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A note on some variations of the maximal inequality for the fractional Schrödinger equation

By

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Abstract

The purpose of this note is to provide a summary of the recent work of the authors on two variations of the pointwise convergence problem for the solutions to the fractional Schrödinger equations; convergence along a tangential line and along a set of lines, as exhibiting some new results in each setting. For the former case, we make a simple observation on a path along a tangential curve of exponential order. We discuss counterexamples for the latter case that show some of the known smooth regularities are essentially optimal.

§ 1. Introduction

Let $d \in \mathbb{N}$ and $m > 1$. On $\mathbb{R}^d \times \mathbb{R}$ the fractional Schrödinger equation is famously known as

$$i\partial_t u + (-\Delta)^{\frac{m}{2}} u = 0 \tag{1.1}$$

for the initial data $u(\cdot, 0) = f$, whose solution may be (formally) expressed as

$$u(x,t) = S_t^m f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^m)} \widehat{f}(\xi)\, \mathrm{d}\xi$$

by using the Fourier transform given by $\widehat{f}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, \mathrm{d}x$.

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The main object of our interest in this note is to determine the optimal regularity $s \geq 0$ for which the following local maximal-in-time estimate with respect to time for the fractional Schrödinger equation holds; for some $q \geq 1$, there exists a constant $C > 0$ such that

$$\|S_t^m f(x)\|_{L_x^q(\mathbb{B}^d)L_t^\infty(\mathbb{I})} \leq C\|f\|_{H^s(\mathbb{R}^d)} \tag{1.2}$$

for all $f \in H^s(\mathbb{R}^d)$, defined by

$$\|f\|_{H^s(\mathbb{R}^d)} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{-\frac{d}{2}} \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}}.$$

Furthermore, let $\mathbb{B}^d$ denote the unit ball in $\mathbb{R}^d$, centered at the origin, and let $\mathbb{I} = [0, 1]$. By locality, once we prove (1.2) for some $q_0 \geq 1$, the inequality (1.2) (under the same conditions but) with q smaller than q_0 is deduced by Hölder's inequality. Of course, when the initial data is smooth enough, for instance, strictly smoother than $\frac{d}{2}$, the validity of (1.2) (with arbitrary q) follows immediately. In fact, a trivial computation reveals that for any $(x, t) \in \mathbb{R}^{d+1}$

$$|e^{it(-\Delta)^{\frac{m}{2}}} f(x)| \lesssim \int_{\mathbb{R}^d} |\widehat{f}(\xi)| \, \mathrm{d}\xi \lesssim \left(\int_{\mathbb{R}} (1+r^2)^{-s} r^{d-1} \, \mathrm{d}r \right)^{\frac{1}{2}} \|f\|_{H^s(\mathbb{R}^d)},$$

which is finite whenever $s > \frac{d}{2}$. Moreover, this computation indicates that the oscillatory cancellation, completely ignored in the first step, may have a crucial role in order to go beyond the smooth regularity.

The maximal inequality (1.2) is motivated by the study of the pointwise convergence behavior of the solution to the fractional Schrödinger equation, sometimes referred to as Carleson's problem. Namely, if there exists $C > 0$ such that (1.2) holds[1] for all $f \in H^s(\mathbb{R}^d)$ for some $q \geq 1$ and $s \geq 0$, then

$$\lim_{t \to 0} S_t^m f(x) = f(x) \quad \text{a.e.} \tag{1.3}$$

holds for all $f \in H^s(\mathbb{R}^d)$. The reduction is a similar spirit to the Lebesgue differentiation theorem ([26]) and it does not lose much information. In fact, (1.2) conversely follows from the pointwise convergence (1.3) provided the *weak-type* maximal estimate for $q \in [1, 2]$ via Nikishin–Stein maximal principle (see, for example, [46] for the details). When the spatial dimension $d = 1$, the problem is relatively easy and was completely solved in the 1980s.

Theorem 1.1 (Carleson [9], Dalberg–Kenig [20], Sjölin [53], Kenig–Ponce–Vega [32]). *Let $d = 1$ and $m > 1$. Then, there exists $C > 0$ such that* (1.2) *holds with $q = 4$ for all $f \in H^s(\mathbb{R})$ if and only if $s \geq \frac{1}{4}$.*

[1] The left-hand side of (1.2) can be weakened by replacing $L_x^q(\mathbb{B}^d)$ by the weak-type space $L_x^{q,\infty}(\mathbb{B}^d)$.

In higher dimensions, it turned out to be extremely difficult and one can find some historical contributions in [6, 7, 33, 41, 44, 53, 60] for example. The breakthrough came with Bourgain's number theoretic counterexample for the standard Schrödinger equation in 2016 [8] (see also an expository summary of his argument [46] due to Pierce and an alternative proof [42] due to Lucà–Rogers) Thanks to the strong connections of Carleson's problem with other important conjectures in harmonic analysis such as Stein's restriction conjecture and Kakeya maximal conjecture, soon later Du–Guth–Li [23] and Du–Zhang [25] applied state-of-the-art tools in harmonic analysis, such as multilinear restriction theorem, decoupling inequality, polynomial partitioning, refined Strichartz estimates, and showed the necessary regularity given by Bougain is essentially sufficient for the maximal estimate (1.3) (except the endpoint).

Theorem 1.2 (Bourgain [8], Du–Guth–Li [23], Du–Zhang [25], Cho–Ko [10])**.** *Let $d \geq 2$ and $m > 1$. For $d = 2$, (1.2) with $q = 3$ holds if $s > \frac{1}{3}$ and, for $d \geq 3$, (1.2) with $q = 2$ holds if $s > \frac{d}{2(d+1)}$. Moreover, $s \geq \frac{d}{2(d+1)}$ is necessary for (1.2) with $m = 2$ and $q \geq 1$.*

It remains an open question whether $s \geq \frac{d}{2(d+1)}$ is necessary in the context of the fractional Schrödinger equation. Recent progress in this area has been made by An–Chu–Pierce [1] and Eceizabarrena–Ponce-Vanegas [28] in this direction. The exponents $q = 4$ when $d = 1$ and $q = 3$ when $d = 2$ are critical for (1.2) to hold with $s = \frac{d}{2(d+1)}$ (or s arbitrary close to $\frac{d}{2(d+1)}$). To see this, applying appropriate change of variables and a time localization lemma ([33, 49]), one may observe that (1.2) is equivalent to

$$\Big\| \sup_{t\in[0,R]} |S_t^m f| \Big\|_{L^q(B_R)} \lesssim R^{d(\frac{1}{q}-\frac{1}{2})+s} \|f\|_{L^2(\mathbb{R}^d)}$$

for $R \gg 1$, f with $\operatorname{supp} \widehat{f} \subset \{\xi : |\xi| \sim R\}$, and a ball B_R of radius R. Therefore, it follows that $q \leq \frac{d}{2(d+1)}$ since $d(\frac{1}{q} - \frac{1}{2}) + s \geq 0$. When $d \geq 3$ it is known that the critical q is strictly smaller than $\frac{d}{2(d+1)}$ but the exact value remains unknown.

§ 2. Fractal dimension of the divergence sets

Although we know that the Lebesgue measure of the *divergence set* $\mathfrak{D}(f) := \{x \in \mathbb{R} : S_t^m f(x) \not\to f(x) \text{ as } t \to 0\}$ is zero for $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$ as a consequence of Theorem 1.1, the set $\mathfrak{D}(f)$ may still be large enough to be "detected" by a fractal measure. This direction in the context of pointwise convergence was first considered by Sjögren–Sjölin [52]. In [2] Barceló–Bennett–Carbery–Rogers recently concerned with this question and measured the divergence set by the use of Frostman's lemma, together with the results about the singularities of the Bessel potential due to Žubrinić.

Theorem 2.1 (Barceló–Bennett–Carbery–Rogers [2], Žubrinić [63]). *Let $d = 1$ and $m > 1$. Then,*

$$\sup_{f \in H^s(\mathbb{R})} \dim_H \mathfrak{D}(f) = 1 - 2s,$$

where $\dim_H$ *denotes the Hausdorff dimension.*

The analogous results in higher dimensions are also available but there are still some remaining unknown cases for small s. Interested readers are encouraged to visit [24, 40, 41, 43] aside from [2]. The key estimate is the maximal inequality (1.2) yet its spatial measure $\mathrm{d}x$ is replaced by a so-called α-dimensional measure $\mathrm{d}\mu$ characterized by the property

$$\sup_{x \in \mathbb{R}^d, r > 0} \frac{\mu(B^d(x,r))}{r^\alpha} < \infty,$$

where $0 < \alpha \le d$ and a ball $B^d(x,r)$ of radius r centered at x.

§ 3. Convergence along a tangential curve

The original convergence of the solutions to the fractional Schrödinger equations can be regarded as the limit along the vertical line to the hyperplane $\mathbb{R}^d \times \{0\}$ at x, i.e.

$$\lim_{t \to 0} S_t^m f(x) = \lim_{\substack{(y,t) \to (x,0) \\ (y,t) \in \{x\} \times (0,1)}} S_t^m f(y).$$

We shall replace the path of the vertical line with more general paths. In this section, let us consider a convergence along a curve. When $d = 1$, we shall define curves by

$$\rho_\kappa(x,t) = x - t^\kappa$$

with $\kappa > 0$ and call it *non-tangential* and *tangential* when $\kappa \ge 1$ and $0 < \kappa < 1$, respectively. The corresponding pointwise convergence problem along a tangential curve is

$$\lim_{\substack{(y,t) \to (x,0) \\ y = x - t^\kappa}} S_t^m f(y) = \lim_{t \to 0} S_t^m f(\rho_\kappa(x,t)) = f(x) \quad \text{a.e. } x. \tag{3.1}$$

By the standard argument mentioned earlier (see also [16] in this particular setting), it suffices to show the maximal estimate:

$$\|S_t^m f(\rho_\kappa(x,t))\|_{L_x^q(\mathbb{I}) L_t^\infty(\mathbb{I})} \le C \|f\|_{H^s(\mathbb{R}^d)} \tag{3.2}$$

for all $f \in H^s(\mathbb{R}^d)$. In the study of pointwise convergence with a harmonic oscillator, Lee–Rogers [34] revealed[2] that the problem of pointwise convergence–associated with

[2] They only dealt with the standard Schrödinger equation but the same conclusion holds for $m > 1$. In order to show this, a similar argument in [16] can be carried since $|t_1 - t_2| \gtrsim |t_1^\kappa - t_2^\kappa|$ by the mean value theorem.

the harmonic oscillator, along a non-tangential curve, and along a vertical line in the classical sense–is fundamentally equivalent (at least when $m = 2$).

Theorem 3.1 (Lee–Rogers [34]). *Let $d = 1$, $m > 1$ and $\kappa \geq 1$. Then,* (3.2) *with $q = 2$ holds for all $f \in H^s(\mathbb{R})$ if $s \geq \frac{1}{4}$.*

On the other hand, when the curve $\rho_\kappa(x, t)$ is tangential ($0 < \kappa < 1$), this may not be true anymore.

Theorem 3.2 (Cho–Lee–Vargas [15], Cho–Lee [14], Cho–Shiraki [16]). *Let $d = 1$, $m > 1$, $0 < \kappa \leq 1$ and μ be an α-dimensional measure. If $s > \max\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\}$, then there exists $C > 0$ such that*

$$\|S_t^m f(\rho_\kappa(x,t))\|_{L_x^2(\mathbb{I},\mathrm{d}\mu)L_t^\infty(\mathbb{I})} \leq C\|f\|_{H^s(\mathbb{R})} \tag{3.3}$$

for all $f \in H^s(\mathbb{R})$.

Here, the condition $s > \max\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\}$ is sharp in the sense that one can find an α-dimensional measure μ and the initial data f such that (3.3) fails whenever $s < \max\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\}$. As we see momentarily, this is based on the Knapp-type argument where we also restrict the domain of $(x, t) \in \mathbb{B}^n \times \mathbb{I}$. For instance, the authors in [16] set $\mathrm{d}\mu(x) = |x|^{\alpha-1}dx$ and for each condition[3], choose the initial data f and the restriction of (x, t) as follows for $\lambda \geq 1$, $A = [0, \lambda^{\frac{1}{m}}]$ and $B = [\lambda^2, \lambda^2 + \lambda]$:

To show	Initial data	x	$t = t(x)$
$s \geq \frac{1}{4}$	$\widehat{f}(\xi) = \lambda^{-1}\chi_B(\xi)$	$(0, \frac{1}{100(m-1)})$	$t(x)$ s.t. $x = t(x)^\kappa + m\lambda^{2m-2}t(x)$
$s \geq \frac{1-\alpha}{2}$	$\widehat{f}(\xi) = \chi_A(\xi)$	$(0, \frac{1}{100}\lambda^{-\frac{1}{m}})$	$t(x) \in (0, \frac{1}{100}\lambda^{-1})$
$s \geq \frac{1-m\alpha\kappa}{2}$	$\widehat{f}(\xi) = \chi_A(\xi)$	$(0, \frac{1}{100}\lambda^{-\kappa})$	$t(x) = x^{\frac{1}{\kappa}}$

By applying a similar argument in the classical situation, Theorem 3.2 implies that the pointwise convergence along the tangential curve (3.1) holds for all $f \in H^s(\mathbb{R})$ if $s > \max\{\frac{1}{4}, \frac{1-m\kappa}{2}\}$, and moreover, $\sup_{f\in H^s(\mathbb{R})} \dim_H \mathfrak{D}(f \circ \rho_\kappa) \leq \max\{1 - 2s, \frac{1-2s}{m\kappa}\}$ if $\max\{\frac{1}{4}, \frac{1-m\kappa}{2}\} < s < \frac{1}{2}$. It is worth noting that the upper bound of the Hausdorff dimension of the divergence sets varies depending on $\kappa \in (0, 1]$: The curve ρ_κ is classified the same as the vertical line when $\kappa \in (\frac{1}{m}, 1]$. When $\kappa < \frac{1}{m}$, the number $\frac{1-2s}{m\kappa}$ is dominant over $1 - 2s$. In particular, for $\kappa \in (\frac{1}{2m}, \frac{1}{m})$, the gap at $s = \frac{1}{4}$ for the upper bound of $\dim_H \mathfrak{D}(f \circ \rho_\kappa)$ remains in existence but reflects the mixture state illustrated in Figure 1 despite the fact that $\frac{1}{4} \geq \frac{1-m\kappa}{2}$. The gap disappears when $\kappa \in (0, \frac{1}{2m})$.

[3] The same spirit for the second row of the table is effectively used in our Proposition 3.3 and Theorem 4.2 in the subsequent sections.

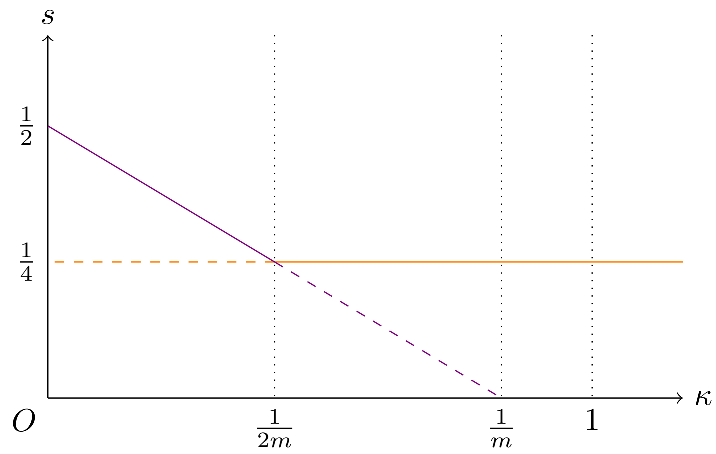

Figure 1. Sharp smooth regularity depending on κ when $m > 1$

In the higher dimensional cases of (3.2), the formulation of the curve is rather abstract. Recently Li–Wang [35, 36] obtained some partial results for a curve γ such that

$$\begin{cases} |\gamma(x,t) - \gamma(x,t')| \le c|t-t'|^{\kappa} \\ \gamma(x,0) = x \end{cases}$$

for $x \in \mathbb{B}^d$, $t, t' \in \mathbb{I}$, $\kappa \in (0,1)$ and for some $c > 0$.

Casually speaking, the smaller the Hölder continuity index becomes, the more the curve $\rho_\kappa(x,t)$ gets "tangent" to the hyperplane. This may be clearer if we write $y = -\rho_\kappa(x,t) = t^\kappa - x$ and re-express it as $t = (y+x)^{\frac{1}{\kappa}}$, the graph of $y \mapsto t$ touching $\mathbb{R} \times \{0\}$ at $y = -x$. While the curve $\rho_\kappa(x,t)$ is of polynomial tangential (of order $\frac{1}{\kappa}$), one may wonder what happens for convergence along a curve of order ∞, or exponentially tangent curve (beyond polynomial order). The typical example of such curve is formed by $t = e^{-\frac{1}{y+x}}$, instead of $t = (y+x)^{\frac{1}{\kappa}}$. Considering the convergence along this "exponentially tangential" curve, which is reformulated and denoted as $\widetilde{\rho}(x,t) = x - (\log \frac{1}{t})^{-1}$, one can show that the smooth regularity for the corresponding maximal inequality is, consistently, as almost bad as a trivial result; $s \ge \frac{1}{2}$.

Proposition 3.3. *Let $m > 1$, $0 < \kappa \le 1$ and $\mathrm{d}\mu(x) = |x|^{\alpha-1}\mathrm{d}x$. Then there exists $C > 0$ such that*

$$\|S_t^m f(\widetilde{\rho}(x,t))\|_{L_x^2(\mathbb{I},\mathrm{d}\mu)L_t^\infty(\mathbb{I})} \le C\|f\|_{H^s(\mathbb{R})} \tag{3.4}$$

fails if $s < \frac{1}{2}$.

Proof. Let $s < \frac{1}{2}$ and suppose (3.4) held. Define the initial data f by

$$\widehat{f}(\xi) = \chi_A(\xi), \quad A = [0, \tfrac{1}{100}\lambda^{\frac{1}{m}}]$$

so that $\|f\|_{H^s(\mathbb{R})} \lesssim \lambda^{\frac{s}{m}}\lambda^{\frac{1}{2m}}$. Then, considering $t = t(x)$ as a fuction of x, it holds that

$$\sup_{t\in\mathbb{I}} |S_t^m f(\widetilde{\rho}(x,t))| \geq \left| \int_A e^{i(x-(\log 1/t(x))^{-1})\xi + t(x)|\xi|^m)} \,\mathrm{d}\xi \right|.$$

For each of the choices of $x \in (0, (\log\lambda)^{-1})$ and $t = t(x) = e^{-\frac{1}{x}}$ let the phase be fairly small, namely,

$$|(x - (\log 1/t(x))^{-1})\xi + t(x)|\xi|^m| \leq \frac{1}{2}$$

for a sufficiently large λ. Therefore, we can choose ε so that $0 < \varepsilon < \frac{1}{2} - s$ and

$$\begin{aligned}\left\| \sup_{t\in\mathbb{I}} \left|S_t^m f(\widetilde{\rho}(x,t))\right| \right\|_{L^2(\mathbb{I},\mathrm{d}\mu)} &\gtrsim \|S_t^m f(\widetilde{\rho}(x, e^{-\frac{1}{x}}))\|_{L^2((0,(\log\lambda)^{-1}),\mathrm{d}\mu)} \\ &\gtrsim \lambda^{\frac{1}{m}}(\log\lambda)^{-\frac{\alpha}{2}} \\ &\gtrsim \lambda^{\frac{1}{m}-\frac{\varepsilon}{m}}\end{aligned}$$

for such large number λ. Combined with the estimate of the right-hand side, it follows that

$$\lambda^{\frac{1}{m}-\frac{\varepsilon}{m}} \lesssim \lambda^{\frac{s}{m}+\frac{1}{2m}},$$

which is a contradiction as $\lambda \to \infty$ under $s < \frac{1}{2}$. □

§ 4. Convergence along a set of lines

In [52], Sjögren–Sjölin (see [31] for the fractional Schrödinger equation) considered the convergence within a conical region over $(x, 0) \in \mathbb{R}^d \times \{0\}$, instead of the limit along the vertical line to the point $(x, 0)$, and proved that the trivial regularity $s > \frac{d}{2}$ (as we observed at the beginning of this note) is actually necessary in this case: It is tempting to unify their result and Theorem 1.1/1.2. To do so, notice that the vertical line is regarded as a line $\{x + t\theta : x \in \mathbb{R}^d,\ t \in \mathbb{I},\ \theta \in \{0\}\}$, while the conical region is regarded as a set $\{x + t\theta : x \in \mathbb{R}^d,\ t \in \mathbb{I},\ \theta \in [-1, 1]\}$ for example. In the one-dimensional case, Lee–Vargas and the first author considered convergence along any path within a region generated by a set $\{x + t\theta : x \in \mathbb{R}^d,\ t \in \mathbb{I},\ \theta \in \Theta\}$ for a given compact set $\Theta \subset \mathbb{R}$ whose Minkowski dimension is lied in $(0, 1)$, such as the third Cantor set. This is an intermediate case since the Minkowski dimension of $\{0\}$ and $[-1, 1]$ are 0 and 1, respectively. In general, the Minkowski dimension of a compact subset $\Theta \subset \mathbb{R}^d$ is defined by

$$\dim_M \Theta = \inf\{\beta > 0 : \limsup_{\delta\to 0} N_\delta(\Theta)\delta^\beta = 0\}$$

for $N_\delta(\Theta)$ denoting the smallest number of δ-ball covering of Θ. By letting Θ generate the path (the conical region with a bunch of linear holes in it), the following sufficient result has been obtained. We give the path explicitly by

$$\varrho(x,t,\theta) = x + t\theta, \quad (x,t,\theta) \in \mathbb{B}^d \times \mathbb{I} \times \Theta$$

and let $\beta(\Theta) = \dim_M \Theta$.

Theorem 4.1 (Cho–Lee–Vargas [15], Shiraki [51])**.** *Let $d = 1$, $m > 1$ and Θ be a compact subset of $\mathbb{R}$. If $s > \frac{1+\beta(\Theta)}{4}$, then*

$$\|S_t^m f(\varrho(x,t,\theta))\|_{L_x^4(\mathbb{I}) L_t^\infty(\mathbb{I}) L_\theta^\infty(\Theta)} \lesssim \|f\|_{H^s(\mathbb{R})} \tag{4.1}$$

for all $f \in H^s(\mathbb{R})$.

This result immediately implies that the pointwise convergence along a set of lines generated by $\Theta \subset \mathbb{R}$

$$\lim_{t\to 0} S_t^m f(\varrho(x,t,\theta)) = f(x) \text{ a.e.}$$

holds for all $f \in H^s(\mathbb{R})$ with $s > \frac{1}{4} + \frac{\beta(\Theta)}{4}$. Very recently the higher dimensional cases are also considered. Li–Wang–Yan [37] adapted an analogous reduction argument in [34] and invoked the results for the pointwise convergence along the vertical line. In particular, when $d = 2$, combining with the result from [23], they showed that there exists $C > 0$ such that

$$\|S_t^2 f(\varrho(x,t,\theta))\|_{L_x^3(\mathbb{B}^2) L_t^\infty(\mathbb{I}) L_\theta^\infty(\Theta)} \lesssim \|f\|_{H^s(\mathbb{R}^2)} \tag{4.2}$$

for all $f \in H^s(\mathbb{R}^2)$ whenever $s > \frac{1+\beta(\Theta)}{3}$, which interpolates the case $s > \frac{1}{3}$ for $\beta(\Theta) = 0$ [23] and $s > 1$ for $\beta(\Theta) = 2$ [52].

As far as the authors are aware, there was no result that indicates whether or not the regularity $s > \frac{d}{2(d+1)}(1 + \beta(\Theta))$ is sharp for any d, unless Θ has either 0 or the full dimension. We construct a counterexample that shows that $s > \frac{1+\beta(\Theta)}{4}$ for (4.1) and $s > \frac{1+\beta(\Theta)}{3}$ for (4.2) are reasonable in the case of $\mathrm{d}\mu(x) = \mathrm{d}x$.

Theorem 4.2. *Let $d \geq 1$ and $m > 1$. Then, there exists $\Theta \subset \mathbb{R}^d$ such that*

$$\|S_t^m f(\varrho(x,t,\theta))\|_{L_x^q(\mathbb{B}^d) L_t^\infty(\mathbb{I}) L_\theta^\infty(\Theta)} \leq C\|f\|_{H^s(\mathbb{R}^d)}$$

fails if $s < \frac{d}{2} - \frac{d}{q} + \frac{\beta(\Theta)}{q}$.

Remarks.

(i) As alluded to earlier, Theorem 4.2 in the case when $(d,q) = (1,4)$ and $m > 1$ shows that (4.1) fails if $s < \frac{1+\beta(\Theta)}{4}$, and the case when $(d,q,m) = (2,3,2)$ shows that (4.2) fails if $s < \frac{1+\beta(\Theta)}{3}$.

(ii) Since we do not know whether the standard step to deduce (pointwise convergence) $\Rightarrow$ (maximal estimate) by Stein's maximal principle in this variant of convergence, there is, unfortunately, no conclusion for pointwise convergence result from Theorem 4.2. (If we assume that Stein's maximal principle was carried in this setting with a set of fractal lines, the valid range of the exponent q might be $1 \leq q \leq 2$ anyway.)

Proof. For a fixed $r \in (0, \frac{1}{2})$ define the r-Cantor set $\mathfrak{C}(r)$ by taking the intersection all generations of the pre-Cantor sets $\mathfrak{C}_k(r)$ for each non-negative $k \in \mathbb{Z}$ (i.e. $\mathfrak{C}(r) = \bigcap_{k=0}^{\infty} \mathfrak{C}_k(r)$), where $\mathfrak{C}_k(r)$ are inductively generated as follows: Starting with $\mathfrak{C}_0(r) = [0,1]$, we remove the interval of length $1-2r$ from the middle of $[0,1]$ and denote the remaining 2 intervals together by $\mathfrak{C}_1(r)$. Similarly, we remove the interval of length $r(1-2r)$ from the middle of each interval of $\mathfrak{C}_1(r)$ and denote the remaining 2^2 intervals together by $\mathfrak{C}_2(r)$, and so on. Note that, by following the construction, $\mathfrak{C}_k(r)$ consists of disjoint 2^k intervals of length r^k, each of which we let $\Omega_{k,j}$ so that $\mathfrak{C}_k(r) = \bigcup_{j=1}^{2^k} \Omega_{k,j}$ ($|\Omega_{k,j}| = r^k$) and $\mathfrak{C}_k(r) \supset \mathfrak{C}_{k+1}(r)$. One of crucial properties of $\mathfrak{C}(r)$ in this context is $\dim_M \mathfrak{C}(r) = \frac{\log 2}{\log 1/r} \in (0,1)$ (Appendix Appendix A).

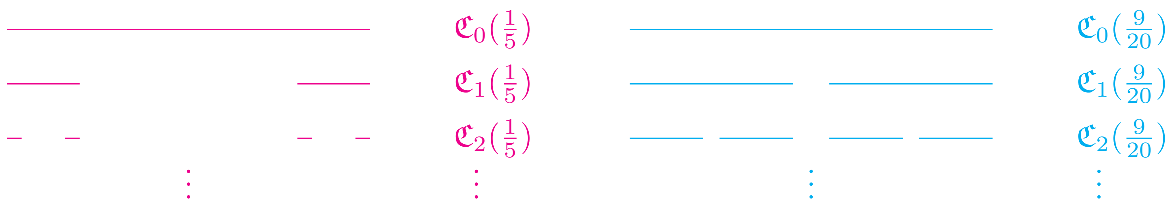

Figure 2. Each generation of pre-Cantor sets associated with $\mathfrak{C}(\frac{1}{5})$ and $\mathfrak{C}(\frac{9}{20})$

For sufficiently large $k \gg 1$, let $\lambda = \lambda_k := r^{-k}$. Let $s < \frac{d}{2} - \frac{d}{q} + \frac{\beta(\Theta)}{q}$ and suppose that the stated inequality held. In order to see the main idea (based on the choice in the second row of the table on page 4), first let us deal with the case when $d = 1$. Set $\Theta = \mathfrak{C}(r)$ and the initial data f by

$$\widehat{f}(\xi) = e^{-i|\xi|^m} \chi_{D_1}(\xi), \quad D_1 = [0, c\lambda] \tag{4.3}$$

so that $\beta(\Theta) = \frac{\log 2}{\log 1/r} \in (0,1)$ and $\|f\|_{H^s(\mathbb{R})} \lesssim \lambda^s |D_1|^{\frac{1}{2}} = \lambda^{s+\frac{1}{2}}$. By the change of variables; $x = -y$ and $t = 1 - \tau$, it would follow that

$$\begin{aligned}\sup_{\substack{t\in\mathbb{I}\\ \theta\in\Theta}} \left|S_t^m f(\varrho(-y,t,\theta))\right| &= \sup_{\substack{t\in\mathbb{I}\\ \theta\in\Theta}} \left|\int_{D_1} e^{i((-y+t\theta)\xi+t|\xi|^m)} e^{-i|\xi|^m)}\, \mathrm{d}\xi\right| \\ &\geq \sup_{\substack{\tau\in[0,1]\\ \theta\in\Theta}} \left|\int_{D_1} e^{i(-(y-\theta)\xi-\tau\theta\xi-\tau|\xi|^m)}\, \mathrm{d}\xi\right| \\ &\geq \left|\int_{D_1} e^{i(-(y-\theta(y))\xi-\tau(y)\theta(y)\xi-\tau(y)|\xi|^m)}\, \mathrm{d}\xi\right|.\end{aligned}$$

Now, to ensure the phase is fairly small, we specify (x,t). If we let $y \in \mathfrak{C}_k(r)$ then we can find $\theta(y) \in \Theta$ satisfying $|y - \theta(y)| < \lambda^{-1}$ (for instance, take the endpoints of each interval $\Omega_{k,j}$, which are also in Θ). Hence, for $\tau \in (0, \lambda^{-m})$, $y \in \mathfrak{C}_k(r)$ and such $\theta(y) \in \Theta$, the phase is bounded above by

$$|(y-\theta(y))\xi + \tau\theta(y)\xi + \tau|\xi|^m| \leq \frac{1}{2},$$

which implies that

$$\begin{aligned}\Big\| \sup_{\substack{t\in\mathbb{I}\\ \theta\in\Theta}} \left|S_t^m f(\varrho(x,t,\theta))\right| \Big\|_{L^q(\mathbb{I})} &\gtrsim \Big\| \sup_{\substack{\tau\in(0,\lambda^{-m})\\ \theta\in\Theta}} \left|S_t^m f(\varrho(y,t,\theta))\right| \Big\|_{L^q(\mathfrak{C}_k(r))} \\ &\gtrsim |\,D_1| \left(\sum_{j=0}^{2^k} \int_{\Omega_{k,j}} \mathrm{d}y \right)^{\frac{1}{q}} \\ &\sim (2^k)^{\frac{1}{q}} \lambda^{1-\frac{1}{q}}.\end{aligned}$$

Since $2^k = (r^{-k})^{\beta(\mathfrak{C}(r))} = \lambda^{\beta(\mathfrak{C}(r))}$, we would obtain $\lambda^{\frac{1}{2}-\frac{1}{q}+\frac{\beta(\Theta)}{q}-s} \leq C$ for some constant C. This is a contradiction as $\lambda \to \infty$.

For the remaining cases where $\beta(\Theta) = 0,\ 1$, one may let $\Theta = \{0\}$, $\mathbb{I}$, respectively, which are easily dealt with. Indeed, the former coincides with the classical well-understood situation, and for the latter one can follow the argument above by setting $\theta(y) = y$ for all $y \in \mathbb{I}$.

Next, we shall consider the case when $d = 2$ and basically modify the above argument. For the case of $\beta(\Theta) \in (0,1)$, we set $\Theta = \mathfrak{C}(r) \times \{0\}$ and the initial data f satisfying

$$\widehat{f}(\xi) = e^{-i|\xi|^m} \chi_{D_2}(\xi), \quad D_2 = [0, c\lambda]^2 \tag{4.4}$$

so that $\beta(\Theta) = \frac{\log 2}{\log 1/r} \in (0,1)$ and $\|f\|_{H^s(\mathbb{R}^2)} \lesssim \lambda^s |\,D_2|^{\frac{1}{2}}$. A similar change of variables gives that

$$\sup_{\substack{t\in\mathbb{I}\\ \theta\in\Theta}} \left|S_t^m f(\varrho(-y,t,\theta))\right| \geq \left|\int_{D_2} e^{i\phi(\xi,y,t,\theta)}\right|,$$

where

$$|\phi(\xi, y, t, \theta)| = |(y_1 - \theta_1(y))\xi_1 + \tau\theta_1(y)\xi_1 + (y_2 - \theta_2(y))\xi_2 + \tau\theta_2(y)\xi_2 + \tau|\xi|^m|.$$

By choosing $y = (y_1, y_2) \in \mathfrak{C}_k(r) \times [0, \lambda^{-1}]$ and $\tau \in (0, \lambda^{-m})$, we may have $|y_i - \theta_i(y)| < \lambda^{-1}$ for $i = 1, 2$, which guarantees that phase is small enough over $\xi \in D_2$, namely, $|\phi(\xi, y, t, \theta)| \leq \frac{1}{2}$. Hence,

$$\begin{aligned}\Big\| \sup_{\substack{t\in\mathbb{I}\\ \theta\in\Theta}} \big|S_t^m f(\varrho(x,t,\theta))\big| \Big\|_{L^q(\mathbb{B}^2)} &\gtrsim \Big\| \sup_{\substack{\tau\in(0,\lambda^{-m})\\ \theta\in\Theta}} \big|S_t^m f(\varrho(y,\tau,\theta))\big| \Big\|_{L^q(\mathfrak{C}_k(r)\times[0,\lambda^{-1}])} \\ &\gtrsim (2^k\lambda^{-2})^{\frac{1}{q}} |\, D_2|.\end{aligned}$$

Therefore, by assuming the sated estimate, we would obtain

$$(2^k\lambda^{-2})^{\frac{1}{q}} |\, D_2| \lesssim \lambda^s |\, D_2|^{\frac{1}{2}}.$$

By recalling $\lambda = r^{-k}$, $2^k = \lambda^{\beta(\Theta)}$ and $|D_2| \sim \lambda^2$, this yields a contradiction as $\lambda \to 0$.

For the case when $\beta(\Theta) \in (1, 2)$, we set $\Theta = \mathfrak{C}(r) \times [0, 1]$ and employ the initial data given by (4.4) so that the same computation reveals that the modulus of the corresponding phase $\phi(\xi, x, t, \theta)$ is bounded above by, say, $\frac{1}{2}$ if we choose $y = (y_1, y_2) \in \mathfrak{C}_k(r) \times [0, 1]$ and $\tau \in (0, \lambda^{-m})$. Therefore, in this case, one may obtain

$$(2^k\lambda^{-1})^{\frac{1}{q}} |\, D_2| \lesssim \lambda^s |\, D_2|^{\frac{1}{2}},$$

which result makes a contradiction.

The remaining are the cases where $\dim_M \Theta$ has the integers such as 0, 1, 2 and may be considered similar to ones in the case of $d = 1$. For instance, let $\Theta = \{0\} \times \{0\}$, $[0, 1] \times \{0\}$, $[0, 1]^2$, respectively.

When $d \geq 3$, a similar argument may be carried as well, which we omit the details. □

Theorem 4.2 encourages us to pursue further generalizations of Theorem 4.1 with respect to the α-dimensional measure $\mathrm{d}\mu$. By combining the argument in [51] with the one in [16], one may deduce the following.

Theorem 4.3. *Let $d = 1$, $m > 1$ and $q \geq 2$. If $s > \min\{\frac{1}{2}, \max\{\frac{1}{4} + \frac{\beta(\Theta)}{4\alpha}, \frac{1}{2} + \frac{\beta(\Theta)-\alpha}{q}\}\}$, then there exists $C > 0$ such that*

$$\|S_t^m f(\varrho(x,t,\theta))\|_{L_x^q(\mathbb{I},\mathrm{d}\mu)L_t^\infty(\mathbb{I})L_\theta^\infty(\Theta)} \leq C\|f\|_{H^s(\mathbb{R})}$$

for all $f \in H^s(\mathbb{R})$.

Note that the lower bound of s is $\frac{1}{2}$ if $\alpha < \beta(\Theta)$ and $\max\{\frac{1}{4}+\frac{\beta(\Theta)}{4\alpha}, \frac{1}{2}+\frac{\beta(\Theta)-\alpha}{q}\}$ if $\beta(\Theta) \le \alpha \le 1$. Since the function $q \mapsto \frac{1}{2}+\frac{\beta-\alpha}{q}$ is increasing when $\alpha \ge \beta$, the choice of $q=2$ minimizes the lower bound of s so that the poinwise convergence

$$\lim_{t\to 0} S_t^m f(\varrho(x,t,\theta)) = f(x), \quad \mu\text{-a.e.}$$

holds for all $f \in H^s(\mathbb{R})$ whenever $s > \min\{\frac{1}{2}, \max\{\frac{1}{4}+\frac{\beta(\Theta)}{4\alpha}, \frac{1+\beta(\Theta)-\alpha}{2}\}\}$. Moreover, by Frostman's lemma one may find that $\sup_{f\in H^s(\mathbb{R})} \dim_H \mathfrak{D}(f\circ\varrho) \le \{\frac{\beta(\Theta)}{4s-1}, 1+\beta(\Theta)-2s\}$ for $s \in [\frac{1+\beta(\Theta)}{4}, \frac{1}{2}]$ (see Figure 3).

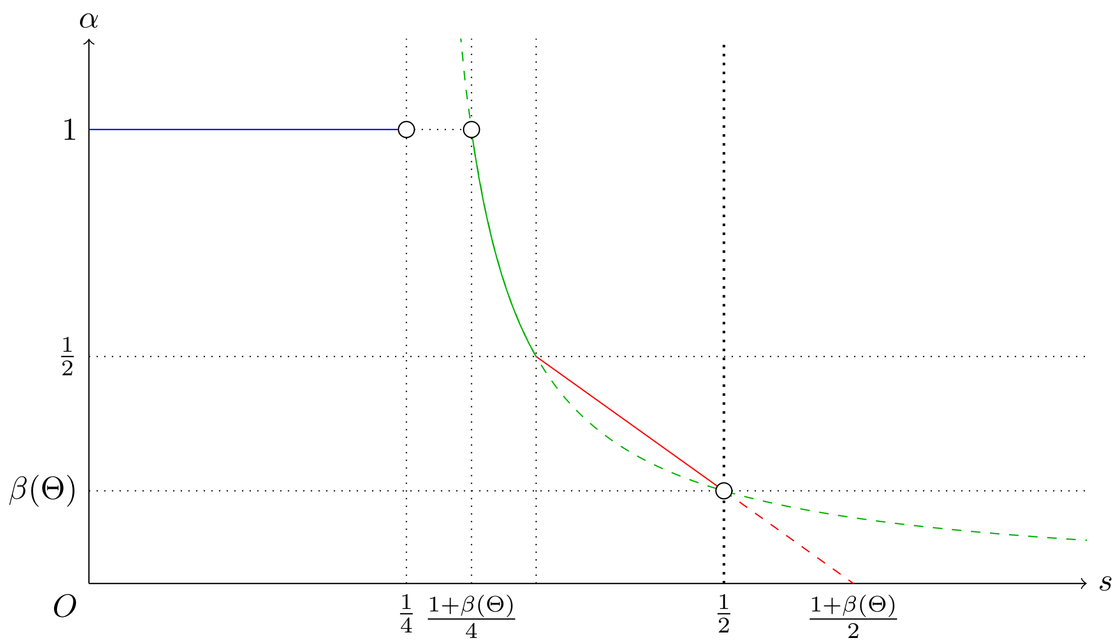

Figure 3. Graph of $\alpha = 1$ when $s \in [0, \frac{1}{4})$ and $\alpha = \min\{\frac{\beta(\Theta)}{4s-1}, 1+\beta(\Theta)-2s\}$ when $s \in (\frac{1+\beta(\Theta)}{4}, \frac{1}{2})$

Proof. For a given Θ, let us write β for the shorthand for $\beta(\Theta)$. Since $s > \frac{1}{2}$ is obtained trivially, it is enough[4] to focus on the case $\alpha \ge \beta$. Let $s_* = \min\{\frac{1}{4}, \frac{\alpha}{q}\}$. The proof is based on a combination of arguments in [51] and [16]. The following proposition has a crucial role in the proof, whose essential idea was introduced in [51].

Proposition 4.4. *Let $\lambda \ge 1$, $q \ge 2$ and Ω be an interval of length $\lambda^{-\frac{qs_*}{\alpha}}$. For arbitrarily small $\varepsilon > 0$, it holds that*

$$\Big\| \sup_{\substack{t\in[-1,1]\\ \theta\in\Omega}} \big|S_t^m f(\varrho(\cdot,t,\theta))\big| \Big\|_{L^q(\mathrm{d}\mu)} \lesssim \lambda^{\frac{1}{2}-s_*+\varepsilon} \|f\|_{L^2} \tag{4.5}$$

[4]As we discussed in above, the conclusion $s > \frac{1}{2}$ when $\alpha < \beta$ also naturally appears by running an analogous argument carefully arranged for the case of $\alpha \ge \beta$.

whenever f is supported in $\{|\xi| \sim \lambda\}$.

First of all, we assume that Proposition 4.4 holds true and prove Theorem 4.3. By using the dyadic decomposition with respect to frequency, we have

$$\Big\| \sup_{\substack{t\in(-1,1)\\ \theta\in\Theta}} \big|S_t^m f(\varrho(x,t,\theta))\big| \Big\|_{L^q} \leq \sum_{k\geq 0} \Big\| \sup_{\substack{t\in(-1,1)\\ \theta\in\Theta}} \big|S_t^m P_k f(\varrho(x,t,\theta))\big| \Big\|_{L^q}.$$

For each k, by the definition of Minkowski dimension, one can find a finite collection of intervals $\{\Omega_{k,j}\}_{j=1}^{N_k}$ such that

$$\Theta \subset \bigcup_{j=1}^{N_k} \Omega_{k,j}, \quad |\Omega_{k,j}| \leq (2^{-k})^{\frac{qs_*}{\alpha}}.$$

Here, N_δ represents the smallest number of intervals of length δ that covers Θ, and $N_k = N_{\delta_k}(\Theta)$ with $\delta_k = (2^{-k})^{\frac{qs_*}{\alpha}}$ in particular. Thus, for each k,

$$\sup_{\substack{t\in(-1,1)\\ \theta\in\Theta}} \big|S_t^m P_k f(\varrho(x,t,\theta))\big|^q \leq \sum_{j=1}^{N_k} \sup_{\substack{t\in(-1,1)\\ \theta\in\Omega_{k,j}}} \big|S_t^m P_k f(\varrho(x,t,\theta))\big|^q,$$

from which it follows that

$$\Big\| \sup_{\substack{t\in(-1,1)\\ \theta\in\Theta}} \big|S_t^m P_k f(\varrho(x,t,\theta))\big| \Big\|_{L^q} \leq \left(\sum_{j=1}^{N_k} \Big\| \sup_{\substack{t\in(-1,1)\\ \theta\in\Omega_{k,j}}} \big|S_t^m P_k f(\varrho(x,t,\theta))\big| \Big\|_{L^q}^q \right)^{\frac{1}{q}}.$$

Invoking Proposition 4.4 with $\lambda = 2^k$, and the fact $N_k \lesssim (2^k)^{\frac{qs_*}{\alpha}\beta+\epsilon}$ for any $\epsilon > 0$ (from the definition of Minkowski dimension), we obtain

$$\begin{aligned}
\Big\| \sup_{\substack{t\in(-1,1)\\ \theta\in\Theta}} \big|S_t^m f(\varrho(x,t,\theta))\big| \Big\|_{L^q} &\leq \sum_{k\geq 0} \left(\sum_{j=1}^{N_k} \Big\| \sup_{\substack{t\in(-1,1)\\ \theta\in\Omega_{k,j}}} \big|S_t^m P_k f(\varrho(\cdot,t,\theta))\big| \Big\|_{L^q}^q \right)^{\frac{1}{q}} \\
&\lesssim \|P_0 f\|_{L^2} + \sum_{k\geq 1} \left(\sum_{j=1}^{N_k} \Big[(2^k)^{\frac{1}{2}-s_*+\varepsilon} \|P_k f\|_{L^2} \Big]^q \right)^{\frac{1}{q}} \\
&\lesssim \|P_0 f\|_{L^2} + \sum_{k\geq 1} (2^k)^{\frac{1}{2}-s_*+\varepsilon+\frac{s_*}{\alpha}\beta} \|P_k f\|_{L^2} \\
&\lesssim \|f\|_{H^{\frac{1}{2}-(1-\frac{\beta}{\alpha})s_*+\varepsilon}},
\end{aligned}$$

as aimed.

Now, we turn to the proof of the proposition. Let an operator $T : L^2(\mathbb{R}) \to L^q_x(\mathbb{I}, d\mu)L^\infty_t(\mathbb{I})L^\infty_\theta(\Omega)$ be given by

$$Tf(x,t,\theta) = \chi(x,t,\theta)\int_{\mathbb{R}} e^{i(\varrho(x,t,\theta)\xi + t|\xi|^m)} f(\xi)\psi(\tfrac{\xi}{\lambda})\,\mathrm{d}\xi,$$

where $\chi = \chi_{I\times I\times\Omega}$ and $\psi \in C_0^\infty((-2,-\frac{1}{2})\cup(\frac{1}{2},2))$. Then, by the Plancherel theorem and duality, (4.5) is equivalent to

$$\|T^*F\|^2_{L^2} \lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L^{q'}_x(\mathrm{d}\mu)L^1_tL^1_\theta}, \tag{4.6}$$

where T^* is the adjoint of T. To estimate (4.6) we decompose $\|T^*F\|_{L^2}$ into $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ such that

$$\|T^*F\|^2_{L^2} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where, under the notations $W = I\times I\times\Omega$, $w = (x,t,\theta)\in W$, $w' = (x',t',\theta')\in W$ and $\mathrm{d}_\mu w = \mathrm{d}\mu(x)\mathrm{d}t\mathrm{d}\theta$, we set

$$\mathcal{I}_\ell := \iint_{V_\ell} \chi(w)\chi(w')\bar{F}(w)F(w')K_\lambda(w,w')\,\mathrm{d}_\mu w\,\mathrm{d}_\mu w',$$

$$K_\lambda(w,w') := \int_{\mathbb{R}} e^{i\phi(\xi,w,w')}\psi(\tfrac{\xi}{\lambda})^2\,\mathrm{d}\xi = \lambda\int_{\mathbb{R}} e^{i\phi(\lambda\xi,w,w')}\psi(\xi)^2\,\mathrm{d}\xi,$$

$$\phi(\xi,w,w') = (\varrho(x,t,\theta)-\varrho(x',t',\theta'))\xi + (t-t')|\xi|^m$$

and

$$\begin{cases} V_1 = \{(w,w')\in W\times W : |x-x'|\le 2\lambda^{-\frac{qs_*}{\alpha}}\},\\ V_2 = \{(w,w')\in W\times W : |x-x'| > 2\lambda^{-\frac{qs_*}{\alpha}} \text{ and } |x-x'|\le 4|t-t'|\},\\ V_3 = \{(w,w')\in W\times W : |x-x'| > 2\lambda^{-\frac{qs_*}{\alpha}} \text{ and } |x-x'| > 4|t-t'|\}.\end{cases}$$

Therefore, it is enough to show that for each $\ell = 1,2,3$

$$\mathcal{I}_\ell \lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L^{q'}_x(\mathrm{d}\mu)L^1_tL^1_\theta}.$$

The case when $\ell = 1$ immediately follows from the trivial kernel estimate $|K_\lambda(w,w')| \lesssim \lambda$ and Lemma Appendix B.2.

For $\mathcal{I}_2$, we shall observe that

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\lambda\xi,w,w')\right| \gtrsim \lambda^m|t-t'||\xi|^{m-2} \gtrsim \lambda|x-x'| \gtrsim \lambda^{1-\frac{qs_*}{\alpha}} \ge 1 \tag{4.7}$$

because of $\frac{qs_*}{\alpha} \le 1$. One can apply van der Corput's lemma (Lemma Appendix B.1) to get

$$|K_\lambda(w,w')| \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}} \lesssim \lambda(\lambda|x-x'|)^{-2s_*} \lesssim \lambda^{1-2s_*+\varepsilon}|x-x'|^{-2s_*+\varepsilon}$$

from the separation assumption and the fact $2s_* \le \frac{1}{2}$. Therefore, applying Lemma Appendix B.2 with $\rho = 2s_* - \varepsilon < \frac{2\alpha}{q}$, it follows that

$$\mathcal{I}_2 \lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L_x^{q'}(\mathrm{d}\mu)L_t^1}.$$

Finally, for $\mathcal{I}_3$, note a key relation

$$|\varrho(w) - \varrho(w')| \sim |x-x'| \tag{4.8}$$

for $(w,w') \in V_3$ since

$$|\theta - \theta'| \le \lambda^{-\frac{qs_*}{\alpha}} < \frac{1}{2}|x-x'|. \tag{4.9}$$

We split K_λ into $\mathcal{K}_1$ and $\mathcal{K}_2$ as follows.

$$\begin{aligned} K_\lambda(w,w') &= \lambda\int_{U_1} e^{i\phi(\lambda\xi,w,w')}\psi(\xi)^2\,\mathrm{d}\xi + \lambda\int_{U_2} e^{i\phi(\lambda\xi,w,w')}\psi(\xi)^2\,\mathrm{d}\xi \\ &=: \mathcal{K}_1 + \mathcal{K}_2, \end{aligned}$$

where

$$\begin{cases} U_1 = \{\xi \in \operatorname{supp}\psi : |x-x'| > 8m\lambda^{m-1}|t-t'||\xi|^{m-1}\}, \\ U_2 = \{\xi \in \operatorname{supp}\psi : |x-x'| \le 8m\lambda^{m-1}|t-t'||\xi|^{m-1}\}. \end{cases}$$

For $\mathcal{K}_1$, we use (4.8) and $\frac{qs_*}{\alpha} \le 1$ to estimate

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\lambda\xi,w,w')\right| &\ge \big|\lambda|\varrho(w)-\varrho(w')| - m\lambda^m|t-t'||\xi|^{m-1}\big| \\ &\gtrsim |\lambda|x-x'| - m\lambda^m|t-t'||\xi|^{m-1}| \\ &\gtrsim \lambda|x-x'| \gtrsim \lambda^{1-\frac{qs_*}{\alpha}} \ge 1. \end{aligned}$$

Notice that the interval U_1 consists of at most two intervals since $\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\lambda\xi,w,w')$ is monotone on each interval $(-\infty,-1]$ and $[1,\infty)$. Then, van der Corput's lemma gives that

$$\mathcal{K}_1 \lesssim \lambda(\lambda|x-x'|)^{-1}. \tag{4.10}$$

On the other hand, for $\mathcal{K}_2$, we have (4.7) again and apply van der Corput's lemma to obtain

$$\mathcal{K}_2 \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}}. \tag{4.11}$$

Combining (4.10) and (4.11), for $(w,w')\in V_3$

$$\begin{aligned}|K_\lambda(w,w')| &\lesssim \lambda(\lambda|x-x'|)^{-\frac12} \lesssim \lambda(\lambda|x-x'|)^{-2s_*}\\ &\lesssim \lambda^{1-2s_*+\varepsilon}|x-x'|^{-2s_*+\varepsilon}\end{aligned}$$

from the separation assumption. By Lemma Appendix B.2 with $\rho = 2s_* - \varepsilon$ we conclude that

$$\mathcal{I}_3 \lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L_x^{q'}(\mathrm{d}\mu)L_t^1L_\theta^1}.$$

□

Remark. It is, of course, reasonable to generalize Theorem 4.2 in the context of α-dimensional measure and hope that $s \geq \frac12 + \frac{\beta-\alpha}{q}$ is also necessary. We found, however, that this may not be straightforward and note here that a weaker condition, $s \geq \frac12 + \frac{\beta-1}{q}$, is necessary for $\alpha\in(0,1]$ when $d=1$: For $\alpha\in(0,1]$ by employing $\mathrm{d}\mu(x) = |x|^{\alpha-1}\mathrm{d}x$, instead of $\mathrm{d}x$, one may reach to

$$\begin{aligned}\|S_t^m f(\varrho(x,t,\theta)\|_{L_x^q(\mathbb{I},\,\mathrm{d}\mu)L_t^\infty(\mathbb{I})L_\theta^\infty(\Theta)} &\gtrsim \lambda\left(\sum_{j=1}^{2^k}\int_{\Omega_{k,j}}\mathrm{d}\mu(x)\right)^{\frac1q}\\ &\sim \lambda\left(\sum_{j=1}^{2^k}\left[(y_{k,j}+\frac{1}{\lambda})^\alpha - y_{k,j}^\alpha\right]\right)^{\frac1q}.\end{aligned}$$

Here, $(y_{k,j})_j$ represents the left-end points of intervals consisting of k-th generation of pre-Cantor set $\mathfrak{C}_k(r)$; i.e. $\Omega_{k,j} = [y_{k,j}, y_{k,j}+\frac{1}{\lambda}]$. Then, the mean value theorem gives that

$$(y_{k,j}+\frac{1}{\lambda})^\alpha - y_{k,j}^\alpha \gtrsim \frac{1}{\lambda},$$

which clearly gives what we claimed.

There are many other variations of the maximal inequality for the (fractional) Schrödinger equation. In the classical higher dimensional cases, the maximal inequality (1.2) for radial initial data was considered by Prestini [48], and later its fractal dimension of the divergence sets was computed by Bennett–Rogers [3]. Some results when $m\in(0,1]$ are also known but appeared to possess a rather different nature than when $m>1$ (the reader may visit [17, 18, 29, 30, 50, 61]). The periodic setting (replacing $\mathbb{R}^d$ by the torus $\mathbb{T}^d$) is also intriguing. Moyua–Vega [45] considered the one-dimensional case and obtained some sufficient conditions and necessary conditions, although there is still a gap between them remaining. For the results in the higher dimensions, for example, see work by Wang–Zhang [62], Eceizabarrena–Lucà [27] as well as Compaan–Lucà–Staffilani [19], where they also discuss the pointwise convergence problem for the solution to certain nonlinear Schrödinger equation. Another interesting variation is due to Bez–Lee–Nakamura [5]; the local maximal inequality for orthonormal systems of initial data $(f_j)_j$. Their results in one spatial dimension teach the pointwise convergence behavior of finitely many fermion particles interacting with each other on a line. Bez–Kinoshita and the second author further computed the Hausdorff dimensions of the corresponding divergence sets as well [4]. Recently, Dimou and Seeger [21] considered pointwise convergence problem along a time sequence that rapidly approaches to 0. It turns out that such sequential convergence may require less smooth regularity than the original convergence. They obtained sharp results for the fractional Schrödinger equations in one dimension, which were extended to higher dimensions by Sjölin [54], Sjölin and Strömberg [55, 56, 57] and Li, Wang, and Yan [38]. Later, Li, Wang, and Yan [39] and Ko, Koh, Lee and first author [11] established sharp results for the fractional Schrödinger equations and more general dispersive equations defined in higher dimensions by using spacial localization.

§ Appendix A.

Recall the r-Cantor set whose construction was given in Section 4.

Lemma Appendix A.1. *For* $r \in (0, \frac{1}{2})$,

$$\dim_M \mathfrak{C}(r) = \frac{\log 2}{\log 1/r},$$

ranged in $(0,1)$.

Proof. Since $\mathfrak{C}_k(r)$ consists of 2^k disjoint intervals of length r^k, we have $N_{r^k}(\mathfrak{C}(r)) \le N_{r^k}(\mathfrak{C}_k(r)) = 2^k$ from which it follows that

$$\limsup_{k\to\infty} \frac{\log N_{r^k}(\mathfrak{C}(r))}{-\log 1/r^k} \le \limsup_{k\to\infty} \frac{\log 2^k}{\log 1/r^k} = \frac{\log 2}{\log 1/r}.$$

On the other hand, by recalling the construction, one can find, at least, a point in $\mathfrak{C}_k(r) \cap \mathfrak{C}(r)$ which is not covered by any 2^k intervals, where the length of each interval is δ satisfying $r^{k+1} \le \delta < r^k$. Therefore, $N_\delta(\mathfrak{C}(r)) \ge 2^k$ holds, and

$$\liminf_{\delta\to 0} \frac{\log N_\delta(\mathfrak{C}(r))}{-\log \delta} \ge \liminf_{k\to\infty} \frac{\log 2^k}{\log 1/r^{k+1}} = \frac{\log 2}{\log 1/r}.$$

Hence, the limit exists and equals

$$\dim_M \mathfrak{C}(r) = \lim_{\delta\to 0} \frac{\log N_\delta(\mathfrak{C}(r))}{-\log \delta} = \frac{\log 2}{\log 1/r}. \tag{Appendix A.1}$$

□

§ Appendix B.

In this section, we note the useful lemmas for the sufficiency in Section 4.

Lemma Appendix B.1 (van der Corput's lemma). *Let a, b be real numbers with $a < b$, ϕ be a sufficiently smooth real-valued function, and ψ be a bounded smooth complex-valued function. Suppose that $|\phi^{(k)}(\xi)| \geq 1$ for all $\xi \in [a, b]$. If $k = 1$ and $\phi'(\xi)$ is monotonic on (a, b), or simply $k \geq 2$, then there exists a constant C_k such that*

$$\left| \int_a^b e^{i\lambda\phi(\xi)} \psi(\xi) \, \mathrm{d}\xi \right| \leq C_k \lambda^{-\frac{1}{k}} \left(\|\psi'\|_{L^1[a,b]} + \|\psi\|_{L^\infty[a,b]} \right)$$

for all $\lambda > 0$.

For a proof of Lemma Appendix B.1, we refer the reader to [59]. Next we introduce a Young convolution/Hardy–Littlewood–Sobolev type-inequalities, which generalizes lemmas in [51, 16].

Lemma Appendix B.2. *Let $0 < \alpha \leq 1$, $q \geq 2$ and μ be an α-dimensional measure. There exists a constant C such that for any interval $[a, b]$ in $\mathbb{R}$ it holds that*

(Appendix B.1)
$$\left| \iint \iint g(x,t) h(x',t') \chi_{[a,b]}(x - x') \, \mathrm{d}\mu(x) \mathrm{d}t \mathrm{d}\mu(x') \mathrm{d}t' \right| \leq C (b-a)^{\frac{2\alpha}{q}} \|g\|_{L_x^{q'}(\mathrm{d}\mu) L_t^1} \|h\|_{L_x^{q'}(\mathrm{d}\mu) L_t^1}.$$

Moreover, if $0 < \frac{q\rho}{2} < \alpha$ then there exists a constant C such that

(Appendix B.2)
$$\left| \iint \iint g(x,t) h(x',t') |x - x'|^{-\rho} \, \mathrm{d}\mu(x) \mathrm{d}t \mathrm{d}\mu(x') \mathrm{d}t' \right| \leq C \|g\|_{L_x^{q'}(\mathrm{d}\mu) L_t^1} \|h\|_{L_x^{q'}(\mathrm{d}\mu) L_t^1}.$$

Here, the both integrals are taken over (x, t), $(x', t') \in \mathbb{I} \times \mathbb{I}$.

This lemma is a generalization of Lemma 4 in [51] and Lemma 7 in [16].

Proof. In order to show (Appendix B.1), it is enough to show that

$$\|g *_\mu \chi_{[a,b]}\|_{L^q(\mathrm{d}\mu)} \lesssim (b-a)^{\frac{2\alpha}{q}} \|g\|_{L^{q'}(\mathrm{d}\mu)}.$$

By applying Hölder's inequality with $\frac{1}{q}+\frac{1}{q'}=1$,

$$\left(\int\left|\int g(x')\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x')\right|^q\,\mathrm{d}\mu(x)\right)^{\frac{1}{q}}$$
$$\leq (b-a)^{\frac{\alpha}{q}}\left(\int\left|\int |g(x')|^{q'}\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x')\right|^{\frac{q}{q'}}\,\mathrm{d}\mu(x)\right)^{\frac{1}{q}},$$

which is further bounded, as a result of Minkowski's inequality since $\frac{q}{q'}\geq 1$, from above by

$$(b-a)^{\frac{\alpha}{q}}\left(\int\left(\int |g(x')|^{q}\chi_{[a,b]}(x-x')\,\mathrm{d}\mu(x)\right)^{\frac{q'}{q}}\,\mathrm{d}\mu(x')\right)^{\frac{1}{q'}}\sim (b-a)^{\frac{2\alpha}{q}}\|g\|_{L^{q'}(\mathrm{d}\mu)}.$$

Then, showing (Appendix B.2) is rather easy via (Appendix B.1) as follows:

$$\left|\iint\iint g(x,t)h(x',t')|x-x'|^{-\rho}\,\mathrm{d}\mu(x)\mathrm{d}t\mathrm{d}\mu(x')\mathrm{d}t'\right|$$
$$\lesssim \sum_{j=0}^{\infty}2^{\rho j}\iint G(x)H(x')\chi_{[2^{-j},2^{-j+1}]}(x-x')\,\mathrm{d}\mu(x)\mathrm{d}\mu(x')$$
$$\lesssim \sum_{j=0}^{\infty}2^{(\rho-\frac{2\alpha}{q})j}\|G\|_{L_x^{q'}(\mathrm{d}\mu)}\|H\|_{L_x^{q'}(\mathrm{d}\mu)}$$
$$\lesssim \|G\|_{L_x^{q'}(\mathrm{d}\mu)}\|H\|_{L_x^{q'}(\mathrm{d}\mu)}$$

whenever $\rho-\frac{2\alpha}{q}<0$. □

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References

[1] C. An, R. Chu, L. B. Pierce, *Counterexamples for high-degree generalizations of the Schrödinger maximal operator*, Int. Math. Res. Not. IMRN. **10** (2023), 8371–8418.

[2] J. A. Barceló, J. Bennett, A. Carbery, K. M. Rogers, *On the dimension of divergence sets of dispersive equations*, Math. Ann. **348** (2011), 599–622.

[3] J. Bennett, K. M. Rogers, *On the size of divergence sets for the Schrödinger equation with radial data*, Indiana Univ. Math. J. **61** (2012), 1–13.

[4] N. Bez, S. Kinoshita, S. Shiraki, *Boundary Strichartz estimates and pointwise convergence for orthonormal systems*, arXiv:2306.14536.

[5] N. Bez, S. Lee, S. Nakamura, *Maximal estimates for the Schrödinger equation with orthonormal initial data*, Selecta Math. **26** (2020), article number 52.

[6] J. Bourgain, *Some new estimates on oscillatory integrals*, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991) (1995), 83–112.

[7] J. Bourgain, *On the Schrödinger maximal function in higher dimension*, Tr. Mat. Inst. Steklova **280** (2013), 46–60.

[8] J. Bourgain, *A note on the Schrödinger maximal function*, J. Anal. Math. **130** (2016), 393–396.

[9] L. Carleson, *Some analytic problems related to statistical mechanics*, In: Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), Lecture Notes in Math. **779** (1980), 5–45.

[10] C. H. Cho, H. Ko, *Pointwise convergence of the fractional Schrödinger equation in* $\mathbb{R}^2$, Taiwanese J. Math. **26** (2022), 177–200.

[11] C. H. Cho, H. Ko, Y. Koh, S. Lee, *Pointwise convergence of sequential Schrödinger means*, J. Inequal. Appl. **54** (2023), 14 pp.

[12] C. H. Cho, Y. Koh, J. Lee, *A global space-time estimate for dispersive operators through its local estimate*, J. Math. Anal. Appl. **514** (2022), 15 pp.

[13] C. H. Cho, Y. Koh, I. Seo, *On inhomogeneous Strichartz estimates for fractional Schrödinger equations and their applications*, Discrete Contin. Dyn. Syst. **36** (2016), 1905–1926.

[14] C. H. Cho, S. Lee, *Dimension of divergence sets for the pointwise convergence of the Schrödinger equation*, J. Math. Anal. Appl. **411** (2014), 254–260.

[15] C. H. Cho, S. Lee, A. Vargas, *Problems on pointwise convergence of solutions to the Schrödinger equation*, J. Fourier Anal. Appl. **18** (2012), 972–994.

[16] C. H. Cho, S. Shiraki, *Pointwise convergence along a tangential curve for the fractional Schrödinger equations*, Ann Fenn Math. **46** (2021), 993–1005.

[17] C. H. Cho, S. Shiraki, *Dimension of divergence sets of oscillatory integrals with concave phase*, arXiv:2212.14330.

[18] M. G. Cowling, *Pointwise behavior of solutions to Schrödinger equations*, Harmonic Analysis (Cortona, 1982), Lecture Notes in Math. **992** (1983), 83–90.

[19] E. Compaan, R. Lucà, G. Staffilani, *Pointwise convergence of the Schrödinger flow*, to appear in Int. Math. Res. Not.

[20] B. E. J. Dahlberg, C. E. Kenig, *A note on the almost everywhere behavior of solutions to the Schrödinger equation.* In: Harmonic analysis (Minneapolis, Minn., 1981), Lecture Notes in Math. **908** (1982), 205–209.

[21] E. Dimou, A. Seeger, *On pointwise convergence of Schrödinger means*, Mathematika. **66** (2020), 356–372.

[22] X. Du, J. Kim, H. Wang, R. Zhang, *Lower bounds for estimates of the Schrödinger maximal function*, Math. Res. Lett. **27** (2020), 687–692.

[23] X. Du, L. Guth, X. Li, *A sharp Schrödinger maximal estimate in* $\mathbb{R}^2$, Ann. of Math. **186** (2017), 607–640.

[24] X. Du, L. Guth, X. Li, R. Zhang, *Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates* Forum Math. Sigma **6** (2018), 18 pp.

[25] X. Du, R. Zhang, *Sharp L^2 estimate of Schrödinger maximal function in higher dimensions*, Ann. of Math. **189** (2019), 837–861.

[26] J. Duoandikoetxea, *Fourier Analysis*, Grad. Studies Math. **29**, Amer. Math. Soc., Providence, 2000.

[27] D. Eceizabarrena, R. Lucà, *Convergence over fractals for the periodic Schrödinger equation*, Anal. PDE **15** (2022), 1775–1805.

[28] D. Eceizabarrena, F. Ponce-Vanegas, *Pointwise convergence over fractals for dispersive equations with homogeneous symbol*, J. Math. Anal. Appl. **515** (2022), 126385.

[29] S. Ham, H. Ko, S. Lee, *Dimension of divergence set of the wave equation*, Nonlinear Anal. **215** (2022), 112631.

[30] T. Jhao, J. Yuan, *Pointwise convergence along a tangential curve for the fractional Schrödinger equation with* $0 < m < 1$, Math. Methods Appl. Sci. **45** (2022), 456–467.

[31] K. Johansson, *A counterexample on nontangential convergence for oscillatory integrals*, Publ. Inst. Math. (Beograd) (N.S.) **87** (2010), 129–137.

[32] C. E. Kenig, G. Ponce, L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33–69.

[33] S. Lee, *On pointwise convergence of the solutions to Schrödinger equations in* $\mathbb{R}^2$, Int. Math. Res. Not. (2006), 1–21.

[34] S. Lee, K. Rogers, *The Schrödinger equation along curves and the quantum harmonic oscillator*, Adv. Math. **229** (2012), 1359–1379.

[35] W. Li, H. Wang, *Pointwise convergence of solutions to the Schrödinger equation along a class of curves*, arXiv: 1807.00292.

[36] W. Li, H. Wang, *On convergence properties for generalized Schrödinger equation along tangential curves*, arXiv:2111.09186.

[37] W. Li, H. Wang, D. Yan, *A note on Non-tangential Convergence for Schrödinger operators*, J. Fourier Anal. Appl. **27** (2021), 14 pp.

[38] W. Li, H. Wang, D. Yan, *Pointwise Convergence for sequences of Schrödinger means in* $\mathbb{R}^2$, arXiv:2010.08701.

[39] W. Li, H. Wang, D. Yan, *Sharp convergence for sequences of Schrödinger means and related generalizations*, to appear in Proc. Roy. Soc. Edinburgh Sect. A

[40] R. Lucà, P. Ponce-Vanegas, *Convergence over fractals for the Schrödinger equation*, Indiana Univ. Math. J. **71** (2022), 2283–2307.

[41] R. Lucà, K. M. Rogers, *Coherence on fractals versus pointwise convergence for the Schrödinger equation*, Comm. Math. Phys. **351** (2017), 341–359.

[42] R. Lucà, K. M. Rogers, *A note on pointwise convergence for the Schrödinger equation*, Math. Proc. Cambridge Philos. Soc. **166** (2019), 209–218.

[43] R. Lucà, K. M. Rogers, *Average decay of the Fourier transform of measures with applications*, J. Eur. Math. Soc. (JEMS) **21** (2019), 465–506.

[44] C. Miao, J. Yang, J. Zheng, *An improved maximal inequality for* $2D$ *fractional order Schrödinger operators*, Studia Mathematica. **230** (2015), 121–165.

[45] A. Moyua, L. Vega. *Bounds for the maximal function associated to periodic solutions of one-dimensional dispersive equations*, Bull. Lond. Math. Soc. **40** (2008), 117–128.

[46] L. B. Pierce, *On Bourgain's Counterexample for the Schrödinger Maximal Function*, Q. J. Math. **71** (2020), 1309–1344.

[47] F. Planchon, *Dispersive estimates and the 2D cubic NLS equation*, J. Anal. Math. **86**

(2002), 319–334.
[48] E. Prestini, *Radial functions and regularity of solutions to the Schrödinger equation*, Monatsh. Math. **109** (1990), 135–143.
[49] K. M. Rogers, *Strichartz estimates via the Schrödinger maximal operator*, Math. Ann. **343** (2009), 603–622.
[50] K. M. Rogers, P. Villarroya, *Sharp estimates for maximal operators associated to the wave equation*, Ark. Mat. **46** (2008), 143–151.
[51] S. Shiraki, *Pointwise convergence along restricted directions for the fractional Schrödinger equation*, J. Fourier Anal. Appl. **26** (2020), 12 pp.
[52] P. Sjögren, P. Sjölin, *Convergence properties for the time dependent Schrödinger equation*, Ann. Acad. Sci. Fenn. A I Math. **14** (1989), 13–25.
[53] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), 699–715.
[54] P. Sjölin, *Two Theorems on Convergence of Schrödinger Means*, J. Fourier Anal. Appl. **25** (2019), 1708–1716.
[55] P. Sjölin, J-O. Strömberg, *Convergence of sequences of Schrödinger means*, J. Math. Anal. Appl. **483** (2020), 23 pp.
[56] P. Sjölin, J-O. Strömberg, *Schrödinger means in higher dimensions*, J. Math. Anal. Appl. **504**, 125353, 32 pp (2021).
[57] P. Sjölin, J-O. Strömberg, *Analysis of Schrödinger means*, Ann. Fenn. Math. **46**, 389–394 (2021).
[58] E. M. Stein, *On limits of sequences of operators*, Ann. Math. **74** (1961), 140–170.
[59] E. M. Stein, *Harmonic Analysis*, Princeton Univ. Press, Princeton (1993).
[60] L. Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), 874–878.
[61] B. G. Walther, *Maximal estimates for oscillatory integrals with concave phase*, Contemp. Math. **189** (1995), 485–495.
[62] X. Wang, C. Zhang, *Pointwise convergence of solutions to the Schrödinger equation on manifolds*, Canad. J. Math. **71** (2019), 983–995.
[63] D. Žubrinić, *Singular sets of Sobolev functions*, C. R. Math. Acad. Sci. Paris **334** (2002), 539–544.