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Remark on blow-up of the threshold solutions to the nonlinear Schrödinger equation with the repulsive Dirac delta potential

By

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Abstract

We consider the focusing L^2-supercritical nonlinear Schrödinger equation with the repulsive Dirac delta potential. The global dynamics below the ground state standing waves was obtained by Ikeda and the author [18]. Recently, Ardila and the author [3] gave a sufficient condition for the threshold solutions to scatter. In the present paper, we are interested in a sufficient condition for the threshold solutions to blow up.

§ 1. Introduction

§ 1.1. Motivation

The following type equation appears in a wide variety of physical models with a point defect on the line (see [15] and references therein).

$$i\partial_t u + \partial_x^2 u + \gamma\delta_0 u + \kappa|u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1.1}$$

where $\gamma \in \mathbb{R}$, δ_0 denotes the Dirac delta with mass at the origin, $\kappa = \pm 1$, and $p > 1$. The operator $H_\gamma := -\partial_x^2 - \gamma\delta_0$ is defined by

$$\begin{aligned}
&\mathcal{D}(H_\gamma) := \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\setminus\{0\}) : \partial_x f(0+) - \partial_x f(0-) = -\gamma f(0)\},\\
&H_\gamma f := -\partial_x^2 f, \quad f \in \mathcal{D}(H_\gamma).
\end{aligned}$$

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Then, it is known that the operator H_γ is self-adjoint on $L^2(\mathbb{R})$ (see [2]) and thus the Schrödinger evolution group e^{-itH_γ} is well defined. The operator H_γ satisfies $\sigma_p(H_\gamma) = \emptyset$ if $\gamma \leq 0$ and $\sigma_p(H_\gamma) = \{-\frac{\gamma^2}{4}\}$ if $\gamma > 0$. Thus, the potential is called repulsive if $\gamma < 0$ and attractive if $\gamma > 0$.

When $\gamma = 0$, which means the equation (1.1) is just the nonlinear Schrödinger equation without linear potential, the equation is invariant under the scaling: $u_\alpha(t,x) := \alpha^{\frac{2}{p-1}} u(\alpha^2 t, \alpha x)$ for $\alpha > 0$. That is, u_α is a solution if and only if u is a solution. This scaling does not change $\dot{H}^{s_c}(\mathbb{R})$-norm where $s_c = 1/2 - 2/(p-1)$. Indeed, we have $\|\alpha^{\frac{2}{p-1}} u(0, \alpha \cdot)\|_{\dot{H}^{s_c}} = \|u(0,\cdot)\|_{\dot{H}^{s_c}}$. If $p > 5$, then $s_c > 0$ and thus the condition is called L^2-supercritical (or mass-supercritical). On the other hand, it is called L^2-subcritical if $p < 5$ and L^2-critical if $p = 5$. Even if $\gamma \neq 0$, we use these notions.

The equation (1.1) has a Hamiltonian form: $i\partial_t u = \mathcal{H}'(u)$, where $\mathcal{H}(u) := \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{\gamma}{2}|u(0)|^2 - \frac{\kappa}{p+1}\|u\|_{L^{p+1}}^{p+1}$. If $\kappa = -1$, then the nonlinearity of the Hamiltonian $\mathcal{H}$ is positive and thus its case is called defocusing. On the other hand, if $\kappa = 1$, it is a negative term and thus it is called focusing.

In the present paper, we are interested in the global behavior of the solutions to the focusing L^2-supercritical nonlinear Schrödinger equation with the repulsive Dirac delta potential:

$$(\delta\text{NLS}) \qquad \begin{cases} i\partial_t u + \partial_x^2 u + \gamma\delta_0 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $\gamma < 0$ and $p > 5$.

It is known (see [14, Section 2], [7, Theorem 3.7.1]) that the equation (δNLS) is locally well-posed in $H^1(\mathbb{R})$ and it conserves the following quantities:

$$(\text{Mass}) \qquad M(u(t)) := \frac{1}{2}\|u(t)\|_{L^2}^2,$$

$$(\text{Energy}) \qquad E_\gamma(u(t)) := \frac{1}{2}\|\partial_x u(t)\|_{L^2}^2 - \frac{\gamma}{2}|u(t,0)|^2 - \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}.$$

In the previous paper [18], we investigated the global dynamics of the solutions below the standing wave solutions. Before stating the previous result, let us recall the definitions of scattering and blow-up. Let u be a solution to (δNLS) on the maximal existence time interval $(-T_-, T_+)$.

Definition 1.1 (scattering). We say that the solution u to (δNLS) scatters if and only if $T_\pm = \infty$ and there exist $u_\pm \in H^1(\mathbb{R})$ such that $\|u(t) - e^{-itH_\gamma}u_\pm\|_{H^1} \to 0$ as $t \to \pm\infty$.

Remark 1. By Mizutani [21], if the solution u scatters, then the solution goes to a free solution. That is, there exist $v_\pm \in H^1(\mathbb{R})$ such that $\|u(t) - e^{it\partial_x^2}v_\pm\|_{H^1} \to 0$ as $t \to \pm\infty$.

Definition 1.2 (blow-up). We say that the solution u to (δNLS) blows up in positive time (resp. negative time) if and only if $T_+ < \infty$ (resp. $T_- < \infty$).

From the view point of the global dynamics, the ground state solution of the nonlinear Schrödinger equation without potential plays an important role. It is denoted by Q and defined by

$$Q(x) := \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} |x| \right) \right\}^{\frac{1}{p-1}},$$

which is a unique positive solution of

$$-\partial_x^2 Q + Q = Q^p, \quad x \in \mathbb{R}. \tag{1.2}$$

Indeed, Ikeda and the author [18] classified the solutions whose mass-energy is below the ground state solution Q. To state it, we define some notations:

$$P(\phi) = P_\gamma(\phi) := \|\partial_x \phi\|_{L^2}^2 - \frac{\gamma}{2}|\phi(0)|^2 - \frac{p-1}{2(p+1)}\|\phi\|_{L^{p+1}}^{p+1},$$

and

$$\mathcal{N}^+ := \{ f \in H^1(\mathbb{R}) : E_\gamma(f) M(f)^\sigma < E_0(Q) M(Q)^\sigma, P(f) \geq 0 \},$$
$$\mathcal{N}^- := \{ f \in H^1(\mathbb{R}) : E_\gamma(f) M(f)^\sigma < E_0(Q) M(Q)^\sigma, P(f) < 0 \},$$

where $\sigma := (p+3)/(p-5)$.

Theorem (Ikeda–Inui [18]). *Let u be a solution to (δNLS) on $(-T_-, T_+)$ with the initial data $u_0 \in H^1(\mathbb{R})$.*

1. *If the initial data u_0 belongs to $\mathcal{N}^+$, then the solution u scatters.*
2. *If the initial data u_0 belongs to $\mathcal{N}^-$, then one of the following four cases holds.*
   (a) *The solution u blows up in both time directions.*
   (b) *The solution u blows up in a positive time, and u is global toward negative time and* $\limsup_{t\to-\infty} \|\partial_x u(t)\|_{L^2} = \infty$ *holds.*
   (c) *The solution u blows up in a negative time, and u is global toward positive time and* $\limsup_{t\to\infty} \|\partial_x u(t)\|_{L^2} = \infty$ *holds.*
   (d) *The solution u is global in both time directions and* $\limsup_{t\to\pm\infty} \|\partial_x u(t)\|_{L^2} = \infty$ *holds.*

Remark 2. In the case $\gamma = 0$, the classification below the ground state is obtained by [16, 10, 17, 13, 1]. They used the argument by Kenig and Merle [19]. See also [8, 9, 4] for new proofs.

Remark 3. It is worth mentioning that Q is not the ground state of (δNLS) when $\gamma < 0$. However, $E_0(Q)M(Q)^\sigma$ is the threshold to classify the global behavior into scattering and blow-up (or grow-up). In the radial setting, we can determine the global behavior of the solutions below the ground state solutions of (δNLS). See [18].

Remark 4. Banica and Visciglia [5] considered the defocusing L^2-supercritical NLS with repulsive delta potential. In the defocusing case, the solution is uniformly bounded and there is no ground state and indeed they proved all the solutions scatter.

Remark 5. If the initial data u_0 belongs to $\mathcal{N}^-$ and satisfies $\int_{\mathbb{R}} x^2|u_0(x)|^2dx < \infty$, then the solution blows up in both time directions. This follows from the standard virial argument.

Since $e^{it}Q$ is not a solution of (δNLS), we expect that the threshold solution, i.e the solution satisfying $E_\gamma(u)M(u)^\sigma < E_0(Q)M(Q)^\sigma$, scatters or blows up. Recently, Ardila and the author [3] showed the following scattering result.

Theorem (Ardila–Inui [3]). *Let u_0 satisfy $E_\gamma(u_0)M(u_0)^\sigma = E_0(Q)M(Q)^\sigma$ and $P(u_0) \geq 0$. Then the solution u with the initial data u_0 scatters.*

Remark 6. The proof of this theorem is based on the idea originated in Duyckaerts, Landoulsi, and Roudenko [11], which is applied to NLS with a repulsive potential by Miao, Murphy, and Zheng [22]. They proved threshold scattering for NLS with general potential and NLS with inverse square potential on 3d. In [3], we combine their result and Campos, Farah, and Roudenko [6], in which they considered global dynamics of threshold solutions for NLS without potential on general dimensions including 1d. (see also Duyckaerts and Roudenko [12] for the original work.)

§ 1.2. Main result

In the present paper, we are interested in the case of $P(u_0) < 0$. Next theorem is the main result of this paper.

Theorem 1.3. *Let u_0 satisfy $\int_{\mathbb{R}} x^2|u_0(x)|^2dx < \infty$, $E_\gamma(u_0)M(u_0)^\sigma = E_0(Q)M(Q)^\sigma$, and $P(u_0) < 0$. Then the solution u with the initial data u_0 blows up in both time directions.*

Our proof is based on blow-up argument by Duyckaerts and Roudenko [12] and the modulation obtained by [3]. The strategy is as follows. Suppose that u is global in the positive time direction. First, we prove that the solution blows up in the negative time direction by the finite variance and a virial argument (see Corollary 2.8). Moreover, then, $\mu(u(t)) := \|\partial_x Q\|_{L^2}^2 - (\|\partial_x u(t)\|_{L^2}^2 - \gamma|u(t,0)|^2)$ satisfies $\int_t^\infty \mu(u(s))ds \lesssim e^{-ct}$ (see

Proposition 2.7). This ensures a modulation argument. The modulation argument is obtained in [3]. In the present paper, we additionally show that the time derivative of the parameter λ is controlled by $\mu(u(t))$ (see Proposition 2.6). This gives us the convergence of the translation parameter y. However, this derives a contradiction to the fact that $e^{-2|y(t)|}/|y(t)|^2$ converges to zero (see (2.2)).

Remark 7. In Theorem 1.3, we only consider the solution with finite variance. In general, it is still an open problem whether the solutions with infinite variance blow up in finite time. Also, in our setting, it is an open problem that the threshold solutions with infinite variance blow up or grow up.

Notation. We denote $A \lesssim B$ or $B \gtrsim A$ by $A \leq CB$ for some positive constant C. $A \sim B$ means $A \lesssim B \lesssim A$.

We often write $L^r(\mathbb{R})$ to denote the Banach space of functions $f : \mathbb{R} \to \mathbb{C}$ with the norm

$$\|f\|_{L^r} = \left(\int_{\mathbb{R}} |f(x)|^r dx \right)^{\frac{1}{r}}.$$

We also use

$$\langle f, g \rangle := \Re \int_{\mathbb{R}} f(x)\overline{g(x)}dx.$$

We denote the norm derived from $H_\gamma^{1/2}$ by

$$\|f\|_H^2 := \|H_\gamma^{\frac{1}{2}} f\|_{L^2}^2 = \|\partial_x f\|_{L^2}^2 - \gamma |f(0)|^2.$$

We define the difference between Q and the solution by

$$\mu(t) = \mu(u(t)) := \|\partial_x Q\|_{L^2}^2 - (\|\partial_x u(t)\|_{L^2}^2 - \gamma |u(t,0)|^2).$$

§ 2. Proof

§ 2.1. Variational setting

We recall the variational properties of the ground state of Q. We have the Gagliardo–Nirenberg inequality

$$\|f\|_{L^{p+1}}^{p+1} \leq C_{GN} \|\partial_x f\|_{L^2}^{\frac{p-1}{2}} \|f\|_{L^2}^{\frac{p+3}{2}}$$

for $f \in H^1(\mathbb{R})$, where the optimal constant C_{GN} is attained by Q and thus

$$C_{GN}^{-1} = \frac{\|\partial_x Q\|_{L^2}^{\frac{p-1}{2}} \|Q\|_{L^2}^{\frac{p+3}{2}}}{\|Q\|_{L^{p+1}}^{p+1}}.$$

By the Pohozaev identity, we have

$$E_0(Q) = \frac{p-5}{2(p+3)}\|Q\|_{L^2}^2 = \frac{p-5}{2(p-1)}\|\partial_x Q\|_{L^2}^2 = \frac{p-5}{4(p+1)}\|Q\|_{L^{p+1}}^{p+1}.$$

It follows from the Gagliardo–Nirenberg inequality that

$$\|f\|_{L^{p+1}}^{p+1} < C_{GN}\left(\|\partial_x f\|_{L^2}^2 - \gamma|f(0)|^2\right)^{\frac{p-1}{4}}\|f\|_{L^2}^{\frac{p+3}{2}} \tag{2.1}$$

for $f \in H^1(\mathbb{R}) \setminus \{0\}$. We emphasize that the equality in (2.1) does not hold since the constant is not attained.

Lemma 2.1. *Let f satisfy $E_\gamma(f)M(f)^\sigma = E_0(Q)M(Q)^\sigma$, where we recall $\sigma = (p+3)/(p-5)$. We have $P(f) < 0$ if and only if $\|f\|_H\|f\|_{L^2}^\sigma > \|\partial_x Q\|_{L^2}\|Q\|_{L^2}^\sigma$.*

Proof. See [3, Lemma 2.6]. □

Lemma 2.2. *If the initial data u_0 satisfies $E_\gamma(u_0)M(u_0)^\sigma = E_0(Q)M(Q)^\sigma$ and $P(u_0) < 0$, then the solution u satisfies $P(u(t)) < 0$ for all existence time.*

Proof. We give a proof for reader's convenience though the proof is well known. Suppose that there exists a time t_0 such that $P(u(t_0)) \geq 0$. Then, we have

$$\|u(t_0)\|_H\|u(t_0)\|_{L^2}^\sigma \leq \|\partial_x Q\|_{L^2}\|Q\|_{L^2}^\sigma.$$

By the continuity of the solution, we may take a time t_1 such that $\|u(t_1)\|_H\|u(t_1)\|_{L^2}^\sigma = \|\partial_x Q\|_{L^2}\|Q\|_{L^2}^\sigma$. The inequality (2.1) implies that

$$E_\gamma(f)M(f)^\sigma > \frac{1}{2}\|f\|_H^2\|f\|_{L^2}^{2\sigma} - \frac{C_{GN}}{p+1}\|f\|_H^{\frac{p-1}{2}}\|f\|_{L^2}^{\frac{p+3}{2}+2\sigma}$$

for any $f \in H^1(\mathbb{R})\setminus\{0\}$. Setting $F(a) := \frac{1}{2}a^2 - \frac{C_{GN}}{p+1}a^{\frac{p-1}{2}}$, the right hand side is denoted by $F(\|u\|_H\|u\|_{L^2}^\sigma)$. It holds that F is increasing on $(0, a_1)$ and decreasing on (a_1, ∞), where $a_1 = \|\partial_x Q\|_{L^2}\|Q\|_{L^2}^\sigma$ (see e.g. [13, Lemma 3.1]). We have

$$F(a_1) = F(\|\partial_x Q\|_{L^2}\|Q\|_{L^2}^\sigma) = \frac{p-5}{2(p-1)}(\|\partial_x Q\|_{L^2}\|Q\|_{L^2}^\sigma)^2 = E_0(Q)M(Q)^\sigma$$

Therefore, we obtain $E_\gamma(u(t_1))M(u(t_1))^\sigma > F(\|u(t_1)\|_H\|u(t_1)\|_{L^2}^\sigma) = E_0(Q)M(Q)^\sigma$. This is a contradiction since the energy and mass are conserved. □

§ 2.2. Lemmas

To prove Theorem 1.3, it is enough to show the following proposition.

Proposition 2.3. *Let u_0 satisfy $\int_{\mathbb{R}} x^2|u_0(x)|^2dx < \infty$, $M(u_0) = M(Q)$, $E_\gamma(u_0) = E_0(Q)$, and $P(u_0) < 0$. Then the solution u with the initial data u_0 blows up in both time directions.*

Proof of Theorem 1.3 under Proposition 2.3. Suppose that Proposition 2.3 is true for any $\gamma < 0$. Consider $u_0 \in H^1(\mathbb{R})$ such that

$$E_\gamma(u_0)M(u_0)^\sigma = E_0(Q)M(Q)^\sigma \quad \text{and} \quad P_\gamma(u_0) < 0.$$

We set $\gamma^* = \alpha\gamma$, $u_{0,\alpha}(x) = \alpha^{\frac{2}{(p-1)}}u_0(\alpha x)$ and $u_\alpha(t,x) = \alpha^{\frac{2}{p-1}}u(\alpha^2 t, \alpha x)$, where $\alpha^{\frac{p-5}{p-1}} = \frac{M(u_0)}{M(Q)}$. Notice that

$$E_{\gamma^*}(u_{0,\alpha}) = \alpha^{\frac{p+3}{p-1}}E_\gamma(u_0) \quad \text{and} \quad P_{\gamma^*}(u_{0,\alpha}) = \alpha^{\frac{p+3}{p-1}}P_\gamma(u_0).$$

Thus, since $\alpha^{\frac{p+3}{p-1}} = (\frac{M(u_0)}{M(Q)})^\sigma$, we obtain

$$E_{\gamma^*}(u_{0,\alpha}) = E_0(Q), \quad M(u_{0,\alpha}) = M(Q) \quad \text{and} \quad P_{\gamma^*}(u_{0,\alpha}) < 0.$$

Then the function u_α satisfies

$$i\partial_t u_\alpha + \partial_x^2 u_\alpha + \gamma^*\delta_0 u_\alpha + |u_\alpha|^{p-1}u_\alpha = 0.$$

Thus, u_α blow up in both time directions by Proposition 2.3. This means that u also blows up in both time directions. □

The following inequality is one type of the inequality of Claim 5.4 in [12].

Lemma 2.4. *Let $\varphi \in C^1(\mathbb{R};\mathbb{R})$ and $f \in H^1(\mathbb{R})$. Assume that $\int_{\mathbb{R}} |\partial_x\varphi|^2|f|dx < \infty$, $M(f) = M(Q)$, and $E_\gamma(f) = E_0(Q)$. Then we have the following inequality.*

$$\left|\Im \int_{\mathbb{R}} \partial_x\varphi(x)\partial_x f(x)\overline{f(x)}dx\right| \lesssim |\mu(f)|^2 \int_{\mathbb{R}} |\partial_x\varphi(x)|^2|f(x)|^2dx.$$

Proof. Let $\lambda \in \mathbb{R}$. By (2.1), we have

$$\|f\|_{L^{p+1}}^{p+1} = \|e^{i\lambda\varphi}f\|_{L^{p+1}}^{p+1} \le C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}(\|(e^{i\lambda\varphi}f)'\|_{L^2}^2 - \gamma|f(0)|^2)^{\frac{p-1}{4}}.$$

Now, we have

$$\|(e^{i\lambda\varphi}f)'\|_{L^2}^2 = \lambda^2\|\varphi'f\|_{L^2}^2 + 2\lambda\Im\int_{\mathbb{R}}\varphi'f'\overline{f}dx + \|f'\|_{L^2}^2.$$

Therefore, we obtain

$$\|f\|_{L^{p+1}}^{p+1} \le C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}\left(\lambda^2\|\varphi'f\|_{L^2}^2 + 2\lambda\Im\int_{\mathbb{R}}\varphi'f'\overline{f}dx + \|f'\|_{L^2}^2 - \gamma|f(0)|^2\right)^{\frac{p-1}{4}}.$$

This means that

$$\lambda^2\|\varphi' f\|_{L^2}^2 + 2\lambda\Im\int_{\mathbb{R}}\varphi' f'\overline{f}dx + \|f'\|_{L^2}^2 - \gamma|f(0)|^2 - \left(\frac{\|f\|_{L^{p+1}}^{p+1}}{C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}}\right)^{\frac{4}{p-1}} \geq 0$$

This is the inequality related to the quadratic equation for λ. Thus, we obtain

$$\left|\Im\int_{\mathbb{R}}\varphi' f'\overline{f}dx\right|^2 \leq \|\varphi' f\|_{L^2}^2\left\{\|f'\|_{L^2}^2 - \gamma|f(0)|^2 - \left(\frac{\|f\|_{L^{p+1}}^{p+1}}{C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}}\right)^{\frac{4}{p-1}}\right\}.$$

We calculate

$$\|f\|_H^2 - \left(\frac{\|f\|_{L^{p+1}}^{p+1}}{C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}}\right)^{\frac{4}{p-1}} = \|\partial_x Q\|_{L^2}^2 - \mu(f) - \left(\frac{\|f\|_{L^{p+1}}^{p+1}}{C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}}\right)^{\frac{4}{p-1}}.$$

Now, it holds from $E_\gamma(f) = E_0(Q)$ that $\|f\|_{L^{p+1}}^{p+1} = \|Q\|_{L^{p+1}}^{p+1} - \frac{p+1}{2}\mu(f)$ and thus we obtain

$$\begin{aligned}&\|\partial_x Q\|_{L^2}^2 - \mu(f) - \left(\frac{\|f\|_{L^{p+1}}^{p+1}}{C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}}\right)^{\frac{4}{p-1}} \\ &= \|\partial_x Q\|_{L^2}^2 - \mu(f) - \left(\|Q\|_{L^{p+1}}^{p+1} - \frac{p+1}{2}\mu(f)\right)^{\frac{4}{p-1}}(C_{GN}\|Q\|_{L^2}^{\frac{p+3}{2}})^{-\frac{4}{p-1}}.\end{aligned}$$

Then, by the Taylor expansion and

$$(C_{GN}\|Q\|_{L^2}^{\frac{p+3}{2}})^{-\frac{4}{p-1}} = \frac{p-1}{2(p+1)}\|Q\|_{L^{p+1}}^{\frac{(p-5)(p+1)}{p-1}},$$

we obtain

$$\begin{aligned}&\|\partial_x Q\|_{L^2}^2 - \mu(f) - \left(\|Q\|_{L^{p+1}}^{p+1} - \frac{p+1}{2}\mu(f)\right)^{\frac{4}{p-1}}(C_{GN}\|Q\|_{L^2}^{\frac{p+3}{2}})^{-\frac{4}{p-1}} \\ &\leq \|\partial_x Q\|_{L^2}^2 - \mu(f) - \left(\|Q\|_{L^{p+1}}^{\frac{4(p+1)}{p-1}} - \frac{2(p+1)}{p-1}\|Q\|_{L^{p+1}}^{-\frac{(p-5)(p+1)}{p-1}}\mu(f) - C|\mu(f)|^2\right)(C_{GN}\|Q\|_{L^2}^{\frac{p+3}{2}})^{-\frac{4}{p-1}} \\ &= C'|\mu(f)|^2\end{aligned}$$

Combining these estimates, we get

$$\begin{aligned}\left|\Im\int_{\mathbb{R}}\varphi' f'\overline{f}dx\right|^2 &\leq \|\varphi' f\|_{L^2}^2\left\{\|f\|_H^2 - \left(\frac{\|f\|_{L^{p+1}}^{p+1}}{C_{GN}\|f\|_{L^2}^{\frac{p+3}{2}}}\right)^{\frac{4}{p-1}}\right\} \\ &\lesssim \|\varphi' f\|_{L^2}^2|\mu(f)|^2\end{aligned}$$

This completes the proof. □

Corollary 2.5. *Under the assumption of Lemma 2.4 with $P(f) < 0$, we have*

$$\left|\Im \int_{\mathbb{R}} \partial_x \varphi(x) \partial_x f(x) \overline{f(x)} dx\right| \lesssim |P(f)|^2 \int_{\mathbb{R}} |\partial_x \varphi(x)|^2 |f(x)|^2 dx.$$

Proof. We have

$$\begin{aligned} P(f) &= \frac{p-1}{2} E_\gamma(f) - \frac{p-5}{4} \|f\|_H^2 + \frac{\gamma}{2} |f(0)|^2 \\ &\leq \frac{p-1}{2} E_0(Q) - \frac{p-5}{4} \|f\|_H^2 \\ &= \frac{p-5}{4} \|\partial_x Q\|_{L^2}^2 - \frac{p-5}{4} \|f\|_H^2 \\ &= \frac{p-5}{4} \mu(f). \end{aligned}$$

Since we have $\|f\|_H^2 \|f\|_{L^2}^{2\sigma} > \|\partial_x Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma}$ by Lemma 2.1 and $M(f) = M(Q)$, we have $\|f\|_H^2 > \|\partial_x Q\|_{L^2}^2$, i.e. $\mu(f) < 0$. Therefore, it holds that $|\mu(f)|^2 \lesssim |P(f)|^2$. This completes the proof. □

§ 2.3. Modulation

Let u be a solution and global in positive time direction. For small $\mu_0 > 0$, we define the set

$$I_0 = \{t \in [0, \infty) : |\mu(t)| < \mu_0\},$$

where we recall $\mu(t) = \mu(u(t))$ for a solution.

The following proposition is essentially obtained in the previous paper [3, Proposition 4.1].

Proposition 2.6. *There exist $\mu_0 > 0$ sufficiently small and functions $\theta : I_0 \to \mathbb{R}$ and $y : I_0 \to \mathbb{R}$ such that we can write*

$$u(t, x) = e^{i\theta(t)} \{g(t) + Q(x - y(t))\} \quad \textit{for all } t \in I_0,$$

and the following holds:

$$\frac{e^{-2|y(t)|}}{|y(t)|^2} + |y'(t)| + (-\gamma |u(t,0)|^2)^{\frac{1}{2}} \lesssim |\mu(t)| \sim \|g(t)\|_{H^1} \quad \textit{for all } t \in I_0. \tag{2.2}$$

Moreover, letting $g = \lambda Q(\cdot - y) + h$, $\lambda := \langle g_1, Q^p(\cdot - y)\rangle / \langle Q, Q^p \rangle$, and $g = g_1 + i g_2$, we have

$$|\lambda'(t)| \lesssim |\mu(t)|.$$

Proof. We only give the last statement $|\lambda'(t)| \lesssim |\mu(t)|$. The others are proved in [3]. We have $h(t) = e^{-i\theta(t)}\{u(t) - e^{i\theta(t)}(1+\lambda(t))Q(\cdot - y(t))\}$. It holds that

$$\begin{aligned}(2.3)\quad i\partial_t h + \partial_x^2 h = \theta' h + e^{-i\theta}(-\gamma\delta_0 u - |u|^{p-1}u) + \theta'(1+\lambda)Q(\cdot - y)\\ - i\lambda' Q(\cdot - y) + i(1+\lambda)y'\partial_x Q(\cdot - y) + (1+\lambda)(-Q(\cdot - y) + Q^p(\cdot - y)).\end{aligned}$$

from the equation (δNLS) and (1.2). Letting $\mathcal{N}(f) := |f|^{p-1}f$, we have the following nonlinear estimate.

$$\begin{aligned}&|\mathcal{N}(u) - (1+\lambda)\mathcal{N}(Q(\cdot - y))|\\ &= |\mathcal{N}(g + Q(\cdot - y)) - \mathcal{N}(Q(\cdot - y))| + |\lambda|\mathcal{N}(Q(\cdot - y))\\ &\lesssim |Q(\cdot - y)|^{p-1}|g| + |g|^p + |\lambda||Q(\cdot - y)|^p\end{aligned}$$

We have $e^{-i\theta}\delta_0 u = \delta_0(g + Q^p(\cdot - y))$ and the orthogonality properties (see the proof of Lemma 4.7 in [3]):

$$\langle h_1, \partial_x Q(\cdot - y)\rangle = \langle h_1, Q^p(\cdot - y)\rangle = \langle h_2, Q(\cdot - y)\rangle = 0,$$

where $h = h_1 + ih_2$. By the orthogonality, we also have $\langle \partial_t h_1, Q^p(\cdot - y)\rangle = 0$. Therefore, multiplying (2.3) with $Q^p(\cdot - y)$ and taking integral and imaginary part, we obtain

$$\begin{aligned}|\lambda'| &\lesssim \int_{\mathbb{R}} \partial_x^2 h_2 Q^p(\cdot - y)dx - \gamma|g_2(t,0)Q^p(-y)|\\ &\quad + \int_{\mathbb{R}} |Q(\cdot - y)|^{2p-1}|g|dx + \int_{\mathbb{R}} |g|^{p-1}|Q^p(\cdot - y)|dx + \int_{\mathbb{R}} |\lambda||Q^{2p}(\cdot - y)|dx\\ &\lesssim \|h\|_{H^1} + \|g\|_{H^1} + |\lambda|\\ &\lesssim |\mu(t)|,\end{aligned}$$

where we used $\|g\|_{H^1} \sim \|h\|_{H^1} \sim |\lambda| \sim |\mu(t)|$ (see the proof of [3, Lemma 4.7]). This shows that $|\lambda'(t)| \lesssim |\mu(t)|$. □

§ 2.4. Proof

Proposition 2.7. *We assume the same assumption in Proposition 2.3. Suppose that u is global in positive time direction. Then we have*

$$\Im \int_{\mathbb{R}} x\partial_x u(t,x)\overline{u(t,x)}dx > 0$$

for all existence time t. Moreover, there exists $c > 0$ such that

$$\int_t^\infty |\mu(s)|ds \lesssim e^{-ct}$$

for any $t > 0$.

Proof. Let $J(t) = \int_{\mathbb{R}} x^2|u(t,x)|^2dx$. Then, it holds by [20, Proposition 10] that

$$J'(t) = c_1\Im\int_{\mathbb{R}} x\partial_x u\overline{u}dx, \quad J''(t) = c_2P(u(t)).$$

We will first show that $J'(t) > 0$ for all existence time. If not, there exists t_1 such that $J'(t_1) \leq 0$. Since $J'' < 0$ for all existence time, we have

$$J'(t_2) - J'(t_1) = \int_{t_1}^{t_2} J''(s)ds = c_2\int_{t_1}^{t_2} P(u(s))ds < 0$$

for $t_2 > t_1$. Thus, we have $J'(t_2) < J'(t_1) \leq 0$. For any $t > t_2$, we also have

$$J'(t) - J'(t_2) = \int_{t_2}^{t} J''(s)ds = c_2\int_{t_2}^{t} P(u(s))ds < 0$$

and thus $J'(t) < J'(t_2) < 0$ for any $t > t_2$. This means there exists t^* such that $J(t^*) = 0$. This is a contradiction to that u is a non-zero forward global solution.

Next, we will show that $J'(t) \lesssim e^{-ct}$ for $t \geq 0$. By Corollary 2.5 as $\varphi(x) = x^2$ and $f(x) = u(t,x)$, we obtain

$$|J'(t)|^2 \lesssim (J''(t))^2J(t)$$

for all existence time t. Since $J > 0$, $J' > 0$, and $J'' < 0$, we obtain

$$\frac{J'(t)}{\sqrt{J(t)}} \lesssim -J''(t). \tag{2.4}$$

Integrating this on $(0,t)$, we get

$$\sqrt{J(t)} - \sqrt{J(0)} \lesssim -J'(t) + J'(0) \lesssim J'(0).$$

This means that J is bounded on $(0,\infty)$. Using this boundedness and (2.4) again, we have $J'(t) \lesssim -J''(t)$. This implies $J'(t) \lesssim e^{-ct}$ for $t \geq 0$. We obtain

$$0 \leq -\int_t^\infty \mu(s)ds \lesssim -\int_t^\infty P(u(s))ds \approx -\int_t^\infty J''(s)ds = -[J'(s)]_{s=t}^{s=\infty} = J'(t) \lesssim e^{-ct},$$

where we use $-\mu \lesssim -P$ (see the proof of Corollary 2.5). This completes the proof. □

Corollary 2.8. *We assume the assumption of Proposition 2.3 and that the solution is global in positive time, then u blows up in negative time.*

Proof. Suppose that u is global in negative time. Set $v(t,x) = \overline{u(-t,x)}$. Then, v is a solution of (δNLS) satisfying the above assumption. Thus, it holds that

$$\Im\int_{\mathbb{R}} x\partial_x v(t,x)\overline{v(t,x)}dx > 0$$

for all t. We get

$$0 < \Im \int_{\mathbb{R}} x\partial_x v(-t,x)\overline{v(-t,x)}dx = \Im \int_{\mathbb{R}} x\overline{\partial_x u(t,x)}u(t,x)dx$$
$$= -\Im \int_{\mathbb{R}} x\partial_x u(t,x)\overline{u(t,x)}dx < 0$$

This is a contradiction. □

Proof of Proposition 2.3. Suppose that u is global in positive time direction. Then, by Corollary 2.8, the solution blows up in negative time. We have $\liminf_{t\to\infty}\mu(t) = 0$ by Proposition 2.7. Thus, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $\mu(t_n) \to 0$ as $n \to \infty$. We will prove that $\mu(t) \to 0$ as $t \to \infty$. If not, there exists $\varepsilon_1 \in (0, \mu_0)$ and $\{t_n'\}$ such that $-\mu(t_n') > \varepsilon_1$. We can take a sequence $\{t_n''\}$ such that

$$t_n < t_n'', \quad -\mu(t_n'') = \varepsilon_1, \quad -\mu(t) < \varepsilon_1 \text{ for all } t \in [t_n, t_n'').$$

On the interval $[t_n, t_n'']$, the parameter λ is well defined. By Proposition 2.6, we have

$$|\lambda(t_n'') - \lambda(t_n)| \leq \int_{t_n}^{t_n''} |\lambda'(t)|dt \lesssim e^{-ct_n} \to 0$$

as $n \to \infty$. By the definition of t_n, we have $|\lambda(t_n)| \sim |\mu(t_n)| \to 0$. However, we have $|\lambda(t_n'')| \sim |\mu(t_n'')| = \varepsilon_1 > 0$ by Proposition 2.6 and the definition of t_n''. This is a contradiction. This means that $\mu(t) \to 0$ as $t \to \infty$.

Therefore, it follows from Proposition 2.6 that

$$|y(t_2) - y(t_1)| = \int_{t_1}^{t_2} |y'(t)|dt \lesssim \int_{t_1}^{t_2} |\mu(t)|dt \lesssim e^{-ct_1}$$

for large $t_2 > t_1$. This implies that $y(t)$ converges to $y_\infty \in \mathbb{R}$ as $t \to \infty$. However, this means that

$$\frac{e^{-2|y(t)|}}{|y(t)|^2} \to \frac{e^{-2|y_\infty|}}{|y_\infty|^2} > 0.$$

This contradicts $e^{-2|y(t)|}/|y(t)|^2 \lesssim |\mu(t)| \to 0$ as $t \to \infty$. As a consequence, the solution is not global in positive time direction. □

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