*

$$\mathrm{hd}\left(\mathsf{M}_s^{\mathrm{Vo}} \otimes \mathsf{S}_{s^-} \otimes \mathsf{S}_s\right) \simeq \mathsf{M}_s^{\mathrm{Vi}}.$$

Proof. By T-system described in (4.5), we have

$$\mathsf{M}_s^{\mathrm{Vo}} \otimes \mathsf{S}_{s^-} \otimes \mathsf{S}_s \twoheadrightarrow \mathsf{M}_s^{\mathrm{Vo}} \otimes \left(\bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-} M[s^-(\jmath)^+, s(\jmath)^-] \right) \simeq X \otimes Y \otimes Z,$$

where

$$\begin{aligned} X &= \bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-,\, s(\jmath)^+ \leqslant b} M[s(\jmath)^+, b\}, \\ Y &= \bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-,\, s(\jmath)^+ \leqslant b} M[s^-(\jmath)^+, s(\jmath)^-], \\ Z &= \bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-,\, s(\jmath)^+ > b} M[s^-(\jmath)^+, s(\jmath)^-]. \end{aligned}$$

We have an epimorphism

$$\begin{aligned} X \otimes Y &\twoheadrightarrow \bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-,\, s(\jmath)^+ \leqslant b} \left(M[s(\jmath)^+, b\} \nabla M[s^-(\jmath)^+, s(\jmath)^-] \right) \\ &\simeq \bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-,\, s(\jmath)^+ \leqslant b} M[s^-(\jmath)^+, b\}. \end{aligned}$$

by [33, Lemma 3.2.22]. On the other hand we have

$$Z \simeq \bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-,\, s(\jmath)^+ > b} M[s^-(\jmath)^+, b\}.$$

Finally we obtain epimorphisms

$$\begin{aligned} \mathsf{M}_s^{\mathrm{Vo}} \otimes \mathsf{S}_{s^-} \otimes \mathsf{S}_s &\twoheadrightarrow X \otimes Y \otimes Z \\ &\twoheadrightarrow \Big(\bigotimes_{\substack{d(\imath,\jmath)=1, \\ s^- < s(\jmath)^-,\, s(\jmath)^+ \leqslant b}} M[s^-(\jmath)^+, b\} \Big) \otimes \Big(\bigotimes_{\substack{d(\imath,\jmath)=1, \\ s^- < s(\jmath)^-,\, s(\jmath)^+ > b}} M[s^-(\jmath)^+, b\} \Big) \\ &\simeq \bigotimes_{d(\imath,\jmath)=1,\, s^- < s(\jmath)^-} M[s^-(\jmath)^+, b\} \simeq \mathsf{M}_s^{\mathrm{Vi}}. \quad \square \end{aligned}$$

Hence we have shown the existence of an epimorphism (7.10), and obtain (iii) in Lemma 7.21. Thus we complete the proof of Lemma 7.21.

Now let us show (7.7) (c).

Lemma 7.22. *For any $s \in \mathsf{K}^{\mathrm{ex}}$, we have*

$$\mathfrak{d}(\mathsf{M}_s, \mathsf{M}'_s) = 1.$$

Proof. Since the set of real modules $\{\mathsf{M}_t\}_{t\in\mathsf{K}}$ is a real commuting family and $\mathsf{M}'_s \simeq \mathsf{M}_s^{\mathrm{Vo}} \nabla \mathsf{S}_{s^-}$, Proposition 2.19, Theorem 5.5 and Proposition 4.20 imply that

$$\begin{aligned}\mathfrak{d}(\mathsf{M}_s, \mathsf{M}'_s) &\leqslant \mathfrak{d}(\mathsf{M}_s, \mathsf{M}_s^{\mathrm{Vo}}) + \mathfrak{d}(\mathsf{M}_s, \mathsf{S}_{s^-}) \\ &= \mathfrak{d}(\mathsf{M}_s, \mathsf{S}_{s^-}) = 1.\end{aligned}$$

On the other hand, we have

$$\mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_{s^-}, \mathsf{M}_{s^-} \otimes \mathsf{M}_s^{\mathrm{Vo}}) = 1 \quad \text{and} \quad \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_{s^-}, \mathsf{M}_{s^+} \otimes \mathsf{M}_s^{\mathrm{Vi}}) = 0,$$

by Proposition 4.6, (7.8) and Lemma 4.23. Since $\mathsf{M}_s \nabla \mathsf{M}'_s \simeq \mathsf{M}_{s^-} \otimes \mathsf{M}_s^{\mathrm{Vo}}$ and $\mathsf{M}'_s \nabla \mathsf{M}_s \simeq \mathsf{M}_{s^+} \otimes \mathsf{M}_s^{\mathrm{Vi}}$, we have

$$\mathsf{M}_s \nabla \mathsf{M}'_s \not\simeq \mathsf{M}'_s \nabla \mathsf{M}_s.$$

Thus, we have the desired result from Theorem 2.10 (d). □

Now we shall show (7.7) (d).

Lemma 7.23. *For $s < b$, M'_s commutes with S_b.*

Proof. Sublemma 3 tells that

$$\mathsf{M}'_s \nabla \mathsf{S}_s \simeq \bigotimes_{d(\imath,\imath_t)=1,\, t^- < s^- < t < s} M[t, b\}.$$

Hence, $\mathsf{M}'_s \nabla \mathsf{S}_s$ commutes with S_b. Since $\mathscr{D}^{-1}\mathsf{S}_s$ commutes with S_b, we conclude that $\mathsf{M}'_s \simeq (\mathscr{D}^{-1}\mathsf{S}_s) \nabla (\mathsf{M}'_s \nabla \mathsf{S}_s)$ commutes with S_b. □

Proposition 7.24. *Let $s \leqslant b$.*

(i) *M'_s is real simple.*
(ii) *M'_s commutes with $M[k, b\}$ for any $k \in \mathbb{Z}$ such that $k \leqslant b$ and $k \neq s$.*

Proof. We argue by induction on $b \geqslant s$.

If $b = s$, then $\mathsf{M}'_s = \mathsf{S}_{s^-}$ is real and commutes with $M[k, b\}$ for $k < b$ by Lemma 4.24.

Now assume that $b > s$. Assuming that (i) and (ii) hold when we replace b with $b - 1$, we shall prove (i) and (ii).

Set

$$Y = \bigotimes_{\substack{t \in [a,b-1];\ d(\imath_s,\imath_t)=1, \\ s^- < t^- < s < t \leqslant b-1}} M[t,b-1\},$$

$$X = Y \nabla \mathsf{S}_{s^-}.$$

Then by the induction hypothesis, X is real and commutes with $M[k,b-1\}$ if $k \leqslant b$ and $k \neq s$. (Here, we understand $M[k,b-1\} = \mathbf{1}$ if $k = b$.)

$$\mathsf{M}_s^{\mathrm{Vo}} \simeq \begin{cases} \mathsf{S}_b \nabla Y & \text{if } d(\imath_s,\imath_b) = 1 \text{ and } s^- < s(\imath_b)^-, \\ Y & \text{otherwise.} \end{cases}$$

Since $(\mathsf{S}_b, \mathsf{S}_{s^-})$ is unmixed we have

$$\mathsf{M}'_s := \mathsf{M}_s^{\mathrm{Vo}} \nabla \mathsf{S}_{s^-} \simeq \begin{cases} \mathsf{S}_b \nabla X & \text{if } d(\imath,\imath_b) = 1 \text{ and } s^- < s(\imath_b)^-, \\ X & \text{otherwise.} \end{cases}$$

We have also

$$M[k,b\} \simeq \begin{cases} \mathsf{S}_b \nabla M[k,b-1\} & \text{if } \imath_k = \imath_b, \\ M[k,b-1\} & \text{otherwise.} \end{cases}$$

We have $\mathfrak{d}(\mathscr{D}\mathsf{S}_b, X) = \mathfrak{d}(\mathscr{D}\mathsf{S}_b, M[k,b-1\}) = 0$. By Lemma 7.23, we have

$$\mathfrak{d}(\mathsf{S}_b, \mathsf{M}'_s) = 0 \quad \text{and} \quad \mathfrak{d}(\mathsf{S}_b, M[k,b\}) = 0.$$

Now we shall apply Lemma 2.24 with $M_1 = X$, $M_2 = M[k,b-1\}$, and $L_1, L_2 = \mathsf{S}_b$ or $\mathbf{1}$. Then we have

$$L_1 \nabla M_1 \simeq \mathsf{M}'_s \quad \text{and} \quad L_2 \nabla M_2 \simeq M[k,b\}.$$

Since M_1 and M_2 are real and commute by the induction hypothesis, $L_1 \nabla M_1 \simeq \mathsf{M}'_s$ and $L_2 \nabla M_2 \simeq M[k,b\}$ are real and commute. □

Thus we complete the proof of Theorem 7.20.

7.6. **The cluster algebra structure on** $K(\mathscr{C}_{\mathfrak{g}}^{<\xi})$**.** We take a quantum affine algebra $U'_q(\mathfrak{g})$ and the associated data $(\triangle, \sigma)$. We freely use terminologies in § 6.

In the sequel, we choose an arbitrary $\widehat{I}_{\mathfrak{g}}$*, and we consider only* Q*-data* $\mathscr{Q}$ *and height functions* ξ *such that* $\widehat{I}_{\mathfrak{g}} = \widehat{I}_{\mathscr{Q}}$ (see Remark 6.8). Note that the choice of $\widehat{I}_{\mathfrak{g}}$ determines $\mathscr{C}_{\mathfrak{g}}^0$. Indeed, $\mathscr{C}_{\mathfrak{g}}^0$ is the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ which contains all $V(\imath,p)$ with $(\imath,p) \in \widehat{I}_{\mathfrak{g}}$ and is stable under taking tensor products, subquotients and extensions.

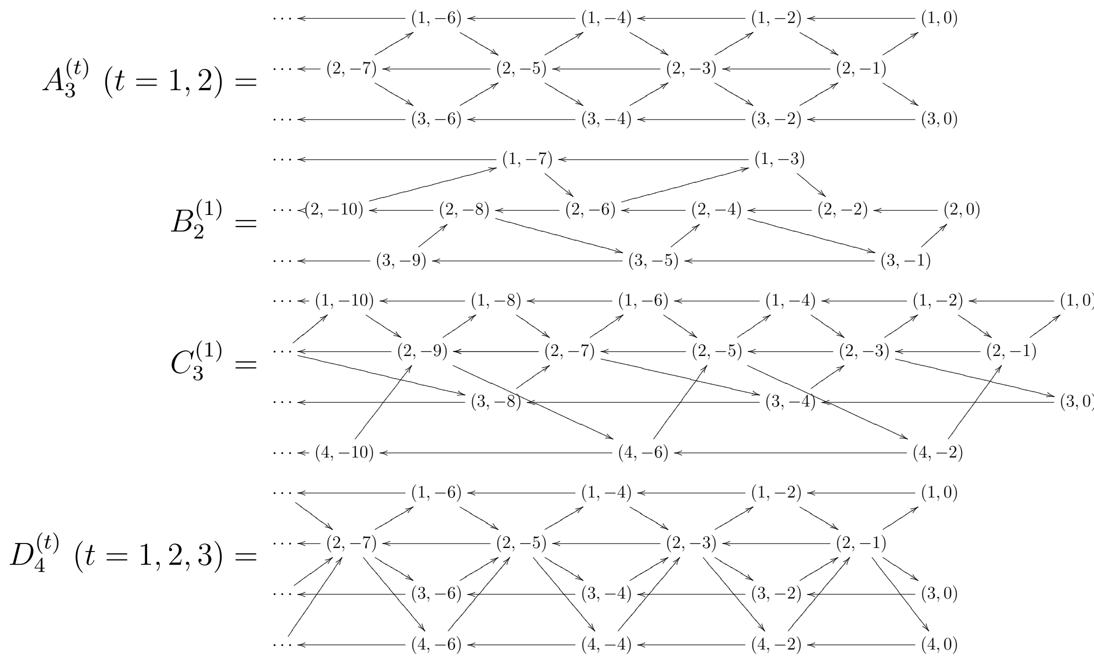


FIGURE 2. Some examples of the quivers $\mathrm{Q}_{\mathrm{HL}}^{<1}$.

Recall that $\widehat{I}_{\mathfrak{g}}^{<\xi} := \{(\imath, p) \in \widehat{I}_{\mathfrak{g}} \mid p < \xi_\imath\}$ and the category $\mathscr{C}_{\mathfrak{g}}^{<\xi}$ is the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing all $V(\imath, p)$ with $(\imath, p) \in \widehat{I}_{\mathfrak{g}}^{<\xi}$ and stable under taking tensor products, subquotients and extensions (see (6.8) and (6.13)).

Definition 7.25. We say that an admissible sequence $\mathfrak{s} = \big((\imath_k, p_k)\big)_{k\in\mathbb{Z}}$ in $\widehat{I}_{\mathfrak{g}}$ is *ξ-adapted* if $\widehat{I}_{\mathfrak{g}}^{<\xi} = \{(\imath_k, p_k) \mid k \in \mathbb{Z}_{\leqslant 0}\}$.

As seen in § 6.3, there exists a ξ-adapted admissible sequence.

Definition 7.26. We define the quiver Q_{HL} whose set of vertices is $I_{\mathrm{fin}} \times \mathbb{Z}$ and the arrows are assigned as follows (cf. (6.4)):

$$(\imath, x) \to (\jmath, y) \qquad \text{(i) if } d(\imath, \jmath) = 1 \text{ and } x = y - 2\mathsf{d}_\jmath + \min(\mathsf{d}_\imath, \mathsf{d}_\jmath), \\ \text{or (ii) } \imath = \jmath \text{ and } x = y + 2\,\mathsf{d}_\imath.$$

We denote by $\mathrm{Q}_{\mathrm{HL}}^{<\xi}$ the quiver on $\widehat{I}_{\mathfrak{g}}^{<\xi}$ induced by Q_{HL}.

The following proposition gives a relation between Q_{HL} and $\mathrm{Q}_{\mathrm{GLS}}$ given in (7.5).

Proposition 7.27. *Let $\mathfrak{s} = \big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$ be an admissible sequence in $\widehat{I}_{\mathfrak{g}}$. Then the quiver $\mathrm{Q}_{\mathrm{GLS}}(\mathfrak{s})$ is isomorphic to the quiver Q_{HL} by the bijection $\mathbb{Z} \xrightarrow{\sim} \widehat{I}_{\mathfrak{g}}$ given by $\mathbb{Z} \ni s \mapsto (\imath_s, p_s) \in \widehat{I}_{\mathfrak{g}}$.*

Proof. Set $(\imath, x) = (\imath_s, p_s)$ and $(\jmath, y) = (\imath_t, p_t)$. By Definition 6.9 (1), it is obvious that

$$\imath = \jmath \quad \text{and} \quad x = y + 2\mathsf{d}_\imath \iff t = s^-.$$

Hence it remains to prove the equivalence under the hypothesis $d(\imath, \jmath) = 1$:

$$p_s = p_t - 2\mathsf{d}_\jmath + \min(\mathsf{d}_\imath, \mathsf{d}_\jmath) \iff s^- < t^- < s < t.$$

Since

$$p_{t^-} + \min(\mathsf{d}_{\imath_{t^-}}, \mathsf{d}_{\imath_s}) = p_t - 2\mathsf{d}_{\imath_t} + \min(\mathsf{d}_{\imath_t}, \mathsf{d}_{\imath_s}),$$

it is enough to show (by replacing t with t^+)

$$p_s = p_t + \min(\mathsf{d}_{\imath_t}, \mathsf{d}_{\imath_s}) \quad \text{if and only if} \quad s^- < t < s < t^+,$$

under the hypothesis that $d(\imath_s, \imath_t) = 1$.
If $d(\imath_s, \imath_t) = 1$ and $s^- < t < s < t^+$, then Definition 6.9 (2) implies $p_s = p_t + \min(\mathsf{d}_{\imath_t}, \mathsf{d}_{\imath_s})$. Conversely assume that $p_s = p_t + \min(\mathsf{d}_{\imath_t}, \mathsf{d}_{\imath_s})$. Then we have

$$p_{s^-} = p_s - 2\,\mathsf{d}_\imath = p_t - 2\,\mathsf{d}_\imath + \min(\,\mathsf{d}_\imath,\,\mathsf{d}_\jmath) < p_t < p_s = p_t + \min(\,\mathsf{d}_\imath,\,\mathsf{d}_\jmath) < p_t + 2\,\mathsf{d}_\jmath = p_{t^+}.$$

Thus we have $s^- < t^- < s < t$ by Lemma 6.10 (i). □

Definition 7.28. For $(\imath, p) \in \widehat{I}_{\mathfrak{g}}$, we define

$$\mathcal{M}^{<\xi}_{(\imath,p)} = \mathrm{hd}\big(V(\imath, p + r\cdot(2\,\mathsf{d}_\imath)) \otimes V(\imath, p + (r-1)(2\,\mathsf{d}_\imath)) \otimes \cdots \otimes V(\imath, p)\big),$$

where r is the largest integer such that $p + r\cdot(2\,\mathsf{d}_\imath) < \xi_\imath$. It is a KR-module.

Remark 7.29. Let us take a ξ-adapted admissible sequence $\mathfrak{s}$ in $\widehat{I}_{\mathfrak{g}}$ and $a \in \mathbb{Z}$ such that $(\imath, p) = (\imath_a, p_a)$. Then we have

$$\mathcal{M}^{<\xi}_{(\imath,p)} \simeq M^{\mathfrak{s}}[a, 0\},\tag{7.11}$$

For $t \in \mathbb{Z}$, we set $\widehat{I}^{<t}_{\mathfrak{g}} := \{(\imath, p) \in \widehat{I}_{\mathfrak{g}} \mid p < t\}$, and let $\mathscr{C}^{<t}_{\mathfrak{g}}$ be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ which contains $V(\imath, p)$ for $(\imath, p) \in \widehat{I}^{<t}_{\mathfrak{g}}$ and stable under taking tensor products, subquotients and extensions. Then there exists a unique height function ξ^t such that

$\widehat{I}_{\mathfrak{g}}^{<t} = \widehat{I}_{\mathfrak{g}}^{<\xi^t}$, namely, ξ^t satisfying $(\xi^t)_\imath \in [t, t+2\mathsf{d}_\imath - 1]$ for all $\imath \in I_{\mathrm{fin}}$. We write $\mathcal{M}^{<t}_{(\imath,p)}$ and $\mathrm{Q}^{<t}_{\mathrm{HL}}$ instead of $\mathcal{M}^{<\xi^t}_{(\imath,p)}$ and $\mathrm{Q}^{<\xi^t}_{\mathrm{HL}}$. Hence we have

$$\mathcal{M}^{<t}_{(\imath,k)} = \mathrm{hd}\Big(V\big(\imath, p + r\cdot(2\mathsf{d}_\imath)\big) \otimes V\big(\imath, p+(r-1)(2\mathsf{d}_\imath)\big) \otimes \cdots \otimes V\big(\imath, p\big)\Big), \tag{7.12}$$

where r is the largest integer such that $p + r\cdot(2\mathsf{d}_\imath) < t$.

In [22], Hernandez-Leclerc proved that the seed arising from $\{\mathcal{M}^{<t}_{(\imath,p)}\}_{(\imath,p)\in\widehat{I}_{\mathfrak{g}}^{<t}}$ and $\mathrm{Q}^{<t}_{\mathrm{HL}}$ gives a cluster algebra structure on the Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}}^{<t})$ as follows: set $z_{\imath,p} = [\mathcal{M}^{<t}_{(\imath,p)}] \in K(\mathscr{C}_{\mathfrak{g}}^{<t})$ for $(\imath,p) \in \widehat{I}_{\mathfrak{g}}^{<t}$ and let $\widetilde{B}^{<t}_{\mathrm{HL}}$ be the exchange matrix associated with $\mathrm{Q}^{<t}_{\mathrm{HL}}$.

Theorem 7.30 ([22, Theorem 5.1])**.** *Take an arbitrary $\widehat{I}_{\mathfrak{g}}$ and an integer t. Let $\mathcal{S}^{<t} := \big(\{z_{\imath,p}\}_{(\imath,p)\in\widehat{I}_{\mathfrak{g}}^{<t}}, \widetilde{B}^{<t}_{\mathrm{HL}}\big)$ be a seed in $K(\mathscr{C}_{\mathfrak{g}}^{<t})$ with the empty set of frozen variables. Then, we have*

$$\mathcal{A}(\mathcal{S}^{<t}) \simeq K(\mathscr{C}_{\mathfrak{g}}^{<t}).$$

8. Monoidal categorification of cluster algebras

In this section, we shall state and prove our main theorem. We fix $\widehat{I}_{\mathfrak{g}}$ associated with $\mathscr{C}_{\mathfrak{g}}^0$. Recall that $\mathscr{C}_{\mathfrak{g}}^0$ is the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ which contains $V(\imath,p)$ ($(\imath,p)\in\widehat{I}_{\mathfrak{g}}$) and is stable under taking tensor products, subquotients and extensions. We only treat height functions ξ such that $(\imath,\xi_\imath) \in \widehat{I}_{\mathfrak{g}}$ for every $\imath \in I_{\mathrm{fin}}$, and admissible sequences $\mathfrak{s}$ in $\widehat{I}_{\mathfrak{g}}$.

8.1. Statement of the main theorem.

The purpose of this section is to prove the following main theorem of this paper.

Theorem 8.1. *Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair with $\mathcal{D} = \mathcal{D}_{\mathscr{Q}}$ for some* Q*-datum $\mathscr{Q}$ and an arbitrary reduced expression $\underline{w}_0$ of w_0, and let $\mathfrak{C}$ be an admissible chain of i-boxes with a range $[a,b]$. Then, there is an exchange matrix $\widetilde{B}(\mathfrak{C}) = (b_{s,t})_{(s,t)\in\mathsf{K}(\mathfrak{C})\times\mathsf{K}^{\mathrm{ex}}(\mathfrak{C})}$ satisfying the following properties.*

(a) *$\mathscr{S}(\mathfrak{C}) := \big(\mathsf{M}(\mathfrak{C}), \widetilde{B}(\mathfrak{C}); \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C})\big)$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ (see (7.4) for the notations in $\mathscr{S}(\mathfrak{C})$).*

(b) *$\mathcal{A}\big([\mathscr{S}(\mathfrak{C})]\big) \simeq K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$,*

Namely, the category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ provides a Λ-monoidal categorification of the cluster algebra $K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$ with the initial monoidal seed $\mathscr{S}(\mathfrak{C})$.

Note that $K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$ is the polynomial algebra generated by $\{[\mathsf{S}_s] \mid s \in [a,b]\}$ as an algebra (see [40, § 6.3]).

This section is devoted to the proof of this theorem. Since the proof is intricate, we afford the plan of the proof.

For $t \in \mathbb{Z}$ and a height function ξ, we define the monoidal seed $\mathscr{S}^{<t}$ in $\mathscr{C}_{\mathfrak{g}}^{<t}$ and the monoidal seed $\mathscr{S}^{<\xi}$ in $\mathscr{C}_{\mathfrak{g}}^{<\xi}$ as follows:

$$\mathscr{S}^{<t} := \big(\{\mathcal{M}^{<t}_{(\imath,p)}\}_{(\imath,p)\in \widehat{I}_{\mathfrak{g}}^{<t}}, \widetilde{B}^{<t}_{\mathrm{HL}}; \widehat{I}_{\mathfrak{g}}^{<t}, \widehat{I}_{\mathfrak{g}}^{<t}\big),$$
$$\mathscr{S}^{<\xi} := \big(\{\mathcal{M}^{<\xi}_{(\imath,p)}\}_{(\imath,p)\in \widehat{I}_{\mathfrak{g}}^{<\xi}}, \widetilde{B}^{<\xi}_{\mathrm{HL}}; \widehat{I}_{\mathfrak{g}}^{<\xi}, \widehat{I}_{\mathfrak{g}}^{<\xi}\big),$$

where $\widetilde{B}^{<t}_{\mathrm{HL}}$ (resp. $\widetilde{B}^{<\xi}_{\mathrm{HL}}$) is the exchange matrix associated with the quiver on $\widehat{I}_{\mathfrak{g}}^{<t}$ (resp. $\widehat{I}_{\mathfrak{g}}^{<\xi}$) induced by Q_{HL} (see (7.12) for $\mathcal{M}^{<t}_{(\imath,p)}$ and Definition 7.28 for $\mathcal{M}^{<\xi}_{(\imath,p)}$).

Remark that, once $\widehat{I}_{\mathfrak{g}}$ is fixed, $\mathscr{S}^{<t}$ depends only on $t \in \mathbb{Z}$ and $\mathscr{S}^{<\xi}$ depends only on ξ. The monoidal seed $\mathscr{S}^{<t}$ is a special case of monoidal seeds $\mathscr{S}^{<\xi}$. Indeed, if ξ is a unique height function satisfying the following condition:

$$\xi_\imath \in [t, t+2\,\mathsf{d}_\imath - 1] \text{ for every } \imath \in I_{\mathrm{fin}}, \tag{8.1}$$

then we have $\mathscr{S}^{<\xi} = \mathscr{S}^{<t}$. We denote by ξ^t the height function satisfying (8.1).

Recall that $\mathfrak{s} = \big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$ is ξ-adapted if $\widehat{I}_{\mathfrak{g}}^{<\xi} = \{(\imath_s, p_s) \mid s \leqslant 0\}$. We have

$$\mathscr{S}^{<\xi} = \mathscr{S}_{-}^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0} = \mathscr{S}(\mathfrak{C}^{[-\infty,0],\mathfrak{s}}),$$

where $\mathscr{Q}$ is the Q-datum associated with ξ, $\widehat{\underline{w}}_0$ is associated with a $\mathscr{Q}$-adapted reduced expression $\underline{w}_0$ of w_0, $\mathfrak{s}$ is a ξ-adapted admissible sequence in $\widehat{I}_{\mathfrak{g}}$ and $\mathfrak{C}_{-}^{[-\infty,0],\mathfrak{s}}$ is the admissible chain of i-boxes $([1-k,0\})_{1\leqslant k}$ (see (7.6) and Theorem 8.1 (a) for the notations).

Consider the following statements, where ξ is a height function:

$$\begin{cases} \text{(I)} & \mathscr{S}^{<t} \text{ is a completely } \Lambda\text{-admissible monoidal seed in } \mathscr{C}_{\mathfrak{g}}^{<t} \text{ for any } t \in \mathbb{Z}, \\ \text{(II)}_\xi & \mathscr{S}^{<\xi} \text{ is a completely } \Lambda\text{-admissible monoidal seed in } \mathscr{C}_{\mathfrak{g}}^{<\xi}, \\ \text{(III)}_\xi & \text{for the Q-datum } \mathscr{Q} \text{ associated with } \xi \text{ and any reduced expression } \underline{w_0} \text{ of } w_0, \text{ the seed } \mathscr{S}_{-}^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0} \text{ is a completely } \Lambda\text{-admissible monoidal seed in } \mathscr{C}_{\mathfrak{g}}^{<\xi}, \\ \text{(IV)}_\xi & \text{for the Q-datum } \mathscr{Q} \text{ associated with } \xi, \text{ any reduced expression } \underline{w_0} \text{ of } w_0, \text{ and any admissible chain } \mathfrak{C} \text{ of } i\text{-boxes, } \mathscr{S}(\mathfrak{C}) = \big(\mathsf{M}(\mathfrak{C}), \widetilde{B}; \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C})\big) \text{ is a completely } \Lambda\text{-admissible monoidal seed in } \mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0} \text{ for a suitable choice of exchange matrix } \widetilde{B}. \end{cases}$$

The first step of the proof of main Theorem 8.1 is to show (I) by using the result of Hernandez-Leclerc (Theorem 7.30). Hence, $(\mathrm{II})_\xi$ holds as soon as ξ satisfies condition (8.1).

Then we will prove

$$(\mathrm{II})_\xi \Longrightarrow (\mathrm{III})_\xi.$$

in Proposition 8.4, by showing the monoidally mutation equivalence between $\mathscr{S}^{<\xi}$ and $\mathscr{S}_{-}^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}$. Then we prove

$$(\mathrm{III})_\xi \Longrightarrow (\mathrm{IV})_\xi.$$

Now, we can see easily that for any ξ there exists a ξ-adapted reduced expression $\underline{w} = s_{\imath_1} \cdots s_{\imath_r}$ such that $\xi' := s_{\imath_r} \cdots s_{\imath_1} \xi$ satisfies condition (8.1). On the other hand, $(\mathrm{II})_{\xi'}$ holds. Hence, in order to see $(\mathrm{II})_\xi$ for an arbitrary ξ, it is enough to show

$$(\mathrm{II})_{s_\imath \xi} \Longrightarrow (\mathrm{II})_\xi \quad \text{if } \imath \text{ is a sink of } \xi.$$

It is performed in Proposition 8.12.

Theorem 8.1 is equivalent to saying that $(\mathrm{IV})_\xi$ holds for any ξ.

8.2. **Monoidal categorification by $\mathscr{C}_{\mathfrak{g}}^{<t}$.** In order to achieve the main goal of this section, we shall first prove the conjecture suggested by Hernandez-Leclerc for untwisted affine $\mathfrak{g}$.

Recall that $\widehat{I}_{\mathfrak{g}}^{<t} := \{(\imath, p) \in \widehat{I}_{\mathfrak{g}} \mid p < t\}$, and $\mathscr{C}_{\mathfrak{g}}^{<t}$ is the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ containing $V(\imath, p)$ for $(\imath, p) \in \widehat{I}_{\mathfrak{g}}^{<t}$ and stable under taking tensor products, subquotients and extensions.

Theorem 8.2 (cf. [22, Conjecture 5.2])**.** *The category $\mathscr{C}_{\mathfrak{g}}^{<t}$ provides a Λ-monoidal categorification of the cluster algebra $K(\mathscr{C}_{\mathfrak{g}}^{<t})$ associated to the initial completely Λ-admissible monoidal seed $\mathscr{S}^{<t}$.*

Proof. Take an admissible sequence $\mathfrak{s} = \big((\imath_s, p_s)\big)_{s\in\mathbb{Z}}$ in $\widehat{I}_{\mathfrak{g}}$ such that $\widehat{I}_{\mathfrak{g}}^{<t} := \{(\imath_k, p_k) \mid k \leqslant 0\}$. Then we have $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathfrak{s}} = \mathscr{C}_{\mathfrak{g}}^{<t}$ and

$$\mathscr{S}^{<t} = \mathscr{S}_{-}^{[-\infty,0],\,\mathfrak{s}} \tag{8.2}$$

by Proposition 7.27. Hence $\mathscr{S}^{<t}$ is a Λ-admissible monoidal seed by Theorem 7.20. Since $\mathcal{A}([\mathscr{S}^{<t}]) \simeq K(\mathscr{C}_{\mathfrak{g}}^{<t})$ by Theorem 7.30 due to Hernandez-Leclerc, the assertion follows from Theorem 7.8. □

Thus we have obtained (I).

8.3. **Mutation equivalence.**

Definition 8.3. Let $\mathscr{S} = (\{M_i\}_{i\in\mathsf{K}}, \widetilde{B}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}})$ and $\mathscr{S}' = (\{M_i'\}_{i\in\mathsf{K}'}, \widetilde{B}'; \mathsf{K}', (\mathsf{K}')^{\mathrm{ex}})$ be admissible monoidal seeds in $\mathcal{C}$.

(1) We say that $\mathscr{S}'$ *is monoidally mutated from* $\mathscr{S}$ if the following condition is satisfied: For any finite subset J of K', there exist
  - (i) a finite sequence $(k_1, k_2, \ldots, k_r)$ in K^{ex} such that $\mu_{k_s} \circ \cdots \circ \mu_{k_1}(\mathscr{S})$ is an admissible monoidal seed for each $1 \leqslant s \leqslant r$,
  - (ii) an injective map $\upsigma\colon \mathsf{J} \to \mathsf{K}$, depending on the choice of J, such that
    - (a) $\upsigma(\mathsf{J}^{\mathrm{ex}}) \subset \mathsf{K}^{\mathrm{ex}}$, where $\mathsf{J}^{\mathrm{ex}} := \mathsf{J} \cap (\mathsf{K}')^{\mathrm{ex}}$,
    - (b) $M_j' = \upmu(M)_{\upsigma(j)}$ for all $j \in \mathsf{J}$,
    - (c) $(\widetilde{B}')_{(i,j)} = \upmu(\widetilde{B})_{\upsigma(i),\upsigma(j)}$ for any $(i,j) \in \mathsf{J} \times \mathsf{J}^{\mathrm{ex}}$,

    where $\upmu := \mu_{k_r} \circ \cdots \mu_{k_1}$.

(2) We say that the admissible monoidal seeds $\mathscr{S}$ and $\mathscr{S}'$ are *monoidally mutation equivalent* if $\mathscr{S}'$ is monoidally mutated from $\mathscr{S}$ and $\mathscr{S}$ is also monoidally mutated from $\mathscr{S}'$.

For a PBW-pair $(\mathcal{D}, \widehat{\underline{w}}_0)$, let $\mathfrak{C}_{-}^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0}$ be the admissible chain of i-boxes $([1-k, 0\})_{1\leqslant k}$ and recall that

$$\mathscr{S}_{-}^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0} := \big(\{M(\mathfrak{C}_{-}^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0}), \widetilde{B}_{\mathrm{GLS}}^{\mathcal{D},\widehat{\underline{w}}_0}(\widehat{\underline{w}}_0); [-\infty,0],\, [-\infty,0]\big). \tag{8.3}$$

When $\mathscr{Q}$ is the Q-datum with the height function ξ^t (see (8.1)) and $\underline{w}_0$ is a $\mathscr{Q}$-adapted reduced expression of w_0, we have

$$\mathscr{S}^{<t} = \mathscr{S}_{-}^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}$$

and it is a completely Λ-admissible monoidal seed by Theorem 8.2.

Proposition 8.4. *Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair and $\underline{w}_0'$ be another reduced expression of w_0. Then the Λ-admissible seeds*

$$\mathscr{S}_-^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0} \text{ and } \mathscr{S}_-^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0'} \text{ are monoidally mutation equivalent.}$$

This subsection is devoted to prove the above proposition.

Remark 8.5. Recall the notion of the affine cuspidal modules $\mathsf{S}_k^{\mathcal{D},\widehat{\underline{w}}_0}$ in Section 4.3.

(a) When $\underline{w}_0' = s_{\jmath_1} \ldots s_{\jmath_\ell}$ is obtained from $\underline{w}_0 = s_{\imath_1} \ldots s_{\imath_\ell}$ by a single *braid move*; i.e., for some k such that $1 < k < \ell$, we have

$$d(\imath_{k-1}, \imath_k) = 1,\ \imath_{k\pm1} = \jmath_k,\ \jmath_{k\pm1} = \imath_k \text{ , and } \imath_s = \jmath_s \text{ for any } s \notin \{k-1, k, k+1\},$$

the affine cuspidal modules are related by

$$\mathsf{S}_s^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \begin{cases} \mathsf{S}_{s+2}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k-1, \\ \mathsf{S}_{s+1}^{\mathcal{D},\widehat{\underline{w}}_0} \nabla \mathsf{S}_{s-1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k, \\ \mathsf{S}_{s-2}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k+1, \\ \mathsf{S}_{s}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{otherwise.} \end{cases} \tag{8.4}$$

Here (8.4) follows from [40, Proposition 5.9]. Indeed, by this proposition, we can reduce to the case where $k = 2$, and it is easy to check this case. Note that $\mathsf{S}_s^{\mathcal{D},\widehat{\underline{w}}_0} \simeq \mathsf{S}_{s-1}^{\mathcal{D},\widehat{\underline{w}}_0} \nabla \mathsf{S}_{s+1}^{\mathcal{D},\widehat{\underline{w}}_0}$ if $s \equiv_\ell k$.

(b) When $\underline{w}_0' = s_{\jmath_1} \ldots s_{\jmath_\ell}$ is obtained from $\underline{w}_0 = s_{\imath_1} \ldots s_{\imath_\ell}$ by a single *commutation move*; i.e., there exists a $1 < k \leqslant \ell$ such that

$$d(\imath_{k-1}, \imath_k) > 1,\ \imath_k = \jmath_{k-1},\ \imath_{k-1} = \jmath_k \text{ and } \imath_s = \jmath_s \text{ for } s \notin \{k-1, k\},$$

we have

$$\mathsf{S}_s^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \begin{cases} \mathsf{S}_{s+1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k-1, \\ \mathsf{S}_{s-1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k, \\ \mathsf{S}_{s}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{otherwise.} \end{cases}$$

(c) All reduced expressions of w_0 are connected via braid moves and commutation moves.

Set $\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0} := M^{\mathcal{D},\widehat{\underline{w}}_0}[s, 0\}$ for a reduced expression $\underline{w}_0$ of w_0 and $s \in \mathbb{Z}_{\leqslant 0}$.

Lemma 8.6. *Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair and $\underline{w}'_0 = s_{\jmath_1} \cdots s_{\jmath_\ell}$ be a reduced expression of w_0 obtained from $\underline{w}_0 = s_{\imath_1} \cdots s_{\imath_\ell}$ by a single commutation move as in* Remark 8.5 (b). *Then we have*

$$\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}'_0} = \begin{cases} \mathsf{M}_{s-1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k, \\ \mathsf{M}_{s+1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k-1, \\ \mathsf{M}_{s}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{otherwise.} \end{cases}$$

Proof. This is a direct consequence of Remark 8.5 (b) □

Lemma 8.7. *Let us keep the notations in the previous lemma. Then the quivers $\mathrm{Q}_{\mathrm{GLS}}^{[-\infty,0]}(\widehat{\underline{w}}_0)$ and $\mathrm{Q}_{\mathrm{GLS}}^{[-\infty,0]}(\widehat{\underline{w}}'_0)$ are isomorphic to each other under the index change $\Phi\colon [-\infty, 0] \to [-\infty, 0]$ given by*

$$\Phi(s) = \begin{cases} s-1 & \text{if } s \equiv_\ell k, \\ s+1 & \text{if } s \equiv_\ell k-1, \\ s & \text{otherwise.} \end{cases}$$

The proof is straightforward.

Lemma 8.8. *Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair and let $\underline{w}'_0 = s_{\jmath_1} \cdots s_{\jmath_\ell}$ be a reduced expression of w_0 obtained from $\underline{w}_0 = s_{\imath_1} \cdots s_{\imath_\ell}$ by a single braid move as in* Remark 8.5 (a). *Then we have*

$$\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}'_0} \simeq \begin{cases} \mathsf{M}_{s+1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k-1, \\ \mathsf{M}_{s-1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } s \equiv_\ell k, \\ \big(\mathsf{M}_{s}^{\mathcal{D},\widehat{\underline{w}}_0}\big)' & \text{if } s \equiv_\ell k+1, \\ \mathsf{M}_{s}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{otherwise,} \end{cases} \tag{8.5}$$

where $\big(\mathsf{M}_{s}^{\mathcal{D},\widehat{\underline{w}}_0}\big)'$ denotes the mutation $\mu_s\big(\mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}\big)_s$ of $\mathsf{M}_{s}^{\mathcal{D},\widehat{\underline{w}}_0}$ described in (7.9).

Note that, when $s \equiv_\ell k+1$, there exists at most one *vertically* outgoing arrow starting from s in $\mathrm{Q}_{\mathrm{GLS}}(\widehat{\underline{w}}_0)$, which is $s \to (s-1)_\imath^+$ if $(s-1)_\imath^+ \leqslant 0$. Since $(s)_\imath^- = s-2$ (see (8.6) below), we have

$$\big(\mathsf{M}_{s}^{\mathcal{D},\widehat{\underline{w}}_0}\big)' \simeq \mathsf{M}_{(s-1)_\imath^+}^{\mathcal{D},\widehat{\underline{w}}_0} \nabla \mathsf{S}_{s-2}^{\mathcal{D},\widehat{\underline{w}}_0} \quad (\text{when } s \equiv_\ell k+1).$$

Proof. We shall argue by the descending induction on s. If $k+1-\ell < s \leqslant 0$, the assertion is obvious by (8.4).

M In the course of the proof, we denote

$$(a)_\imath^+ = \max\{k > a \mid \imath_k = \imath_a\}, \qquad (a)_\jmath^+ = \max\{k > a \mid \jmath_k = \jmath_a\}$$
$$(a)_\imath^- = \max\{k < a \mid \imath_k = \imath_a\}, \tag{8.6}$$

Note that we have

$$\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0} \simeq \mathsf{M}_{(s)_\imath^+}^{\mathcal{D},\widehat{\underline{w}}_0} \nabla \mathsf{S}_s^{\mathcal{D},\widehat{\underline{w}}_0} \quad \text{and} \quad \mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \mathsf{M}_{(s)_\jmath^+}^{\mathcal{D},\widehat{\underline{w}}_0'} \nabla \mathsf{S}_s^{\mathcal{D},\widehat{\underline{w}}_0'}.$$

Here we understand

$$\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0} \simeq \mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \mathbf{1} \quad \text{if } s > 0.$$

Note that if we set $t = (s)_\jmath^+ > s$, then we have

$$\mathsf{M}_t^{\mathcal{D},\widehat{\underline{w}}_0'} = \begin{cases} \mathsf{M}_{t+1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } t \equiv_\ell k-1, \\ \mathsf{M}_{t-1}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } t \equiv_\ell k, \\ \mathsf{M}_{t}^{\mathcal{D},\widehat{\underline{w}}_0} & \text{if } t \not\equiv_\ell k, k\pm 1. \end{cases}$$

(i) Let us consider the case $s \equiv_\ell k$. Then $\jmath_s = \imath_{s+1}$. Set $t = (s)_\jmath^+$. First, let us show

$$\mathsf{M}_t^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \mathsf{M}_{(s+1)_\imath^+}^{\mathcal{D},\widehat{\underline{w}}_0}. \tag{8.7}$$

(a) Case $t \equiv_\ell k-1$. Then $\jmath_s = \imath_{s+1} = \jmath_t = \imath_{t+1}$, and hence $(s+1)_\imath^+ = t+1$. Hence we have (8.7).
(b) Case $t \equiv_\ell k$. Then $\jmath_s = \imath_{s+1} = \jmath_t = \imath_{t-1}$, and hence $(s+1)_\imath^+ = t-1$. Therefore we have (8.7).
(c) The case $t \equiv_\ell k+1$ does not occur.
(d) Case $t \not\equiv_\ell k, k\pm 1$. Then $t = (s+1)_\imath^+$ and we have (8.7).

Thus we complete the proof of (8.7).

Then, we have

$$\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \mathsf{M}_t^{\mathcal{D},\widehat{\underline{w}}_0'} \nabla (\mathsf{S}_s^{\mathcal{D},\widehat{\underline{w}}_0'}) \simeq \mathsf{M}_{(s+1)_\imath^+}^{\mathcal{D},\widehat{\underline{w}}_0} \nabla (\mathsf{S}_{s+1}^{\mathcal{D},\widehat{\underline{w}}_0} \nabla \mathsf{S}_{s-1}^{\mathcal{D},\widehat{\underline{w}}_0}) \simeq \mathsf{M}_{s-1}^{\mathcal{D},\widehat{\underline{w}}_0}.$$

(ii) Consider the case $s \equiv_\ell k-1$. In this case, we have $(s)_\jmath^+ = s+2$. Set $t = (s+2)_\jmath^+$. As in (i), let us show

$$\mathsf{M}_t^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \mathsf{M}_{(s+1)_\imath^+}^{\mathcal{D},\widehat{\underline{w}}_0}. \tag{8.8}$$

(a) Case $t \equiv_\ell k-1$. Then $\jmath_s = \imath_{s+1} = \jmath_t = \imath_{t+1}$, and hence $(s+1)_\imath^+ = t+1$. Hence we have (8.8).

(b) Case $t \equiv_\ell k$. Then $\jmath_s = \imath_{s+1} = \jmath_t = \imath_{t-1}$, and hence $(s+1)^+_\imath = t-1$. Therefore we have (8.8).

(c) The case $t \equiv_\ell k+1$ does not occur.

(d) Case $t \not\equiv_\ell k, k \pm 1$. Then $t = (s+1)^+_\imath$ and we have (8.8).

Thus we complete the proof of (8.8).

Hence, we have

MH (8.7) ↦ (8.8) 3 times

$$\begin{aligned}\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0'} &\simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0'}_{(s+2)^+_\jmath} \nabla \mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0'}_{s+2} \nabla \mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0'}_{s} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{(s+1)^+_\imath} \nabla (\mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0}_{s} \nabla \mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0}_{s+2}) \\ &\simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{(s+1)^+_\imath} \nabla (\mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0}_{s+1}) \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{s+1},\end{aligned}$$

which implies the assertion for this case.

(iii) Let us consider $s \equiv_\ell k+1$. Similarly to the proof of (8.7), we can prove

$$\mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0'}_{(s)^+_\jmath} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{(s-1)^+_\imath}$$

Hence, we have

$$\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0'}_{(s)^+_\jmath} \nabla \mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0'}_{s} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{(s-1)^+_\imath} \nabla \mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0}_{s-2} \simeq (\mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{s})',$$

which implies the assertion in this case.

(iv) Finally, assume that $s \not\equiv_\ell k, k \pm 1$. Similarly to the proof of (8.7), we can prove

$$\mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0'}_{(s)^+_\jmath} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{(s)^+_\imath}$$

Hence, we have

$$\mathsf{M}_s^{\mathcal{D},\widehat{\underline{w}}_0'} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0'}_{(s)^+_\jmath} \nabla \mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0'}_{s} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{(s)^+_\imath} \nabla \mathsf{S}^{\mathcal{D},\widehat{\underline{w}}_0}_{s} \simeq \mathsf{M}^{\mathcal{D},\widehat{\underline{w}}_0}_{s}. \qquad \square$$

Lemma 8.9. *Let us keep the notations in the previous lemma. Then there exists a quiver isomorphism*

$$\Phi : \mu_{k+1-\ell\mathbb{Z}_{>0}}\big(\mathrm{Q}^{[-\infty,0]}_{\mathrm{GLS}}(\widehat{\underline{w}}_0)\big) \to \mathrm{Q}^{[-\infty,0]}_{\mathrm{GLS}}(\widehat{\underline{w}}_0'),$$

where

$$\mu_{k+1-\ell\mathbb{Z}_{>0}}(\mathrm{Q}^{[-\infty,0]}_{\mathrm{GLS}}(\widehat{\underline{w}}_0)) := \cdots \circ \mu_{k+1-2\ell} \circ \mu_{k+1-\ell}(\mathrm{Q}^{[-\infty,0]}_{\mathrm{GLS}}(\widehat{\underline{w}}_0)).$$

Proof. Note that $\mu_{k+1-t\ell} \circ \mu_{k+1-s\ell}\big(\mathrm{Q}_{\mathrm{GLS}}^{[-\infty,0]}(\widehat{\underline{w}}_0)\big) = \mu_{k+1-s\ell} \circ \mu_{k+1-t\ell}\big(\mathrm{Q}_{\mathrm{GLS}}^{[-\infty,0]}(\widehat{\underline{w}}_0)\big)$ for any $t, s \in \mathbb{Z}_{>0}$. For $s \in \mathbb{Z}_{\leqslant 0}$, define the index change map $\Phi : [-\infty, 0] \to [-\infty, 0]$ as follows:

$$\Phi(s) = \begin{cases} s & \text{if } s \not\equiv_\ell k, k-1 \\ s-1 & \text{if } s \equiv_\ell k, \\ s+1 & \text{if } s \equiv_\ell k-1. \end{cases}$$

For each $t \in \mathbb{Z}_{\geqslant 0}$ and $\imath \in I_{\mathrm{fin}}$, set $u_\imath^t := \min\{x \in [-(t+1)\ell+1, -t\ell] \mid \imath_x = \imath\}$. Then, there is no arrow between

$$[-(t_1+1)\ell+1, -t_1\ell] \setminus \{u_\imath^{t_1} \mid \imath \in I_{\mathrm{fin}}\} \text{ and } [-(t_2+1)\ell+1, -t_2\ell]$$

in $\mathrm{Q}_{\mathrm{GLS}}^{[-\infty,0]}(\widehat{\underline{w}}_0)$ when $t_1 > t_2 \in \mathbb{Z}_{\geqslant 0}$. Thus it is enough to show that the restriction of Φ to $[a, 0]$ is a quiver isomorphism if a is sufficiently small. Then, our assertion follows from the lemma above together with Proposition 7.14 and Lemma 7.16. $\square$

Proof of Proposition 8.4. From the above four lemmas, we can conclude the followings:

(i) If $\underline{w}_0'$ can be obtained from $\underline{w}_0$ by a single commutation move $\mathscr{S}_-^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0}$ and $\mathscr{S}_-^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0'}$ are monoidally mutation equivalent.

(ii) If $\underline{w}_0'$ can be obtained from $\underline{w}_0$ by a single braid move, the sequence $\mu_{k+1-\ell\mathbb{Z}_{>0}}$ gives a monoidally mutation equivalence between $\mathscr{S}_-^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0}$ and $\mathscr{S}_-^{[-\infty,0],\mathcal{D},\widehat{\underline{w}}_0'}$ (see Definition 8.3).

Thus our assertion follows from Remark 8.5 (c). $\square$

As a corollary of Proposition 8.4, we obtain the following assertion.

Corollary 8.10. Let $(\mathcal{D}, \widehat{\underline{w}}_0)$ be a PBW-pair and $\underline{w}_0'$ be another reduced expression of w_0. If the Λ-admissible seed $\mathscr{S}_-^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{<\xi}$, then so is $\mathscr{S}_-^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0'}$.

Thus we obtain (III)$_\xi$ for ξ satisfying condition (8.1).

8.4. **Proof of the main theorem.** By the result of the preceding subsection, we have already proved that the monoidal seed $\mathscr{S}_-^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}$ is a completely Λ-admissible monoidal seed when $\mathscr{Q}$ is a Q-datum with the height function ξ^t and $\underline{w}_0$ is an arbitrary reduced expression of w_0. The next step is to generalize this result to an arbitrary Q-datum (with an arbitrary height function ξ) and an arbitrary interval $[a, b]$.

Proposition 8.11. *Let $t \in \mathbb{Z}$ and assume that $\mathscr{S}_{-}^{[-\infty,t],\,\mathcal{D},\,\widehat{\underline{w}}_0}$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{0}$. Then, for any admissible chain $\mathfrak{C} = (\mathfrak{c}_k)_{k\in[1,l]}$ of i-boxes with a range $[a,b]$, $\mathscr{S}(\mathfrak{C}) = \big(\mathsf{M}(\mathfrak{C}), \widetilde{B}(\mathfrak{C}); \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C})\big)$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\,\mathcal{D},\widehat{\underline{w}}_0}$ with a suitable exchange matrix $\widetilde{B}(\mathfrak{C})$.*

Proof. We shall prove first the case $-\infty < a \leqslant b = t$. Set $\mathsf{K}_0 = [a,t]$ and $\mathsf{K}_0^{\mathrm{ex}} = \{s \in \mathsf{K}_0 \mid a \leqslant s^-\}$. Then there is no arrow in $\mathsf{Q}_{\mathrm{GLS}}^{[-\infty,t],\mathcal{D},\widehat{\underline{w}}_0}$ between $\mathsf{K}_0^{\mathrm{ex}}$ and $[-\infty,t] \setminus \mathsf{K}_0$. Hence $\mathscr{S}_{-}^{[a,t],\,\mathcal{D},\,\widehat{\underline{w}}_0} = \Big(\mathscr{S}_{-}^{[-\infty,t],\,\mathcal{D},\widehat{\underline{w}}_0}\Big)\Big|_{(\mathsf{K}_0,\,\mathsf{K}_0^{\mathrm{ex}})}$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{0}$ by Lemma 7.15. Hence $\mathscr{S}_{-}^{[a,t],\,\mathcal{D},\widehat{\underline{w}}_0}$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,t],\,\mathcal{D},\widehat{\underline{w}}_0}$ by Proposition 7.19. Then, Corollary 7.18 implies that, for any admissible chain $\mathfrak{C}$ of i-boxes with the range $[a,t]$, $\mathscr{S}(\mathfrak{C}) = \big(\mathsf{M}(\mathfrak{C}), \widetilde{B}(\mathfrak{C}); \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C})\big)$ is also a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,t],\,\mathcal{D},\widehat{\underline{w}}_0}$ for some exchange matrix $\widetilde{B}(\mathfrak{C})$.

Now we treat the case $-\infty < a \leqslant b \leqslant t$. First we treat the special case $\mathfrak{C}_0 = \mathfrak{C}_{-}^{[a,b]} := \big([b-k+1,b\}\big)_{1\leqslant k\leqslant b-a+1}$. Then the monoidal seed

$$\mathscr{S}(\mathfrak{C}_0) = \mathscr{S}_{-}^{[a,b],\,\mathcal{D},\widehat{\underline{w}}_0} := \big(\{M^{\mathcal{D},\widehat{\underline{w}}_0}[s,b\}\}_{s\in[a,b]}, \widetilde{B}_{\mathrm{GLS}}^{[a,b],\,\mathcal{D},\widehat{\underline{w}}_0}; \mathsf{K}(\mathfrak{C}_0), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C}_0)\big)$$

is Λ-admissible by Theorem 7.20. Then we can enlarge $\mathfrak{C}_0$ to an admissible chain $\mathfrak{C}' = (\mathfrak{c}_k)_{1\leqslant k\leqslant l'}$ of i-boxes with the range $[a,t]$. For example, we can take $l' = l + (t-b)$, $\mathfrak{c}_k = \{a, b+k-l]$ for $l < k \leqslant l'$. Since $\mathscr{S}(\mathfrak{C}_0)$ is Λ-admissible, Lemma 7.14 implies that condition (7.3) is satisfied for $\mathsf{K}(\mathfrak{C}')$, $\mathsf{K}(\mathfrak{C}_0)$ and $\mathsf{K}^{\mathrm{ex}}(\mathfrak{C}_0)$. Moreover it implies $\mathscr{S}(\mathfrak{C}_0) = \mathscr{S}(\mathfrak{C}')|_{(\mathsf{K}(\mathfrak{C}_0),\,\mathsf{K}^{\mathrm{ex}}(\mathfrak{C}_0))}$. We know already that $\mathscr{S}(\mathfrak{C}')$ is a completely Λ-admissible monoidal seed with a suitable exchange matrix. Hence we can apply Lemma 7.15 to conclude that $\mathscr{S}(\mathfrak{C}_0)$ is also a completely Λ-admissible monoidal seed. Finally, Proposition 7.19 implies that $\mathscr{S}(\mathfrak{C})$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\,\mathcal{D},\widehat{\underline{w}}_0}$ for any admissible chain $\mathfrak{C}$ with a finite range $[a,b]$ provided that $-\infty < a \leqslant b \leqslant t$.

Now let $[a,b]$ be an arbitrary *finite* interval, and let $\mathfrak{C}$ be an admissible chain of i-boxes with the range $[a,b]$. There is $m \in \mathbb{Z}_{>0}$ such that $b - 2m\ell < t$. Here ℓ is the length of the longest element of the Weyl group of $\mathfrak{g}_{\mathrm{fin}}$. Define the automorphism δ of $\mathbb{Z}$ by $\delta(s) = s + \ell$. Then we have $\mathsf{S}_{\delta(s)} \simeq \mathscr{D}\mathsf{S}_s$ by Definition (4.5). We set $\delta^{-2m}\mathfrak{C} = \{\delta^{-2m}\mathfrak{c}_k\}|_{1\leqslant k\leqslant l}$. Then the monoidal autofunctor $\mathscr{D}^{-2m}$ transforms $\mathscr{S}(\mathfrak{C})$ to $\mathscr{S}(\delta^{-2m}\mathfrak{C})$ with the range $[a-2m\ell, b-2m\ell]$. Since $\mathscr{S}(\delta^{-2m}\mathfrak{C})$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a-2m\ell,b-2m\ell],\mathcal{D},\widehat{\underline{w}}_0}$ with a suitable exchange matrix, we conclude that $\mathscr{S}(\mathfrak{C})$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$.

Now we take an arbitrary interval $[a,b]$, possibly infinite. Let $\mathfrak{C}=(\mathfrak{c}_k)_{1\leqslant k\leqslant l}$ with envelopes $\widetilde{\mathfrak{c}}_k=[\widetilde{a}_k,\widetilde{b}_k]$. For each m such that $m\leqslant l$, set $\mathfrak{C}_{\leqslant m}=(\mathfrak{c}_k)_{1\leqslant k\leqslant m}$ with the range $[\widetilde{a}_m,\widetilde{b}_m]$. Then $\mathscr{S}(\mathfrak{C}_{\leqslant m})$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ for any m, and hence $\mathscr{S}(\mathfrak{C})$ is also a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$. □

Proposition 8.12. *Let ξ be a height function, and let $\imath$ be a sink of ξ. If $\mathscr{S}^{<s_\imath\xi}$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^0$, then $\mathscr{S}^{<\xi}$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^0$*

Proof. Take a ξ-adapted admissible sequence $\mathfrak{s}$ in $\widehat{I}_{\mathfrak{g}}$ such that $\imath_1=\imath$. Then we have $\mathscr{S}^{<s_\imath\xi}=\mathscr{S}_-^{[-\infty,1],\mathfrak{s}}$ and $\mathscr{S}^{<\xi}=\mathscr{S}_-^{[-\infty,0],\mathfrak{s}}$. Then the result follows from Proposition 8.11. □

Proof of Theorem 8.1. Let ξ be the height function associated with the admissible sequence $\mathfrak{s}$. Take $t\in\mathbb{Z}$ such that $\xi_\imath<t$ for any $\imath\in I_{\rm fin}$. Then, there is a ξ-adapted reduced expression $\underline{w}=s_{\imath_1}\cdots s_{\imath_r}$ such that $\xi':=s_{\imath_r}\cdots s_{\imath_1}\xi$ satisfies $\xi'_\imath\in[t,t+2\mathsf{d}_\imath-1]$ for all $\imath\in I_{\rm fin}$. Hence $\widehat{I}^{<\xi'}=\widehat{I}^{<t}$. Then Theorem 8.2 implies that $\mathscr{S}^{<\xi'}=\mathscr{S}^{<t}$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^0$. Hence by a successive application of Proposition 8.12, the monoidal seed $\mathscr{S}^{<\xi}$ is completely Λ-admissible in $\mathscr{C}_{\mathfrak{g}}^0$.

By Corollary 8.10, the result for arbitrary $\mathscr{S}^{<\xi}$ in Proposition 8.12 implies that $\mathscr{S}_-^{[-\infty,0],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}$ is also a completely Λ-admissible monoidal seed for any Q-datum $\mathscr{Q}$ and any reduced expression $\underline{w}_0$ of w_0.

Hence we conclude that for any admissible chain $\mathfrak{C}$ of i-boxes with the range $[a,b]$, $\mathscr{S}:=\mathscr{S}(\mathfrak{C})$ is a completely Λ-admissible monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}$ by Proposition 8.11.

It remains to prove that $\mathcal{A}([\mathscr{S}])\simeq K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0})$. By truncating $\mathfrak{C}$, we may assume that $[a,b]$ is a finite interval. Set $\mathscr{S}=\big(\{M_k\}_{1\leqslant k\leqslant l},\widetilde{B},;\mathsf{K},\mathsf{K}^{\rm ex}\big)$.

Let $\{X_k\}_{1\leqslant k\leqslant l}$ be the cluster variables, and set $f:=\prod_{k=1}^{l}[M_k]\in K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0})$. Then we have

$$\mathcal{A}([\mathscr{S}])\subset\mathbb{Z}[X_1^{\pm1},\ldots,X_l^{\pm1}]\to K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0})[f^{-1}]$$

by $X_k\mapsto[M_k]$. Since $\mathscr{S}$ is completely Λ-admissible in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}$, it induces a ring homomorphism

$$\mathcal{A}([\mathscr{S}])\longrightarrow K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0}).$$

For any $s \in [a,b]$, after successive box moves, the moved $\mathfrak{C}$ contains $\{[s]\}$. Hence, the image of $\mathcal{A}([\mathscr{S}])$ contains $[\mathsf{S}_s]$. Since $K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\,\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0})$ is the polynomial ring with the system of generators $\{[\mathsf{S}_s]\}_{s\in[a,b]}$, $\mathcal{A}([\mathscr{S}]) \to K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\,\mathcal{D}_{\mathscr{Q}},\widehat{\underline{w}}_0})$ is surjective. Since their dimensions are equal to $|[a,b]|$, it is an isomorphism. $\square$

8.5. **Conjecture.** We give the following conjecture which asserts that Theorem 8.1 holds for an arbitrary PBW-pair $(\mathcal{D},\widehat{\underline{w}}_0)$.

Conjecture 8.13.

(i) For any PBW-pair $(\mathcal{D},\widehat{\underline{w}}_0)$ and any admissible chain $\mathfrak{C} = (\mathfrak{c}_k)_{1\leqslant k\leqslant l}$ of i-boxes with an arbitrary range $[a,b]$, the monoidal category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ is a Λ-monoidal categorification of $K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$ with an initial Λ-admissible monoidal seed $(\mathsf{M}(\mathfrak{C}), \widetilde{B}; \mathsf{K}(\mathfrak{C}), \mathsf{K}^{\mathrm{ex}}(\mathfrak{C}))$ for some exchange matrix $\widetilde{B}$.

(ii) If $-\infty \leqslant a \leqslant b < +\infty$, the monoidal category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}$ is a Λ-monoidal categorification of $K(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0})$ with an initial Λ-admissible monoidal seed

$$\mathscr{S}_{-}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0} := \big(\{M^{\mathcal{D},\widehat{\underline{w}}_0}[s,b]\}_{s\in[a,b]}, \widetilde{B}_{\mathrm{GLS}}^{[a,b],\mathcal{D},\widehat{\underline{w}}_0}; \mathsf{K}, \mathsf{K}^{\mathrm{ex}}\big).$$

Problem. *For any PBW-pair and an admissible chain $\mathfrak{C}$ of i-boxes, determine explicitly the exchange matrix of $\mathscr{S}(\mathfrak{C})$.*

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